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On Zeros of Holomorphic Functions

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The aim of the article is to find conditions on the coefficients of the Taylor expansion of a holomorphic function in $\mathbb{C}$ that guarantee a absence of zeros.

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The aim of the article is to find conditions on the coefficients of the Taylor expansion of a holomorphic function in $\mathbb{C}$ that guarantee an absence of zeros.

Let a function $f = f(z)$ with respect to complex variable z be holomorphic in a neighborhood of zero in the complex plane $\mathbb{C}$:

$$f(z) = \sum_{k=0}^{\infty} b_k z^k, \quad f(0) = b_0 = 1. \tag{1}$$

Let γ_r be a circle of the form

$$\gamma_r = \{z : \ |z| = r\}, \quad r > 0.$$

Theorem 1. *For function f to be an entire function of finite order of growth which has no zeros, it is necessary and sufficient that for sufficiently small r there exists $k_0 \in \mathbb{N}$ such that*

$$\int_{\gamma_r} \frac{1}{z^k} \frac{df}{f} = 0 \quad \text{при всех} \quad k \geqslant k_0. \tag{2}$$

In this case the minimum k_0 is equal to the order of function.

Recall that the entire function $f(z)$ has a finite order (of growth) if there exists a positive number A such that

$$f(z) = O(e^{r^A}) \quad \text{for} \quad |z| = R \to +\infty.$$

The infimum of such numbers A is called the *order* of function (see, e.g., [2,3]).

Proof. Let the function f be a function of finite order of growth, which has no zeros in $\mathbb{C}$ then it is well known that it has the form: $f(z) = e^{\varphi(z)}$, where $\varphi(z)$ is a polynomial of some degree k_0 (see, e.g., [2, Ch. 7, Sec. 1.5]). Then

$$\int_{\gamma_r} \frac{1}{z^k} \frac{df}{f} = \int_{\gamma_r} \frac{1}{z^k} \varphi'(z)\, dz = 0 \quad \text{при} \quad k > k_0.$$

Conversely, suppose that condition (2) is fulfilled. Since $f(z)$ is holomorphic function in a neighborhood of zero and $f(0) \neq 0$ then values of $f(z)$ lie in a neighborhood of $f(0)$ and this

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neighborhood does not contain the point 0 for sufficiently small $|z|$. Therefore, the holomorphic function $\varphi(z) = \ln f(z)$, $\ln 1 = 0$ is defined in the neighborhood of zero.

Let

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_0 = \ln f(0) = \ln b_0.$$

Then, for sufficiently small r we have

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{z^k} \frac{df}{f} = \frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{z^k} \varphi'(z)\, dz = k a_k. \tag{3}$$

When condition (2) is fulfilled we see that $a_k = 0$ under $k > k_0$. Therefore, $\varphi(x)$ is a polynomial of degree k_0. Consequently, $f(z) = e^{\varphi(z)}$ is an entire function of finite order k_0. □

There exists a recursive relationship between coefficients of f and $\varphi(z)$ (see, e.g., [1, §2, Lemma 2.3]).

Lemma 1. *The following relations are true:*

$$a_k = \frac{(-1)^{k-1}}{k b_0^k} \begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix}$$

and

$$b_k = \frac{b_0}{k!} \begin{vmatrix} a_1 & -1 & 0 & \dots & 0 \\ 2a_2 & a_1 & -2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ka_k & (k-1)a_{k-1} & (k-2)a_{k-2} & \dots & a_1 \end{vmatrix}.$$

Therefore, we have the following statement.

Corollary 1. *For function f to be an entire function of finite order k_0 which has no zeros, it is necessary and sufficient that the determinant*

$$\begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix} = 0 \quad \textit{under} \quad k > k_0, \tag{4}$$

where k_0 is the minimum number with this property.

Example 1. Let

$$f(z) = e^z = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!},$$

i.e, $b_0 = 1$, $b_k = \dfrac{1}{k!}$, $k > 1$.

Let us substitute these values into (4). When $k = 1$ determinant is not equal to zero. For $k > 1$ all determinants are equal to zero since the first two columns are the same. Then function $f(z)$ is of order 1 and it has no zeros in the complex plane.

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О нулях голоморфных функций

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Цель статьи: найти уловия на коэффициенты Тейлора голоморфной функции $\mathbb{C}$, которые гарантируют отсутствие у нее нулей.

Ключевые слова: голоморфная функция, нули функции, целые функции. .