On the Approximation of a Parabolic Inverse Problem by Pseudoparabolic One

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The properties of the solution to the inverse problem on the identification of the leading coefficient of the multi-dimensional pseudoparabolic equation are studied. It is proved that the inverse problem for the pseudoparabolic equation approximates the appropriate inverse problem for the parabolic equation of filtration. The existence and uniqueness of the solution to the parabolic inverse problem is established.

Keywords: filtration, inverse problems for PDE, pseudoparabolic equation, parabolic equation, existence and uniqueness theorems.

Introduction

An inverse problem for the pseudoparabolic equation

\[(u + L_1 u)_t + L_2 u = f\]  \hspace{1cm} (0.1)

with the differential operators \(L_1\) and \(L_2\) of the second order in spatial variables is discussed in this paper. We are interested in finding the leading coefficients of \(L_2\) in (0.1) from the additional boundary data. Applications of this problem deal with the recovery of unknown parameters indicating physical properties of a natural stratum which should be determined on the basis of the investigation of its behaviour under the natural non-steady-state conditions (see [1] for details). This leads to the interest in studying the inverse problems for (0.1) and its analogue.

The investigation of inverse problems for pseudoparabolic equations goes back into 1980s. The first result obtained by Rundell in [2] is concerned with the inverse problems of the identification of an unknown source \(f\) in the (0.1) with linear elliptic operators \(L_1\) and \(L_2\), \(L_1 = L_2\). Rundell proved the global existence and uniqueness theorems in the case that \(f\) depends only on \(x\) or \(t\). Another kind of inverse problems is considered in [3,4]. These works are devoted to problems of reconstructing the kernels in integral term of (0.1) with the integro-differential operator \(L_2\). As for the determination of unknown coefficients in (0.1) we mention the results of Mamayusupov [5], Lubanova and Tani [6]. Mamayusupov proved the uniqueness theorem and found an algorithm for solving the inverse problem with respect to \(u(t,x)\), functions \(b(y), c(y)\) and a constant \(a\) for the equation

\[u_t - \Delta u_t = a \Delta u + b(y)u_y + c(y) + \delta(t,x,y), \quad \text{for } (x,y) \in \mathbb{R}^2, \quad t > 0\]

provided that \(u(t,x,0), \quad u_y(t,x,0)\) and \(u(0,x,y)\) are given. Here \(\delta(t,x,y)\) is the Dirac delta function.

In [6] an inverse problem of identification of an unknown leading coefficient in the operator \(L_2\) for (0.1) was discussed (see Problem 1 below). The existence, uniqueness and regularity of

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the solution to the inverse problem were established there. The statement of the inverse problem was motivated in [1].

A main goal of this paper is to investigate the behavior of the solution to the inverse problem considered in [6] as \( \eta \to 0 \). It is well known [7] that when passing to the limit \( \eta \to 0 \) equation (0.1) formally tends to coincide with the standard linear equation of filtration in a porous medium

\[
    u_t + L_2 u = f, \tag{0.2}
\]

The direct initial boundary value problem for pseudoparabolic equation in a bounded domain \( \Omega \subset \mathbb{R}^n \) approximates the appropriate problem for parabolic equation [8]. In particular, under certain assumptions the solution \( u^\eta \) of equation (0.1) with the initial data \( u^\eta(0, x) = u_0(x) \) tends to the solution \( u \) of (0.2) with the same initial condition in the \( L^2 \)-norm for all \( t \geq 0 \) as \( \eta \to 0 \). It was established in [1] that the inverse problem for the pseudoparabolic equation also approximates weakly the appropriate inverse problem for the parabolic one in the case when \( L_1 = \eta \partial^2 / \partial x^2 \), \( L_2 = k(t) \partial^2 / \partial x^2 \), \( \eta \) is a positive real number and \( k(t) \) is an unknown coefficient. In the present paper this result will be extended to the inverse problems for (0.1) and (0.2) with any number of space variables. Such an investigation is also of an interest in studying the inverse problems for evolution equations whose principal terms contain unknown coefficients. The considerable results in this sphere are obtained for parabolic equations (see [9–12] and references given there).

The paper is organized as follows. In Section 1 for the convenience of the reader we repeat the main results of the paper. Section 2 contains the conclusions and comments to the main results of the paper.

### 1. Preliminaries

Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with a boundary \( \partial \Omega \in C^2 \), \( T \) an arbitrary real number and \( Q_T = \Omega \times (0, T) \). Throughout this paper we use the notation:

\( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) are the norm and the inner product of \( L^2(\Omega) \), respectively;

\( \| \cdot \|_j \) and \( \langle \cdot, \cdot \rangle_j \) are the norm of \( W^j_2(\Omega) \) and the duality relation between \( W^j_2(\Omega) \) and \( W^{-j}_2(\Omega) \), respectively (\( j = 1, 2 \)); as usual \( W^j_2(\Omega) = L^2(\Omega) \).

Let \( M : W^1_2(\Omega) \to (W^1_2(\Omega))^* \) be a linear differential operator of the form

\[
    Mv = -\text{div}(\mathcal{M}(x)\nabla v) + m(x)v, \tag{1.1}
\]

where \( \mathcal{M}(x) \equiv (m_{ij}(x)) \) is a matrix of functions \( m_{ij}(x) \), \( i, j = 1, 2, \ldots, n \). We assume that the following conditions are fulfilled.

I. \( m_{ij}(x) \), \( \partial m_{ij} / \partial x_l \), \( i, j, l = 1, 2, \ldots, n \), and \( m(x) \) are bounded in \( \Omega \). \( M \) is an operator of elliptic type, that is, there exist positive constants \( m_1 \) and \( m_2 \) such that for any \( v \in W^1_2(\Omega) \)

\[
    m_1 \| v \|^2_1 \leq \langle Mv, v \rangle_1 \leq m_2 \| v \|^2_1. \tag{1.2}
\]

II. There exists a positive constant \( m_3 \) such that for any \( v \in W^2_2(\Omega) \)

\[
    \| Mv \| \leq m_3 \| v \|_2. \tag{1.3}
\]

III. \( m_{ij}(x) = m_{ji}(x) \) for \( i, j = 1, 2, \ldots, n \) and \( m(x) \geq 0 \) for \( x \in \Omega \).

We proceed to study the following inverse problem [6].
Theorem 1.1. Let the assumptions I–III be fulfilled and \( \eta \) be a positive constant. Assume that
(i) \( f \in C([0,T]; L^2(\Omega)), \beta \in C^1([0,T]; W^{3/2}_2(\partial \Omega)), U_0 \in L^2(\Omega), g \in C(\overline{Q}_T), \)
\( \omega \in C^1([0,T]; W^{3/2}_2(\partial \Omega)), \varphi_1 \in C^2([0,T]), \varphi_2 \in C([0,T]); \)
(ii) \( f, U_0, \beta, \omega, \varphi_1 \) are nonnegative and
\[
\int_{\Omega} h^n \, dx \geq h_0 = \text{constant} > 0, \quad t \in [0,T];
\]  
(iii) there exist constants \( \alpha_i, i = 0, 1, 2, \) such that \( 0 \leq \alpha_0, \alpha_2 \leq 1, \alpha_0 + \alpha_1 < 2, \)
\[
(1 - \alpha_0) \varphi_1(t) + (1 - \alpha_1) \varphi_1(t) \geq \alpha_2 = \text{constant} > 0, \quad t \in [0,T],
\]
\[ \chi(0) + a(0, x) - U_0(x) \geq 0 \text{ for almost all } x \in \Omega, \]
\[ g(t, x)\chi(t) + \chi'(t) + F(t, x) \geq 0 \text{ for almost all } (t, x) \in QT, \]
where \( \chi(t) = \eta (\alpha_0 \varphi_1(t) + \alpha_1 \Psi(t)) \left[ \int_{\Omega} h^\eta \, dx \right]^{-1}; \)
(iv) for any \( t \in [0, T] \)
\[ \Phi^\eta(t) \geq \Phi^\eta_0 = \text{constant} > 0 \]
holds and \( g(t, x) \) satisfies the inequality
\[ \max_{Q_T} g(t, x) \leq \frac{\Phi^\eta_0}{\eta} \left[ \overline{\varphi}_1 + \overline{\Psi} + 2\eta^{-1} \max_{[0, T]} (a, h^\eta) \right]^{-1} = \frac{k_0}{\eta}. \]

Then Problem 1 has a unique solution \( (u, k) \in C^1([0, T]; W^2_2(\Omega)) \times C([0, T]). \) Moreover, \( u \) and \( k \) satisfies the estimates
\[ 0 \leq u(t, x) \leq \chi(t) + a(t, x) \text{ for almost all } (t, x) \in QT, \quad (1.12) \]
\[ \|u(t)\|_1^2 + \|u_t(t)\|_2^2 + \eta \left( \|u(t)\|_2^2 + \|u_t(t)\|_2^2 \right) \leq C, \quad t \in [0, T], \quad (1.13) \]
\[ k_0 \leq k(t) \leq k_1 \quad (1.14) \]
with positive constants \( C \) and \( k_1 = \alpha_2^{-1} \max_{t \in [0, T]} \{ \Phi^\eta(t) + ([g(a + \chi(t)), h^\eta]) \}. \]

2. Approximation of Parabolic Inverse Problem

As mentioned above, when passing to the limit \( \eta \to 0 \) equation (0.2) formally tends to coincide with the linear parabolic equation and Problem 1 transforms to the following parabolic inverse problem.

PROBLEM 2. Given \( f(t, x), g(t, x), \beta(t, x), u_0(x), \phi_1(t), \phi_2(t); \) find the pair of functions \( (u(t, x), k(t)) \) satisfying the equation
\[ u_t + k(t) Mu + g(t, x)u = f(t, x), \quad (t, x) \in QT, \quad (2.1) \]
and the conditions
\[ u|_{t=0} = u_0(x), \quad x \in \Omega, \quad (2.2) \]
\[ u|_{\partial \Omega} = \beta(t, x), \quad t \in [0, T], \quad (2.3) \]
\[ k(t) \int_{\Omega} \frac{\partial u}{\partial \nu} \omega \, ds + \phi_1(t) k(t) = \phi_2(t), \quad t \in (0, T). \quad (2.4) \]

Hereafter, by the solution of Problem 2 we mean a pair \( (u(t, x), k(t)) \) such that
a) \( u \in V = \{ v \mid v \in L^\infty(0, T; W^2_2(\Omega)), v_t \in L^\infty(0, T; L^2(\Omega)) \}, k(t) \in L^\infty(0, T); \)
b) system (2.1)–(2.4) is satisfied.

We shall denote the solutions of Problem 1 with the initial data
\[ (u^\eta + \eta Mu^\eta)|_{t=0} = u_0 + \eta Mu_0 \equiv U_0 \quad (2.5) \]
and Problem 2 by \( (u^\eta, k^\eta) \) and \( (u, k) \), respectively.

In this section we make use of the inequality
\[ \left\| \frac{\partial u}{\partial \nu} \right\|_{L^\infty(\partial \Omega)} \leq C_2 \left( \|v\|_2^2 + \|v\|_1^{1-\alpha} + \|v\|_1 \right) \quad (2.6) \]
valid for any $v \in W^2_{0}(\Omega)$ where $\alpha = \frac{n}{2} - \frac{n - 1}{q}$, $q \in \left[\frac{2(n-1)}{n}, \frac{2(n-1)}{n-2}\right]$ for $n \leq 3$ and $q \in [1, \infty]$ for $n = 2$. (2.6) is easily derived from the multiplicative inequality [13]. The constant $C_2$ depends on $n$, $q$, $\text{mes} \Omega$, $m_2$ and $m_3$. We also use the property of the function $h^\eta$ established by the following lemma.

**Lemma 2.1.** Let $\omega \in C([0, T]; W^{3/2}_{0}(\Omega))$. Then the solution of the problem (1.9) satisfies the estimate
\[
\|h^\eta\|^2 + \eta\|h^\eta\|^2_1 \leq \eta C_3
\]
where a positive constant $C_3$ depends on $n_2$, $q_1$, $\text{mes} \Omega$, $\|h\|$ and does not depend on $\eta$.

**Proof.** To obtain the estimate (2.7) we multiply the equation (1.9) by $h^\eta$ in terms of $L^2(\Omega)$ and integrate by parts in the left-hand side. This gives
\[
\|h^\eta\|^2 + \eta \langle M h^\eta, h^\eta \rangle_1 = \eta \int_{\partial \Omega} \frac{\partial h^\eta}{\partial \nu} \omega \, ds.
\]
By Hölder’s inequality for $n \geq 2$
\[
\left| \int_{\partial \Omega} \frac{\partial h^\eta}{\partial \nu} \omega \, ds \right| \leq \left\| \frac{\partial h^\eta}{\partial \nu} \right\|_{L^p(\partial \Omega)} \left\| \omega \right\|_{L^{p/(p-1)}(\partial \Omega)}
\]
where $p = 2(n-1)/n$. From (2.6) and the embedding theorem [13] it follows that for any $v \in W^2_{0}(\Omega)$
\[
 \left\| \frac{\partial v}{\partial \nu} \right\|_{L^p(\partial \Omega)} \leq C_4 \|v\|_1, \quad \|v\|_{L^p/(p-1)(\partial \Omega)} \leq C_5 \|v\|_2.
\]
Here constants $C_4$ and $C_5$ depend on $m_2$, $m_3$, $n$ and $\text{mes} \Omega$. Applying (2.10) to (2.9) yields
\[
\left| \int_{\partial \Omega} \frac{\partial h^\eta}{\partial \nu} \omega \, ds \right| \leq C_4 C_5 \|h^\eta\|_1 \|b\|_2 \equiv C_6 \|h^\eta\|_1.
\]
Estimating the right-hand side of (2.8) with the help of this inequality, one can obtain the estimate (2.7). The lemma is proved.

The main result of this section is formulated in the next theorem.

**Theorem 2.2.** Let $\eta \in (0, \eta_0]$, $n \geq 2$, the condition (ii) of Theorem 1.1 and the assumptions I–III are fulfilled. Let
\[
(i) \quad f \in L^2(0, T; W^2_{0}(\Omega)) \cap C(\overline{Q}_T), \quad \beta \in C^1([0, T]; W^{3/2}_{0}(\partial \Omega)), \quad u_0 \in W^2_{0}(\Omega), \quad g \in C(\overline{Q}_T),
\]
\[
(\omega \in C^1([0, T]; W^{3/2}_{0}(\partial \Omega)), \quad \varphi_1 \in C^1([0, T]), \quad \varphi_2 \in C([0, T]);
\]
\[
(ii) \quad u_0 \text{ and } \beta \text{ obey the compatibility condition } u_0(x) |_{\partial \Omega} = \beta(0, x),
\]
\[
a(0, x) - u_0(x) - \eta_0 M u_0 \geq 0, \quad x \in \Omega,
\]
\[
F(t, x) \geq 0, \quad (t, x) \in Q_T,
\]
\[
\phi_1(t) + \Psi(t) \geq \alpha_2 = \text{const} > 0, \quad t \in [0, T].
\]
\[
(iv) \quad \text{there exist positive constants } \overline{\phi}_2, \underline{\phi}_2 \text{ such that}
\]
\[
\overline{\phi}_2 \leq \phi_2(t) \leq \underline{\phi}_2, \quad t \in [0, T],
\]

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Then
\[ u^0 \to u \quad \text{weakly in } L^\infty(0, T; W^2_2(\Omega)), \]
\[ u_t^0 \to u_t \quad \text{weakly in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; W^1_2(\Omega)), \]
\[ k^0 \to k \quad \text{weakly in } L^\infty(0, T) \]
as \eta \to 0. Moreover,
\[ 0 < r(\eta) \leq k^0(t) \leq \alpha_2^{-1}(\bar{\phi}_2 + \max_{t \in [0, T]} (ga, h^0)) \equiv k_2 \quad (2.14) \]
where \( r(\eta) \) is a continuous function of \( \eta \) on \([0, \eta_0]\) and \( r(0) > 0 \).

Proof. Without loss of generality we can assume \( \eta_0 \) to be chosen so that \( \eta_0 \leq 1 \),
\[ 0 < \Phi_0^{\eta_0} \equiv \phi_2 - \max_{t \in [0, T]} \left\{ \frac{\eta_0}{2} ||Ma_t||_1 ||b|| + ||a_t|| ||h^\infty|| \right\} \leq \Phi^0(t) \leq \phi_2, \quad (2.15) \]
because of \((2.7)\). Therefore the hypotheses of the theorem imply that all assumptions of Theorem 1.1 are fulfilled with \( \alpha_0 = \alpha_1 = 0 \). This shows that Problem 1 has a unique solution \((u^\eta(t, x), k^\eta(t)) \in C^1([0, T]; W^2_2(\Omega)) \times C([0, T])\) and the estimates \((1.12)\)–\((1.14)\) hold for any \( \eta, 0 < \eta \leq \eta_0 \). Our next step is to get a uniform lower bound \((2.14)\) for \( k^\eta \) and then uniform estimates for the derivatives of \( u^\eta \).

Let us set
\[ w^\eta(t, x) = a(t, x) - u^\eta(t, x). \quad (2.16) \]
The function \( w(t, x) \) satisfies the equation
\[ w_t^\eta + \eta M w_t^\eta + k^\eta(t) M w^\eta + g(t, x) w^\eta = F(t, x), \quad (t, x) \in Q_T, \quad (2.17) \]
and the conditions
\[ (w^\eta + \eta M w^\eta)|_{t=0} = a(0, x) - U_0(x), \quad x \in \Omega, \quad (2.18) \]
\[ w^\eta|_{\partial \Omega} = 0, \quad t \in [0, T], \quad (2.19) \]
\[ \int_{\partial \Omega} \left\{ \eta \frac{\partial w^\eta}{\partial \nu} + k^\eta \frac{\partial w^\eta}{\partial \nu} \right\} \omega dS = (\varphi_1 + \Psi) k^\eta + \eta (Ma_t, h^\eta)|_{t=0} - \varphi_2, \quad t \in [0, T]. \quad (2.20) \]
As was shown in [6], multiplying \((2.17)\) by \( h^\eta(t, x) \) in terms of \( L^2(\Omega) \), the integration by parts in the left side and substituting \((2.20)\) into the resulting equation leads to the equation
\[ k^\eta(t) \left( \varphi_1(t) + \Psi(t) + \frac{1}{\eta} (w^\eta, h^\eta) \right) = \Phi^\eta(t) - (g(t, x)(a - w^\eta), h^\eta) \quad (2.21) \]
by virtue of \((1.8), (1.9), (1.10), (1.11)\).

According to Theorem 1.1 the pair \((u^\eta, k^\eta) \in C^1([0, T]; W^2_2(\Omega))\) for every \( 0 < T < +\infty \). Since the problems \((2.17)\)–\((2.20)\) and \((2.17)\)–\((2.19)\) are equivalent, the pair \((u, k)\) also solves the problem \((2.17)\)–\((2.19)\), \((2.21)\).

Let us set
\[ k_0^\eta = \min_{t \in [0, T]} k^\eta(t). \quad (2.22) \]
We multiply (2.17) by \( Mw^\eta \) in terms of the inner product of \( L^2(\Omega) \) and integrate by parts in the following way:

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{1,M}^2 + \frac{\eta}{2} \frac{d}{dt} \|Mw^\eta\|^2 + k_0^\eta \|Mw^\eta\|^2 = -(gw^\eta, Mw^\eta)_{1,M} - \int_{\partial \Omega} F \frac{\partial w^\eta}{\partial \nu} \, ds + (F, Mw^\eta)_{1,M}.
\]  

(2.23)

By (1.2), (1.3), (2.6) and the Young inequality,

\[
\left| \int_{\partial \Omega} F \frac{\partial w^\eta}{\partial \nu} \, ds \right| \leq C_7 \left( \left( \frac{1}{(k_0^\eta)^{1/2}} + 1 \right) \|F\|_{1}^2 + \|w^\eta\|_{1,M}^2 \right) + \frac{k_0^\eta}{4} \|Mw^\eta\|^2
\]

(2.24)

where \( C_7 = \text{const} > 0 \) depends on \( C_2, \text{mes}\Omega, m_i, i = 1, 2, 3 \). Then (1.1), (2.23), (2.24) give

\[
\|w\|_{1,M}^2 + \eta \|Mw^\eta\|^2 + k_0^\eta \int_0^t \|Mw^\eta\|^2 \, d\tau \leq \frac{C_8}{(k_0^\eta)^{1/2}} + C_9 \int_0^t \|w\|_{1,M}^2 \, d\tau.
\]

(2.25)

The positive constants \( C_8 \) and \( C_9 \) depends on \( C_7, \|g\|_{C^1(\Omega^*_{T})}, m_i, i = 1, 2, 3 \). In accordance with Gronwall’s lemma, it follows from (2.25) that

\[
\|w\|_{1,M}^2 + \eta \|Mw^\eta\|^2 + k_0^\eta \int_0^t \|Mw^\eta\|^2 \, d\tau \leq \frac{C_{10}}{(k_0^\eta)^{1/2}}.
\]

(2.26)

Here \( C_{10} = \text{const} > 0 \) depends on \( C_8, C_9 \) and does not depend \( \eta \) and \( k_0^\eta \).

Let us come back to the equation (2.21). We first note that the numerator of (2.21) is bounded below by a positive constant independent of \( k_0^\eta \) when \( \eta_0 \) is small enough. Indeed, by (2.7) and (2.26),

\[
\left| \{gw^\eta, h^\eta\} \right| \leq C_{11} \eta^{1/2}
\]

(2.27)

The constant \( C_{11} > 0 \) depends on \( C_2, C_{10} \) and does not depend on \( \eta \) and \( k_0^\eta \). Thus, (2.13), (2.15) and (2.27) give

\[
\phi_2 - \eta (Ma_i, h^\eta)_{1,M} - (F, h^\eta) + \{gw^\eta, h^\eta\} \geq \phi_2 - C_{12} \eta^{1/2}.
\]

(2.28)

Here the positive constant \( C_{12} \) depends on \( C_{11}, \|g\|_{C^1(\Omega^*_{T})}, \max_{\|a\|\leq \|g\|} \|a\| \) and does not depend on \( \eta \) and \( k_0^\eta \). If we choose \( \eta_0 < (\phi_2 C_{12}^{-1})^2 \), then \( \phi_2 - C_{12} \eta^{1/2} > 0 \). Furthermore, by (2.10), (2.26),

\[
\frac{1}{\eta} \left( w^\eta, h^\eta \right) \leq \left| \int_{\partial \Omega} \frac{\partial w^\eta}{\partial \nu} \, \omega \, ds \right| + \left( Mw^\eta, h^\eta \right) \leq \frac{C_{13}}{(k_0^\eta)^{1/4}}
\]

(2.29)

where \( C_{13} = \text{const} > 0 \) depends on \( C_4, C_5, C_{10}, \eta_0, m_i, i = 1, 2, 3 \), and does not depend on \( \eta \) and \( k_0^\eta \). Thus, by (2.12), (2.21), (2.22), (2.28), (2.29), we have

\[
k_0^\eta \geq C_{14} \left( k_0^\eta \right)^{1/4} \left[ \alpha_2 (k_0^\eta)^{1/4} + C_{13} \right]^{-1},
\]

whence

\[
\alpha_2 k_0^\eta + C_{13} (k_0^\eta)^{3/4} - C_{14} \geq 0.
\]

(2.30)

Here \( C_{14} = \phi_2 - C_{12} \eta^{1/2} \). Since there exists a unique positive real root \( \eta_0 \) of the equation

\[
G(\eta) \equiv \alpha_2 \eta^4 + C_{13} \eta^3 - C_{14} = 0,
\]

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where the constants $C$ depend on $\eta$ on $(\eta)$ (from (2.33)). Here the constant $C$ we can readily derive the estimate $C$ from (2.34). Let $f$ be the smoothness of $y$ such that $\eta \mapsto r(\eta)$.

The uniform estimates of $u^n$ and $Mw^n$ become evident. By (2.26) and (2.31),

$$\|u^n\|_1^2 + \eta\|Mw^n\|^2 + r(\eta)\int_0^t \|Mw^n\|^2 d\tau \leq C_{15}(r(\eta))^{-1/2} + C_{16}.$$  

The constants $C_{15}, C_{16}$ depend on $C_{10}$, $\|u\|_{C([0,T];W^2_1(\Omega))}$ and do not depend on $\eta$. The uniform estimate of $u^n$ can be derived from (2.17)–(2.19), (2.21), (2.31) and (2.32). Multiplying (2.17) by $Mw^n$ in terms of the inner product of $t^n$ and integrating by parts in the resulting equation we obtain

$$\frac{1}{k^n(t)}\|w^n\|_{2,M}^2 + \frac{\eta}{k^n(t)}\|Mw^n\|^2 + \frac{1}{2}\frac{d}{dt}\|Mw^n\|^2$$

$$= -\frac{1}{k^n(t)}\int_{\partial\Omega} F \frac{\partial w^n}{\partial \nu} \, ds - \frac{1}{k^n(t)}\langle F + gw^n, Mw^n \rangle_{1,M}. \quad (2.33)$$

By the smoothness of $f$, the embedding theorem and (2.10)?

$$\left| \int_{\partial\Omega} F \frac{\partial w^n}{\partial \nu} \, ds \right| \leq C_{17}(\|f\|_{L^\infty} + \|u\|_{W^3_2(\Omega)}) \|w^n\|_1 \quad (2.34)$$

The constant $C_{17}$ depends on $C_1, n$ and $\text{mes}\Omega$. Therefore, taking into account (2.31), (2.32), (2.34) we can readily derive the estimate

$$\int_0^t \|w^n\|_{2,M}^2 \, d\tau + \eta \int_0^t \|Mw^n\|^2 \, d\tau + \|Mw^n\|^2 \leq \frac{C_{18}}{(r(\eta))^{1/2}}$$

from (2.33). Here the constant $C_{18}$ depends on $k_1, c$, $\|Mw_0\|$, $\|F\|_{L^\infty(\Omega)}$ and does not depend on $\eta$. (2.17), (2.19) and (2.35) lead to the estimate

$$\|w^n\|_{2,M}^2 + \eta\|w^n\|_{1,M}^2 \leq \frac{C_{19}}{(r(\eta))^{1/2}} + C_{20} \quad (2.36)$$

where the constants $C_{19}$ and $C_{20}$ depend on $C_{18}, k_2$, $\|F\|_{C([0,T];L^2(\Omega))}$, $\|g\|_{C(\partial\Omega)}$ and does not depend on $\eta$. Thus, from (2.14), (2.32), (2.35), (2.36) it follows that there exists a subsequence $(u^n, k^n)$ of $(u^n, k^n)$ and a pair of functions $(u, k)$ such that

$$u^n \rightharpoonup u \quad \text{weakly in } L^\infty(0, T; W^2_2(\Omega)), \quad (2.37)$$

$$u^n_{\alpha} \rightharpoonup u_{\alpha} \quad \text{weakly in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; W^1_2(\Omega)), \quad (2.38)$$

$$k^n \rightharpoonup k \quad \text{weakly in } L^\infty(0, T) \quad (2.39)$$
as \( \eta_l \to 0 \). By the compactness theorem [14], (2.37)–(2.39) implies
\[
\begin{align*}
  u^n & \to u \quad \text{in} \quad L^4(0,T;W^{1,4}_2(\Omega)), \\
  k^n & \to k \quad \text{weakly in} \quad L^4(0,T) \quad \text{as} \quad \eta_l \to 0.
\end{align*}
\] (2.40) (2.41)

We are now in a position to show that the pair \((u,k)\) is a solution of Problem 2. In fact, the pair \((u^n,k^n)\) satisfies the identity
\[
\int_0^T \left\{ \left( u^n_t + gu^n, v \right) + \eta_t \left( Mu^n, v \right)_1 + k^n \left( Mu^n, v \right)_1 \right\} \, dt = \int_0^T (f,v) \, dt
\] (2.42)
for every \( v \in L^2(0,T;W^{1,4}_2(\Omega)) \). In view of (2.37)–(2.41) we can pass to the limit in (2.42). Since
\[
\eta_l \int_0^T \left( Mu^n, v \right)_1 \, dt \to 0
\]
as \( \eta_l \to 0 \) (because of (2.36)), we have
\[
\int_0^T \left\{ \left( u_t, v \right) + k(t) \left( Mu, v \right)_1 + \left( gu, v \right) \right\} \, dt = \int_0^T (f,v) \, dt
\] (2.43)
for every \( v \in L^2(0,T;W^{1,4}_2(\Omega)) \). Moreover, by (1.12), (2.14), (2.16), (2.31), (2.32), (2.35) and (2.36), the estimates
\[
\begin{align*}
  r(0) & \leq k(t) \leq \overline{r}_2 \overline{a}_2^{-1}, \\
  0 & \leq u(t,x) \leq a(t,x), \\
  \int_0^T \| u_t \|^2 \, d\tau + \| Mu \|^2_{L^\infty(0,T;L^2(\Omega))} & \leq \frac{C_{18}}{m_1(r(0))^{3/2}} + \int_0^T \| u_t \|^2 \, d\tau, \\
  \| u_t \|_{L^\infty(0,T;L^2(\Omega))} & \leq \left( \frac{C_{19}}{r(0)} \right)^{1/2} + C_{20} + \| a_t \|_{L^\infty(0,T;L^2(\Omega))}^{1/2}
\end{align*}
\] (2.44) (2.45) (2.46) (2.47)
are valid. From (2.43)–(2.47) it follows that the pair \((u,k)\) satisfies equation (2.1) for almost all \((t,x) \in Q_T\). Furthermore, by (1.6), (2.5), (2.37), (2.38) \( u(t,x) \) obeys (2.2), (2.3).

It remains to prove that the condition (2.4) is also fulfilled. Let \( v(t,x) = \hat{v}(t,x)h(t) \) where \( \hat{v}(t,x) \) and \( h(t) \) are arbitrary functions of classes \( L^\infty(0,T;W^{1,4}_2(\Omega)) \) and \( L^2(0,T) \), respectively, \( \hat{v}\big|_{\partial\Omega} = \omega \). Then the identity
\[
\int_0^T \left\{ \left( u^n_t + gu^n, \hat{v} \right) + \eta_t \left( Mu^n, \hat{v} \right)_{1,M} + k^n(t) \left( \left( Mu^n, \hat{v} \right)_{1,M} + \phi_1(t) \right) \right\} h \, dt
\]
holds because of (1.7). A passage to the limit in (2.48) similar to the above yields
\[
\int_0^T \left\{ \left( u_t + gu, \hat{v} \right)_0 + k(t) \left( \left( Mu, \hat{v} \right)_{1,M} + \phi_1 \right) \right\} h \, dt = \int_0^T (f, \hat{v}) + \phi_2 h \, dt.
\] (2.49)

By virtue of (2.1), integrating by parts in the second term of the left-hand side of (2.49) gives
\[
\int_0^T \left\{ k(t) \int_{\partial\Omega} \frac{\partial u}{\partial \nu} ds + \phi_1(t)k(t) - \phi_2(t) \right\} h(t) \, dt = 0
\]
for any \( h(t) \in L^2(0, T) \), which implies that the pair \((u, k)\) satisfies (2.4) for almost all \( t \in (0, T) \). The theorem is proved.

Under the hypotheses of Theorem 2.2 the solution to Problem 2 is unique in the class \( V \times L^\infty(0, T) \).

**Theorem 2.3.** Let the conditions of Theorem 2.2 be fulfilled. Then Problem 2 has a unique solution \((u(t, x), k(t))\). The pair \((u(t, x), k(t))\) satisfies the estimates (2.44)–(2.47) and \( u_2 \in L^2(0, T; W^1_2(\Omega)) \).

**Proof.** The existence of the solution to Problem 2 and the estimates (2.44)–(2.47) were proved in Theorem 2.2. It remains to establish the uniqueness.

Let \((u_1(t, x), k_1(t))\) and \((u_2(t, x), k_2(t))\) be two solutions of Problem 2. Then the pair \((w(t, x), p(t)) = (u_1 - u_2, k_1(t) - k_2(t))\) solutions the problem

\[
\begin{align*}
w_1 - k_1(t)Mw &= -p(t)Mu_2, \\
w \rvert_{t=0} &= w \rvert_{\partial\Omega} = 0, \\
k_1(t) \int_\partial\Omega \frac{\partial w}{\partial \nu} \omega \, ds &= - \phi_2(t) k_2(t)p(t),
\end{align*}
\]

Multiplying (2.50) by \( Mw \) in terms of the inner product of \( L^2(\Omega) \) and integrating by parts, we can easily obtain

\[
\frac{1}{2} \frac{d}{dt} \| w \|^2_{L^2(\Omega)} + k_1(t) \| Mw \|^2 \leq |p(t)| \| Mu_2 \| \| Mw \|.
\]

From (2.6) with \( q = 2, (2.14), (2.52) \) and the Young inequality it follows that

\[
|p(t)| \| Mu_2 \| \| Mw \| \leq C_{21} \| Mu_2 \| \| w \|^{1/2}_{L^2(\Omega)} \| Mw \|^{3/2} \\
\leq \frac{r(0)}{2} \| Mw \|^2 + \frac{C_{21}^2}{2^{r^2(0)}} \| Mu_2 \|^4 \| w \|^2_{L^2(\Omega)}
\]

where \( C_{21} = \text{const} > 0 \) depends on \( C_2, \phi_2, r(0), \alpha_i, m_i, i = 1, 2, 3 \). Since \( u_2 \in L^\infty(0, T; W^2_2(\Omega)) \), according to Gronwall’s lemma, (2.51), (2.53) and (2.54) implies that \( w = 0 \) for almost all \((t, x) \in QT\) and \( p = 0 \) for almost all \( t \in (0, T) \). The theorem is proved.

**Conclusions**

In this paper we discussed the behavior of the solution to the Problem 1 as \( \eta \to 0 \). It was established that Problem 1 for the pseudoparabolic equation approximates weakly Problem 2 for the parabolic one under the hypotheses of Theorem 2.2 when \( \eta \to 0 \). Theorems 1.1 and 2.2 remain true if \( \omega \in C([0, T]; W^2(\partial\Omega)) \) and \( \varphi_1 \in C([0, T]) \).

In general Problem 1 does not approximate Problem 2. As was shown in [1], if the initial and boundary data do not satisfy (2.11), then Problem 1 may be unsolvable.

Theorem 2.2 implies that Problem 2 for the relevant parabolic equation is solved relying on the results on Problem 1. The uniqueness of the solution to Problem 2 is provided by Theorem 2.3.

**References**


Об аппроксимации параболической обратной задачи псевдопараболической задачей

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Исследуется обратная задача идентификации одного из старших коэффициентов псевдопараболического уравнения. Доказывается, что обратная задача для псевдопараболического уравнения аппроксимирует соответствующую обратную задачу для параболического уравнения. Устанавливается также существование и единственность решения параболической обратной задачи.

Ключевые слова: фильтрация, обратные задачи для уравнений в частных производных, псевдопараболическое уравнение, параболическое уравнение, теоремы существования и единственности.