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CONTINUATION OF POWER SERIES BY ENTIRE AND MEROMORPHIC INTERPOLATION OF COEFFICIENTS

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Contents

INTRODUCTION

Analytic functions play a very important role in mathematics and its applications in science. These functions bridge the gap between exact and approximate computations.

One way to identify an analytic function is based on its power series expansion (Weierstrass' approach). The coefficients of a power series expansion of an analytic function carry all the information about properties of this function, including the property of its analytic continuation. This problem and the closely related problem of relationships between singularities of power series and its coefficients have been extensively studied in the last century by Hadamard [1], Lindelöf [2], Pólya [3], Szegö [4], Carlson [5] and many other prominent mathematicians (see the literature list in monograph by Biberbach [6]).

The most effective and complete results were obtained for simple (one-dimensional) series with coefficients interpolated by values $\varphi(k)$ of an entire function $\varphi(z)$ at the natural numbers $k \in N$ (see, for example, [7], [8], [9]).

According to Abel's theorem, the domain of convergence for a one-dimensional series is a disk, therefore, if its sum extends analytically beyond this disk, then it extends across some boundary arc. This arc is called the *arc of regularity.* A description of an open arc of regularity was given in the papers by Arakelian [10], [11]. He gave a criterion for a given arc of a unit circle to be an arc of regularity for a given power series in terms of the indicator function of the interpolating entire function.

Pólya found conditions for analytic continuability of a series to the whole complex plane except some boundary arc [12].

The other side of the problem of analytic continuation is the problem of distribution of singularities of a power series, i.e. points such that the sum of the series does not extend across them [13], [14], [6]. In this context, the cases where all the boundary points are singular are of special interest [15], [16]. Such analytically non extendable series are mainly "strongly lacunar", in other words, these series have "many" monomials with zero coefficients. Examples of such series are

$$\sum_{n=0}^{\infty} z^{n!}, \qquad \sum_{n=0}^{\infty} z^{2^n}, \qquad \sum_{n=0}^{\infty} z^{n^n}.$$

In 1891 Fredholm [17] constructed examples of "moderately lacunar" non extendable series representing infinitely differentiable functions in the closure of the disk of convergence. These series depend on a parameter a and have the following form

$$\sum_{n=0}^{\infty} a^n z^{n^2}, \qquad 0 < a < 1.$$

Here n^2 has the power order 2 respective to the summation index n, therefore we say that Fredholm's series have the lacunarity order 2.

A more general result on non extendable series in terms of lacunarity belongs to Fabry (see [18] or [6]). It claims that if the sequence of natural numbers m_n increases faster than n (i.e. $n = o(m_n)$), then there is a series

$$\sum_{n=0}^{\infty} a_n z^{m_n},$$

converging in the unit disk and not extending across its boundary.

It should be emphasized that the approach to the study of analytic continuation formulated above has been mainly applied to functions of one variable. In the case of multivariate power series many similar problems remain open. Moreover, the applications of multivariate complex analysis in mathematical physics, for example in quantum field theory [19] and thermodynamics [20],[21], motivate further research in this area.

The goal of this thesis is to find multidimensional analogs of theorems by Arakelian and Pólya on the analytic continuability of a power series across parts of the boundary of the domain of convergence. We also aim to describe the conditions for analytic continuability of a power series whose coefficients are interpolated by entire or meromorphic function, and to construct multidimensional examples of Fredholm's moderately lacunar power series with natural boundaries of convergence domains.

In the research we use methods of multivariate complex analysis, in particular, integral representations (Cauchy, Mellin, and Lindelöf representations), multidimensional residues, properties of power series. An important role in the study is played by the interpolation of power series coefficients by analytic functions from such classes as entire functions of exponential type or special meromorphic functions. Accordingly, we use some facts on the growth of interpolating functions, i.e. elements of complex potential theory.

In the problem of natural boundary of the domain of convergence we use the Kovalevskaya phenomenon on unsolvability of the Cauchy problem for the heat equation with temperature initial data.

The first chapter deals with analytic continuation of one-dimensional power series. Here we establish conditions for analytic continuability (or uncontinuability) of series across a given boundary arc. Such conditions are crucial for the development of methods of data and digital signal processing [40]. To be specific, the radius of the convergence disk is assumed to be equal to 1. We distinguish four types of problems related to a boundary arc:

1) continuability to a sector defined by the arc;

2) continuability to a neighborhood of the arc;

3) continuability to the complex plane except some boundary arc;

4) uncontinuability across every boundary point.

Problems 1 and 2 were studied among others by Arakelian, Problem 3 by Pólya. They obtained criteria for continuability of series in terms of entire functions interpolating the coefficients.

In the first section we give conditions for continuability of a power series, whose coefficients are interpolated by values of a meromorphic function. First, let us formulate the results by Arakelian and Pólya.

Consider a power series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{0.1}$$

in $z \in \mathbb{C}$, whose domain of convergence is the unit disk $D_1 := \{z \in \mathbb{C} : |z| < 1\}$. The Cauchy-Hadamard theorem yields that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|f_n|} = 1.$$

We say that a function φ *interpolates* the coefficients of the series (0.1), if

$$\varphi(n) = f_n \text{ for all } n \in \mathbb{N}.$$

Recall (see, Appendix A.1 or [22]) that the *indicator* function $h_\varphi(\theta)$ for an entire function φ is defined as the upper limit

$$h_\varphi(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln|\varphi(re^{i\theta})|}{r}, \quad \theta \in \mathbb{R}.$$

Let Δ_σ be the sector $\{z = re^{i\theta} \in \mathbb{C} : |\theta| \le \sigma\}$, $\sigma \in [0, \pi)$. We denote the open arc $\partial D_1 \setminus \Delta_\sigma$ by γ_σ.

Theorem ([24], [25]) *The sum of the series* (0.1) *extends analytically to the open sector* $\mathbb{C} \setminus \Delta_\sigma$ *if and only if there is an entire function* $\varphi(\zeta)$ *of exponential type interpolating the coefficients* f_n *whose indicator function* $h_\varphi(\theta)$ *satisfies the condition*

$$h_\varphi(\theta) \le \sigma|\sin\theta| \text{ for } |\theta| < \frac{\pi}{2}.$$

We say that the boundary arc γ_σ is an *arc of regularity* for the series (0.1) if it extends analytically to a neighborhood of γ_σ.

Theorem ([10], [11]) *The open arc* $\gamma_\sigma = \partial D_1 \setminus \Delta_\sigma$ *is an arc of regularity of the series* (0.1) *if and only if there is an entire function* $\varphi(\zeta)$ *of exponential type interpolating the coefficients* f_n *whose indicator function* $h_\varphi(\theta)$ *satisfies the conditions:*

$$h_\varphi(0) = 0 \quad \text{and} \quad \overline{\lim_{\theta \to 0}} \frac{h_\varphi(\theta)}{|\theta|} \le \sigma.$$

Problem 3 deals with continuation to the complex plane except the arc $\partial D_1 \cap \Delta_\sigma$. This problem is solved by the following Pólya's theorem.

Theorem ([12]) *The series* (0.1) *extends analytically to* $\mathbb{C}$*, except possibly the arc* $\partial D_1 \cap \Delta_\sigma$*, if and only if there exists an entire function of exponential type* $\varphi(\zeta)$ *interpolating the coefficients* f_n *such that*

$$h_\varphi(\theta) \le \sigma |\sin \theta| \;\; \textit{for} \;\; |\theta| \le \pi.$$

As mentioned above, in the first section we obtain sufficient conditions for analytic continuability of the power series (0.1) in Problems 1-3. This conditions are formulated in terms of meromorphic interpolations of the form

$$\psi(\zeta) = \phi(\zeta) \frac{\prod_{j=1}^{p} \Gamma(a_j \zeta + b_j)}{\prod_{k=1}^{q} \Gamma(c_k \zeta + d_k)}, \tag{0.2}$$

where $\phi(\zeta)$ is entire, $a_j \ge 0, \;\; j = 1, ..., p,$ and

$$\sum_{j=1}^{p} a_j = \sum_{k=1}^{q} c_k. \tag{0.3}$$

Our choice of the interpolation function (0.2) with conditions (0.3) is motivated, in particular, by the fact that the inverse Mellin transformations of some such functions belong to the class of nonconfluent hypergeometric functions [26].

Denote

$$l = \sum_{k=1}^{q} |c_k| - \sum_{j=1}^{p} a_j.$$

An expression of the form

$$\varphi(\zeta) := \phi(\zeta)\frac{\prod_{j=1}^{p} a_j{}^{a_j\zeta}}{\prod_{k=1}^{q} |c_k|^{c_k\zeta}}$$

is called *the associated entire function* for the meromorphic function (0.2).

We prove the following statements.

Theorem 1.1. *The series* (0.1) *extends analytically to the open sector* $\mathbb{C}\setminus\Delta_\sigma$ *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (0.2) *interpolating the coefficients* f_n *such that the indicator of the associated with* $\psi(\zeta)$ *entire function* $\varphi(\zeta)$ *satisfies the conditions*

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \max\{h_\varphi(-\frac{\pi}{2}) + \frac{\pi}{2}l, h_\varphi(\frac{\pi}{2}) + \frac{\pi}{2}l\} \le \sigma.$$

Theorem 1.2. *The open arc* $\gamma_\sigma = \partial D_1 \setminus \Delta_\sigma$ *is an arc of regularity for the series* (0.1) *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (0.2) *interpolating the coefficients* f_n *such that the indicator of the associated with* $\psi(\zeta)$ *entire function* $\varphi(\zeta)$ *satisfies the conditions*

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \overline{\lim_{\theta\to 0}}\frac{h_\varphi(\theta)}{|\theta|} + \frac{\pi}{2}l \le \sigma.$$

Theorem 1.3. *The series* (0.1) *extends analytically to* $\mathbb{C} \setminus (\partial D_1 \cap \Delta_\sigma)$ *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (0.2) *interpolating the coefficients* f_n *such that the indicator of the associated with* $\psi(\zeta)$ *entire function* $\varphi(\zeta)$ *satisfies the conditions*

$$h_\varphi(\theta) + \frac{\pi}{2}l|\sin\theta| \le \sigma|\sin\theta| \ \textit{for} \ |\theta| \le \pi.$$

In section 2 we consider two examples clarifying why interpolation of the coefficients by meromorphic functions, and not by entire, may be more effective. The first example is given by the series

$$f(z)=\sum_{n=0}^{\infty}\frac{(2n-2)(2n-5)...(2n-3(n+2))}{2^{\frac{2}{3}n}n!}z^n,$$

whose coefficients are interpolated by the meromorphic function

$$\psi(\zeta)=\frac{3^{\zeta-1}}{2^{\frac{2}{3}\zeta}}\frac{\Gamma(\frac{2}{3}\zeta+\frac{1}{3})}{\Gamma(\zeta+1)\Gamma(-\frac{1}{3}\zeta+\frac{4}{3})}\ .$$

The associated with $\psi(\zeta)$ entire function $\varphi(z)$ is

$$\varphi(\zeta):=\frac{3^{\zeta-1}}{2^{\frac{2}{3}\zeta}}\frac{(\frac{2}{3})^{\frac{2}{3}\zeta}}{|-\frac{1}{3}|^{\frac{1}{3}\zeta}}\equiv\frac{1}{3}.$$

Here $l=1+\frac{1}{3}-\frac{2}{3}=\frac{2}{3}$. According to Theorem 1.1, the series extends analytically to the open sector $\mathbb{C}\setminus\Delta_{\frac{\pi}{3}}$.

In the third section we study Problem 4. We construct a family of "moderately lacunar" non extendable series whose sums are infinitely differentiable functions in the closure of the convergence disk.

One of the main results in this section is given by Theorem 1.4. It demonstrates that Fredholm's example may be strengthened by reducing the power order of lacunarity from 2 to $1+\varepsilon$. The precise formulation is the following:

If the increasing sequence of natural numbers n_k satisfies the inequality $n_k\ \geq const\times k^{1+\varepsilon}$ with $\varepsilon>0$, then the power series

$$\sum_{k=0}^{\infty}a^k z^{n_k},\qquad 0<a<1$$

is not extendable across the boundary circle and represents infinitely differentiable function in the closed disk.

In chapter 2 we study continuability of power series in several variables. For multiple power series there are significantly less results describing singular subsets on the boundary of the convergence domain, or, in other words, subsets on the boundary such that series analytically extends across them. In the first section we extend Arakelian's result [10] on the arc of regularity formulated above to the case of multiple series.

Consider a multiple power series

$$f(z) = \sum_{k \in \mathbb{N}^n} f_k z^k, \tag{0.4}$$

with the property

$$\overline{\lim_{|k| \to \infty}} \sqrt[|k|]{|f_k| R^k} = 1, \tag{0.5}$$

where $R^k = R_1^{k_1} ... R_n^{k_n}$, and $|k| = k_1 + ... + k_n$. According to the n-dimensional Cauchy-Hadamard theorem ([27], Section 7), the property (0.5) means that R_j constitute the family of *conjugate radii of polydisk of convergence* of the series (0.4).

A subset G on the boundary of the convergence domain is said to be *a regularity set of the series* (0.4) if the sum of the series can be analytically continued across any point of this set.

Let $D_\rho(a) := \{z \in \mathbb{C} : |z - a| < \rho\}$ be an open circle with the centre $a \in \mathbb{C}$ and radius $\rho > 0$. Denote $D_\rho := D_\rho(0)$, and for $\sigma \in (0, \pi]$ by $\gamma_{\sigma,\rho}$ we denote the open arc $\partial D_\rho \setminus \Delta_\sigma$.

In the multivariate case there is no universal definition for the growth indicator of an entire function. Moreover, the information of the growth of an entire function is frequently represented in geometric terms. Following Ivanov [28] (see also [22], Section 3, §3), we introduce the following set which implicitly contains the notion of the growth indicator of an entire function $\varphi(z) \in \mathcal{O}(\mathbb{C}^n)$:

$$T_\varphi(\theta) = \{\nu \in \mathbb{R}^n : \ln |\varphi(re^{i\theta})| \leq \nu_1 r_1 + ... + \nu_n r_n + C_{\nu,\theta}\},$$

where the inequality is satisfied for any $r \in \mathbb{R}^n_+$ with some constant $C_{\nu,\theta}$. Here $re^{i\theta}$ stands for the vector $(r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n})$. Thus, $T_\varphi(\theta)$ is the set of *linear majorants* (up to a shift by $C_{\nu,\theta}$)

$$\nu = \nu(r) = \nu_1 r_1 + ... + \nu_n r_n$$

for the logarithm of the modulus of function φ.

Define the set

$$\mathcal{M}_\varphi(\theta) := \{\nu \in \mathbb{R}^n : \ \nu + \varepsilon \in T_\varphi(\theta), \ \ \nu - \varepsilon \notin T_\varphi(\theta) \quad \text{for any} \ \ \varepsilon \in \mathbb{R}^n_+\},$$

which can be called *a boundary set of linear majorants*.

Let $D \subset \mathbb{C}^n$ be the domain of convergence of the series (0.4). Consider the family of polyarcs $\gamma_{\sigma,R}$:

$$G = \bigcup_R \gamma_{\sigma,R} = \bigcup_R (\gamma_{\sigma_1,R_1} \times ... \times \gamma_{\sigma_n,R_n}) \subset \partial D \tag{0.6}$$

where R runs over the surface of conjugate radii of the convergence of series (0.4), and $\sigma = \sigma(R) = (\sigma_1(R), ..., \sigma_n(R))$.

Theorem 2.1. *A family G of polyarcs (0.6) is the regularity set for the series* (0.4) *if and only if there exists an entire function $\varphi(z)$ interpolating the coefficients f_k such that the following conditions are fulfilled:*
1) $0 \in \mathcal{M}_{R^z\varphi}(0)$,
2) there exists a vector-function $\nu_R(\theta)$ with values in $\mathcal{M}_{R^z\varphi}(\theta)$ to satisfy

$$\overline{\lim_{(\theta_1,..\hat{j}..,\theta_n)\to 0}} \ \overline{\lim_{\theta_j \to 0}} \frac{\nu_j(\theta)}{|\theta_j|} \le \sigma_j(R), \quad j = 1, ..., n.$$

In the second section of Chapter 2 we give conditions for continuability to a sector of a power series whose coefficients are interpolated by values of an entire or a meromorphic function.

Denote

$$T_\varphi := \bigcap_{\theta_j = \pm\frac{\pi}{2}} T_\varphi(\theta_1, ..., \theta_n),$$

$$\mathcal{M}_\varphi := \{\nu \in [0,\pi]^n : \ \nu + \varepsilon \in T_\varphi, \ \ \nu - \varepsilon \notin T_\varphi \quad \text{for all} \ \ \varepsilon \in \mathbb{R}^n_+\}.$$

Let G be a sectorial set of the form

$$G = \bigcup_{\nu \in \mathcal{M}_\varphi} G_\nu, \tag{0.7}$$

where

$$G_\nu = (\mathbb{C} \setminus \Delta_{\nu_1}) \times ... \times (\mathbb{C} \setminus \Delta_{\nu_n}).$$

Theorem 2.2. *The sum of the series* (0.4) *extends analytically to a sectorial set* G *of the form* (0.7) *if there is an entire function* $\varphi(\zeta)$ *of exponential type interpolating the coefficients* f_n *and a vector-function* $\nu(\theta)$ *on* $[-\frac{\pi}{2}, \frac{\pi}{2}]^n$ *with values in* $\mathcal{M}_\varphi(\theta)$ *to satisfy*

$$\nu_j(\theta) \le a|\sin\theta_j| + b\cos\theta_j, \ \ j = 1, ..., n,$$

with some constants $a \in [0,\pi), \ \ b \in [0,\infty)$.

As an example, consider a double power series

$$f(z_1,z_2) = \sum_{k_1,k_2 \in \mathbb{N}^2} \cos\sqrt{k_1k_2}\; z_1^{k_1} z_2^{k_2}, \tag{0.8}$$

whose coefficients are interpolated by values of the entire function

$$\varphi(\zeta_1,\zeta_2) = \cos\sqrt{\zeta_1\zeta_2}.$$

According to Theorem 2.2 the series (0.8) extends to a sectorial domain (0.7), where ν runs over a part of the hyperbola $\nu_1\nu_2 = \frac{1}{4}$:

$$\mathcal{M}_\varphi = \{\nu \in [0,\pi]^2 : \nu_1\nu_2 = \frac{1}{4}\}.$$

In the fourth final section we construct double power series which are not extendable across the boundary of the convergence bidisk

$$U^2 = \{(z_1,z_2) : \ |z_1| < 1, \ |z_2| < 1\}$$

and represent infinitely differentiable functions in $\bar{U}^2 \setminus T^2$, where $T^2 = \{(z_1, z_2) : |z_1| = 1, |z_2| = 1\}$.

These series have the form

$$\sum_{(k_1,k_2)\in A} {z_1}^{k_1} {z_2}^{k_2},$$

where $A = \{(k_1, k_2) \in {Z_+}^2 : k_2 \geq {k_1}^{1+\varepsilon}\} \cup \{(k_1, k_2) \in {Z_+}^2 : k_1 \geq {k_2}^{1+\varepsilon}\}$, $\varepsilon > 0$.

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Chapter 1. Analytic continuation of one-dimensional power series

1.1 Continuation by means of meromorphic interpolation of coefficients

Consider a power series

$$f(z)=\sum_{n=0}^{\infty} f_n z^n \tag{1.1}$$

in $z\in\mathbb{C}$ whose domain of convergence is the unit disk $D_1 := \{z\in\mathbb{C} : |z|<1\}$. The Cauchy-Hadamard theorem yields that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|f_n|}=1. \tag{1.2}$$

We say that a function φ *interpolates* the coefficients of the series (1.1) if

$$\varphi(n)=f_n \;\text{ for all }\; n\in\mathbb{N}. \tag{1.3}$$

Recall (see, e.g. [22]) that the *indicator* function $h_\varphi(\theta)$ for an entire function φ is defined as the upper limit

$$h_\varphi(\theta)=\overline{\lim_{r\to\infty}} \frac{\ln|\varphi(re^{i\theta})|}{r}, \;\; \theta\in\mathbb{R}.$$

Let Δ_σ be the sector $\{z=re^{i\theta}\in\mathbb{C} : |\theta|\le\sigma\},\ \sigma\in[0,\pi)$. By γ_σ we denote the open arc $\partial D_1\setminus\Delta_\sigma$.

We consider interpolating meromorphic functions of the form

$$\psi(\zeta)=\phi(\zeta)\frac{\prod_{j=1}^{p}\Gamma(a_j\zeta+b_j)}{\prod_{k=1}^{q}\Gamma(c_k\zeta+d_k)}, \tag{1.4}$$

where $\phi(\zeta)$ is entire, $a_j\ge 0,\;\; j=1,...,p,$ and

$$\sum_{j=1}^{p} a_j=\sum_{k=1}^{q} c_k. \tag{1.5}$$

Our choice of the interpolation function (1.4) with conditions (1.5) is motivated, in particular, by the fact that the inverse Mellin transformations of some such functions belong to the class of nonconfluent hypergeometric functions [26].

1.1.1 Conditions for continuability to a sector

Denote

$$l := \sum_{k=1}^{q} |c_k| - \sum_{j=1}^{p} a_j.$$

An expression of the form

$$\varphi(\zeta) := \phi(\zeta)\frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta}}{\prod_{k=1}^{q} |c_k|^{c_k\zeta}} \tag{1.6}$$

is called *the associated entire function* for the meromorphic function (1.4).

Theorem 1.1. *The series* (1.1) *extends analytically to the open sector* $\mathbb{C}\setminus\Delta_\sigma$ *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (1.4) *interpolating the coefficients* f_n *such that the indicator of the associated with* $\psi(\zeta)$ *entire function* $\varphi(\zeta)$ *satisfies the conditions*

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \max\{h_\varphi(-\frac{\pi}{2}) + \frac{\pi}{2}l, h_\varphi(\frac{\pi}{2}) + \frac{\pi}{2}l\} \leq \sigma.$$

Proof.

To begin with, we prove Theorem 1.1 in the case when all c_k are positive, i.e. $l = 0$. Then the statement is the following:

The series (1.1) *extends analytically to the open sector* $\mathbb{C}\setminus\ \Delta_\sigma$, *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (1.4) *interpolating the coefficients* f_n *such that the indicator of the associated with* $\psi(\zeta)$ *entire function* $\varphi(\zeta)$ *satisfies the conditions*

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \max\{h_\varphi(-\frac{\pi}{2}), h_\varphi(\frac{\pi}{2})\} \leq \sigma. \tag{1.7}$$

Let φ be an entire function of the form (1.6) satisfying the conditions (1.7). Let us assume that the series (1.1) extends to the open sector $\mathbb{C} \setminus \Delta_\sigma$. It follows from the definition of an indicator that

$$|\varphi(re^{i\theta})| \leq e^{h_\varphi(\theta)r + o(r)} \text{ for } \theta \in \mathbb{R},$$

where $o(r)$ is infinitesimally small compared to r as $r \to \infty$. We need the property of *the trigonometric convexity* of indicator function of entire function of exponential type (see Appendix A.1):

$$h_\varphi(\theta)\sin(\theta_2 - \theta_1) \leq h_\varphi(\theta_1)\sin(\theta_2 - \theta) + h_\varphi(\theta_2)\sin(\theta - \theta_1),$$

where $\theta_1 < \theta < \theta_2$ and $\theta_2 - \theta_1 < \pi$. Taking $\theta_1 = 0, \theta_2 = \alpha$ or $\theta_1 = -\alpha, \theta_2 = 0$ we obtain that for $h_\varphi(0) = 0$ and $\alpha \in (0, \pi)$

$$h_\varphi(\theta) \leq c_\alpha |\sin\theta| \quad \textit{for} \quad |\theta| \leq \alpha$$

with the coefficients

$$c_\alpha = \frac{1}{\sin\alpha} \max\{h_\varphi(\alpha), h_\varphi(-\alpha)\}.$$

If in this estimate for $h_\varphi(0) = 0$ we let $\alpha = \frac{\pi}{2}$, then taking into account (1.7) we get the following estimate for the growth of φ

$$|\varphi(re^{i\theta})| \leq e^{\sigma|\sin\theta|r + o(r)} \text{ for } |\theta| \leq \frac{\pi}{2}.$$

Since $\varphi(\zeta)$ has the form (1.6), we obtain the inequality

$$|\phi(re^{i\theta})| \frac{\prod_{j=1}^{p} |a_j^{a_j re^{i\theta}}|}{\prod_{k=1}^{q} |c_k^{c_k re^{i\theta}}|} \leq e^{\sigma|\sin\theta|r + o(r)} \text{ or } |\theta| \leq \frac{\pi}{2},$$

in the variable $\zeta = \xi + i\eta = r(\cos\theta + i\sin\theta)$ it can be written as

$$|\phi(\zeta)| \leq \left(\frac{\prod_{j=1}^{p} |a_j^{a_j\zeta}|}{\prod_{k=1}^{q} |c_k^{c_k\zeta}|} \right)^{-1} e^{\sigma|\eta| + o(|\zeta|)} \text{ for } \zeta \in \Delta_{\frac{\pi}{2}}. \tag{1.8}$$

We need the following estimate.

Lemma 1. *For all* $\zeta \in \Delta_{\frac{\pi}{2}}$

$$\left|\frac{\prod_{j=1}^{p} \Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q} \Gamma(c_k\zeta + d_k)}\right| \le \frac{\prod_{j=1}^{p} |a_j^{a_j\zeta}|}{\prod_{k=1}^{q} |c_k^{c_k\zeta}|} e^{o(|\zeta|)} \; \textit{for} \; |\zeta| \to \infty. \tag{1.9}$$

Proof. It is easy to see that for $|\zeta| \to \infty$ one has

$$|a\zeta|^{a\xi}(1 - \frac{|b|}{|a\zeta|})^{|a\zeta|} e^{-a\eta \arg(\zeta)} \le |a\zeta + b|^{a\xi} \le |a\zeta|^{a\xi}(1 + \frac{|b|}{|a\zeta|})^{|a\zeta|} e^{-a\eta \arg(\zeta)}.$$

This fact together with Stirling's formula (applicable in the right half plane $\Delta_{\frac{\pi}{2}}$ since $a_j, c_k > 0$) gives

$$\frac{\prod_{j=1}^{p} |\Gamma(a_j\zeta + b_j)|}{\prod_{k=1}^{q} |\Gamma(c_k\zeta + d_k)|} \sim \frac{\prod_{j=1}^{p} |(a_j\zeta + b_j)^{(a_j\zeta+b_j)} e^{-(a_j\zeta+b_j)} (2\pi(a_j\zeta + b_j))^{\frac{1}{2}}|}{\prod_{k=1}^{q} |(c_k\zeta + d_k)^{(c_k\zeta+d_k)} e^{-(c_k\zeta+d_k)} (2\pi(c_k\zeta + d_k))^{\frac{1}{2}}|} \le$$

$$\le \frac{\prod_{j=1}^{p} |a_j\zeta|^{a_j\xi}(1 + \frac{|b_j|}{|a_j\zeta|})^{|a_j\zeta|} e^{-a_j\eta \arg(\zeta)} |(a_j\zeta + b_j)^{b_j} e^{-(a_j\zeta+b_j)} (2\pi(a_j\zeta + b_j))^{\frac{1}{2}}|}{\prod_{k=1}^{q} |c_k\zeta|^{c_k\xi}(1 - \frac{|d_k|}{|c_k\zeta|})^{|c_k\zeta|} e^{-c_k\eta \arg(\zeta)} |(c_k\zeta + d_k)^{d_k} e^{-(c_k\zeta+d_k)} (2\pi(c_k\zeta + d_k))^{\frac{1}{2}}|} \le$$

$$\le \frac{\prod_{j=1}^{p} |a_j^{a_j\zeta}|}{\prod_{k=1}^{q} |c_k^{c_k\zeta}|} |\zeta^{\xi(\sum_{j=1}^{p} a_j - \sum_{k=1}^{q} c_k)}| |e^{-\zeta(\sum_{j=1}^{p} a_j - \sum_{k=1}^{q} c_k)}| \times$$

$$\times \frac{\prod_{j=1}^{p} (1 + \frac{|b_j|}{|a_j\zeta|})^{a_j\xi} e^{-a_j\eta \arg(\zeta)}}{\prod_{k=1}^{q} (1 - \frac{|d_k|}{|c_k\zeta|})^{c_k\xi} e^{-c_k\eta \arg(\zeta)}} \times \frac{\prod_{j=1}^{p} |a_j\zeta + b_j|^{b_j} e^{-b_j} |2\pi(a_j\zeta + b_j)|^{\frac{1}{2}}}{\prod_{k=1}^{q} |c_k\zeta + d_k|^{d_k} e^{-d_k} |2\pi(c_k\zeta + d_k)|^{\frac{1}{2}}}.$$

In view of (1.5), this turns into

$$\left|\frac{\prod_{j=1}^{p} \Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q} \Gamma(c_k\zeta + d_k)}\right| \le \frac{\prod_{j=1}^{p} |a_j^{a_j\zeta}|}{\prod_{k=1}^{q} |c_k^{c_k\zeta}|} |A\zeta + B|^{C},$$

where A, B and C are some constants (independent of ζ). Since $|A\zeta + B|^C = e^{\ln|A\zeta+B|^C}$ and

$$\lim_{|\zeta| \to \infty} \frac{\ln|A\zeta + B|^C}{|\zeta|} = 0,$$

we get $|A\zeta + B|^C = e^{o(|\zeta|)}$ as $\zeta \to \infty$, i.e. the lemma's statement.

It follows from (1.8) and (1.9) that for a meromorphic function $\psi(\zeta)$ defined by (1.4) we have

$$|\psi(\zeta)| \le e^{\sigma|\eta| + o(|\zeta|)} \quad \text{for} \quad \zeta \in \Delta_{\frac{\pi}{2}}. \tag{1.10}$$

Consider the following function

$$g(\zeta, z) := \frac{z^\zeta}{e^{2\pi i\zeta} - 1}$$

of two complex variables $\zeta = \xi + i\eta,\ z = x + iy$. It is meromorphic in $\zeta \in \mathbb{C}$ and holomorphic in $z \in \mathbb{C} \setminus \mathbb{R}_+$.

Denote $D^* := \cup_{m\in\mathbb{Z}} D_{1/4}(m)$. Notice that there exists a constant $c > 0$ such that

$$|e^{2\pi i\zeta} - 1| > \frac{e^{\pi(|\eta|-\eta)}}{c} \quad \text{for} \quad \zeta \in \mathbb{C} \setminus D^*.$$

From this we get the estimate

$$|g(\zeta, z)| < ce^{\xi \log|z| - (\pi - |\pi - \arg z|)|\eta|}$$

for $\zeta \in \mathbb{C} \setminus D^*$ and $z \in \mathbb{C} \setminus \mathbb{R}_+$. Using (1.10) for $\zeta \in \Delta_{\frac{\pi}{2}} \setminus D^*$ and $z \in \mathbb{C} \setminus \mathbb{R}_+$ we see that

$$|\psi(\zeta)||g(\zeta, z)| < ce^{\xi \log|z| - (\pi - \sigma - |\pi - \arg z|)|\eta| + o(|\zeta|)}. \tag{1.11}$$

For $\zeta \in \Delta_{\frac{\pi}{2}} \setminus D^*$ and $z \in \mathbb{C} \setminus \Delta_{\sigma+\delta}$ there is the following bound

$$|\psi(\zeta)||g(\zeta, z)| < ce^{\xi \log|z| - \delta|\eta| + o(|\zeta|)}.$$

Consider the integral

$$I_m = \int\limits_{\partial G_m} \psi(\zeta) g(\zeta, z) d\zeta$$

over the oriented boundary of G_m that consists of the segments (see Fig.1)

$$\Gamma_m^1 = [a - i(m + \frac{1}{2}), a + i(m + \frac{1}{2})],$$

$$\Gamma_m^2 = [a + i(m + \frac{1}{2}), a + m + i(m + \frac{1}{2})],$$

$$\Gamma_m^3 = [a + m + i(m + \frac{1}{2}), a + m - i(m + \frac{1}{2})],$$

$$\Gamma_m^4 = [a + m - i(m + \frac{1}{2}), a - i(m + \frac{1}{2})],$$

where $\frac{1}{4} < a < \frac{3}{4}$.

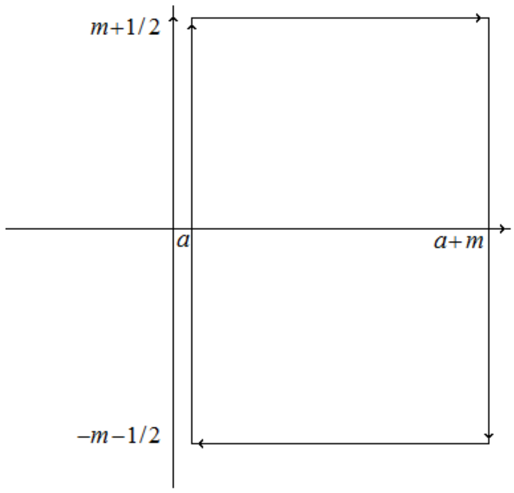

Figure 1

The integral I_m is the sum of four integrals $I_m^1, I_m^2, I_m^3, I_m^4$ over $\Gamma_m^1, \Gamma_m^2, \Gamma_m^3, \Gamma_m^4$, respectively. For $\zeta \in \Delta_{\frac{\pi}{2}} \setminus D^*$ and $z \in \mathbb{C} \setminus \Delta_{\sigma+\delta}$ there hold the following estimates

$$I_m^2 = \int\limits_{\Gamma_m^2} |\psi(\zeta) g(\zeta, z)||d\zeta| \leq ce^{-\delta(m+\frac{1}{2})} \int\limits_{a}^{a+m} e^{\xi \ln |z| + o(|\zeta|)} d\xi,$$

$$I_m^3 = \int\limits_{\Gamma_m^3} |\psi(\zeta) g(\zeta, z)||d\zeta| \leq ce^{(a+m)\ln |z| + o(m)} \int\limits_{-i(m+\frac{1}{2})}^{i(m+\frac{1}{2})} d\eta,$$

$$I_m^4 = \int\limits_{\Gamma_m^4} |\psi(\zeta) g(\zeta, z)||d\zeta| \leq ce^{-\delta(m+\frac{1}{2})} \int\limits_{a+m}^{a} e^{\xi \ln |z| + o(|\zeta|)} d\xi.$$

We see that for $z \in D_1 \setminus \Delta_{\sigma+\delta}$ the integrals I_m^2, I_m^3, I_m^4 tend to 0 as $m \to \infty$.

Thus,

$$\lim_{m\to\infty} I_m = \lim_{m\to\infty} \int\limits_{\partial G_m} \psi(\zeta)g(\zeta,z)d\zeta = \lim_{m\to\infty} \int\limits_{\Gamma_m^1} \psi(\zeta)g(\zeta,z)d\zeta = \lim_{m\to\infty} I_m^1.$$

In the domain G_m the integrand has simple poles at real integer points and finitely many poles at points $\frac{-\nu-b_j}{a_j} \in G_m, \quad \nu = 0, 1, ...$ (recall that a_j, b_j are parameters in the definition (1.4) of $\psi(\zeta)$).

The residue theorem yields

$$\int\limits_{\partial G_m} \varphi(\zeta)g(\zeta,z)d\zeta = \sum_{n=1}^{m} \varphi(n)z^n + P(z),$$

where $P(z)$ is a polynomial.

Consider the integral

$$I = \int\limits_{a-i\infty}^{a+i\infty} \varphi(\zeta)g(\zeta,z)d\zeta.$$

For $\zeta = a + i\eta$ and $z \in \mathbb{C} \setminus \Delta_{\sigma+\delta}$ we have

$$|\varphi(\zeta)||g(\zeta,z)| < ce^{a\ln|z|-\delta|\eta|+o(|\zeta|)}.$$

It follows from this inequality that the integral I converges absolutely and uniformly on any compact subset $K \subset \mathbb{C} \setminus \Delta_{\sigma+\delta}$, and defines a holomorphic function on the set of interior points of K. For $z \in D_1 \setminus \Delta_{\sigma+\delta}$

$$\int\limits_{\Gamma_m^1} \varphi(\zeta)g(\zeta,z)d\zeta \to \int\limits_{a-i\infty}^{a+i\infty} \varphi(\zeta)g(\zeta,z)d\zeta, \text{ as } m \to \infty.$$

Since $I_m \to I$ as $m \to \infty$, $I(z) = f(z) + P(z)$ for $z \in D_1 \cap K^o$. This means that $f(z)$ extends analytically to K^o. Because K is an arbitrary compact set in $\mathbb{C} \setminus \Delta_{\sigma+\delta}$ for any small δ, the function $f(z)$ extends to the open sector $\mathbb{C} \setminus \Delta_{\sigma}$. Thus, Theorem 1.1 is proved if all c_k are positive.

Now we prove Theorem 1.1 in the case when c_k may be negative. To avoid cumbersome notation let us show the idea of the proof under the assumption that only one of c_ks is negative, let it be c_q. It follows easily that in this case $\frac{l}{2} = -c_q$. The expression for $\psi(\zeta)$ becoming

$$\psi(\zeta) = \phi(\zeta)\frac{\prod_{j=1}^{p}\Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q-1}\Gamma(c_k\zeta + d_k)\Gamma(-\frac{l}{2}\zeta + d_q)}.$$

The associated with $\psi(\zeta)$ entire function is

$$\varphi(\zeta) := \phi(\zeta)\frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta}}{\prod_{k=1}^{q}|c_k|^{c_k\zeta}} = \phi(\zeta)\frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta}(\frac{l}{2})^{\frac{l}{2}\zeta}}{\prod_{k=1}^{q-1} {c_k}^{c_k\zeta}}.$$

Note that the function $\psi(\zeta)$ may be rewritten in the form (1.4) such that all c_k are positive

$$\psi(\zeta) = \phi(\zeta)\frac{\prod_{j=1}^{p}\Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q-1}\Gamma(c_k\zeta + d_k)}\Gamma(1 + \frac{l}{2}\zeta + d_q)\sin\pi(-\frac{l}{2}\zeta - d).$$

The associated entire function for this form of $\psi(\zeta)$ is

$$\tilde{\varphi}(\zeta) := \psi(\zeta)\sin\pi(-\frac{l}{2}\zeta - d_q).$$

Its indicator is bounded

$$h_{\tilde{\varphi}}(\theta) \leq h_{\varphi}(\theta) + \pi\frac{l}{2}|\sin(\theta)|, \quad |\theta| \leq \frac{\pi}{2}.$$

According to the hypothesis of Theorem 1.1

$$\max\{h_{\varphi}(-\frac{\pi}{2}) + \frac{\pi}{2}l, h_{\varphi}(\frac{\pi}{2}) + \frac{\pi}{2}l\} \leq \sigma.$$

Thus

$$h_{\tilde{\varphi}}(0) = 0, \quad h_{\tilde{\varphi}}(\pm\frac{\pi}{2}) \leq \sigma.$$

The function $\tilde{\varphi}(\zeta)$ satisfies the conditions (1.7), hence the sum of the series (1.1) extends analytically to the open sector $\mathbb{C} \setminus \Delta_{\sigma}$. Theorem 1.1 is proved.

1.1.2 Conditions for continuability to some neighborhood of an open arc

Theorem 1.2. *The open arc* $\gamma_\sigma = \partial D_1 \setminus \Delta_\sigma$ *is an arc of regularity for the series* (1.1) *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (1.4) *interpolating the coefficients* f_n *such that the indicator of the associated with* $\psi(\zeta)$ *entire function* $\varphi(\zeta)$ *satisfies the conditions*

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \overline{\lim_{\theta\to 0}} \frac{h_\varphi(\theta)}{|\theta|} + \frac{\pi}{2} l \le \sigma.$$

The proof of Theorem 1.2 is largely similar to that of Theorem 1.1. Namely, it follows from condition 2) of Theorem 1.2 that for any $\alpha > 0$ there exists $\delta > 0$ such that $h_\varphi(\theta) \le (\sigma + \delta)|\sin\theta|$ for $|\theta| \le \alpha$. Consequently, the bounds (1.10) and (1.11) for the absolute values of $\psi(\zeta)$ and $\psi(\zeta)g(\zeta, z)$ hold for $\zeta \in \Delta_\alpha$. The domains G and G_m become (see Fig. 2)

$$G = D_1 \cup \Delta_\alpha^o \ \text{ and } \ G_m = \{\zeta = \xi + i\eta \in G : \xi \le m + \frac{1}{2}\},$$

i.e. $\partial G_m = \Gamma_m^1 \cup \Gamma_m^2$.

The integral I_m is then the sum of I_m^1 and I_m^2 over Γ_m^1, Γ_m^2, and for $z \in K \cap D_1^o$ the integral $I_m^2 \to 0$ as $m \to \infty$.

The integral I over ∂G converges for $\zeta \in \Delta_\alpha$, $z \in K$, (see Fig. 3) where

$$K = D_{e^\varepsilon} \setminus (\Delta_{\sigma+2\delta}^o \cup D_{\frac{1}{2}}), \ \ \varepsilon = \frac{\delta \sin\alpha}{2}.$$

The rest of the proof is the same.

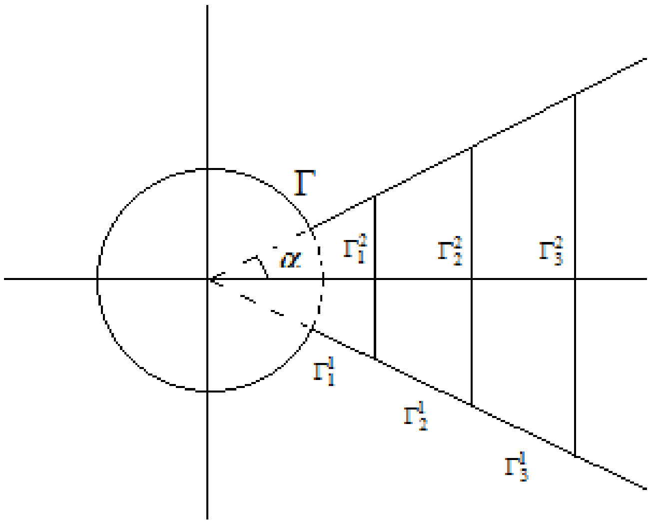

Figure 2

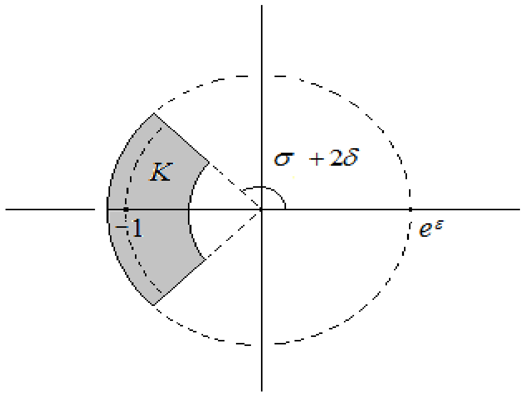

Figure 3

1.1.3 Conditions for continuability to complex plane except some arc

Theorem 1.3. *The series* (1.1) *extends analytically to* $\mathbb{C} \setminus (\partial D_1 \cap \ \Delta_\sigma)$, *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (1.4) *interpolating the coefficients* f_n *such that the indicator of the associated with* $\psi(\zeta)$ *entire function* $\varphi(\zeta)$ *satisfies the conditions*

$$h_\varphi(\theta) + \frac{\pi}{2} l |\sin\theta| \le \sigma |\sin\theta| \ \textit{for} \ \ |\theta| \le \pi.$$

As for the proof of Theorem 1.3, it is enough to note that the main estimates (1.10) and (1.11) hold for all $\zeta \in \mathbb{C}$. Therefore, by choosing appropriate contours of integrations (see Fig. 4) we prove analytic continuation of the sum of the series to $\mathbb{C} \setminus (\partial D_1 \cap \Delta_\sigma)$.

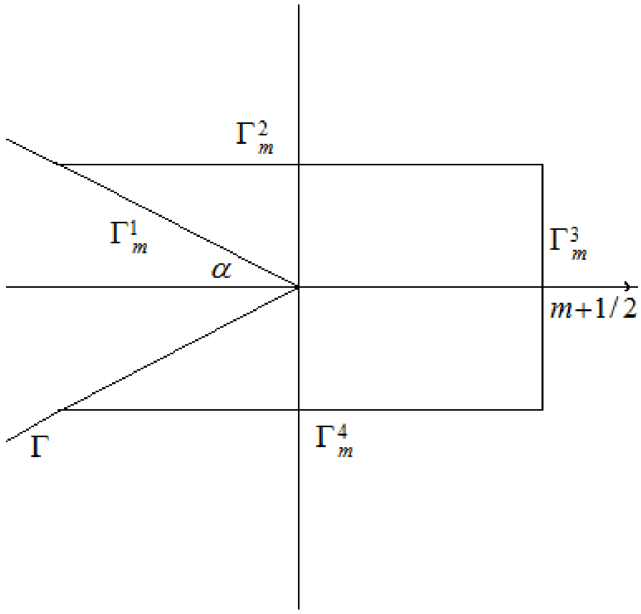

Figure 4

The integral I converges for $z \in K$, where $K = \mathbb{C} \setminus (D_1 \cap \ \Delta_{\sigma+2\delta} \ \cup D_{e^{-\varepsilon}})$ (see Fig. 5) and the sum of the series (1.1) equal to the integral I as $z \in \ K \cap D_1^o$. Thus, the series extends to the whole complex plane $\mathbb{C}$ except some arc of the boundary D_1.

1.2 Examples

Consider two examples clarifying why interpolation of the coefficients by meromorphic functions, and not by entire, may be more effective.

Example 1. Consider the series

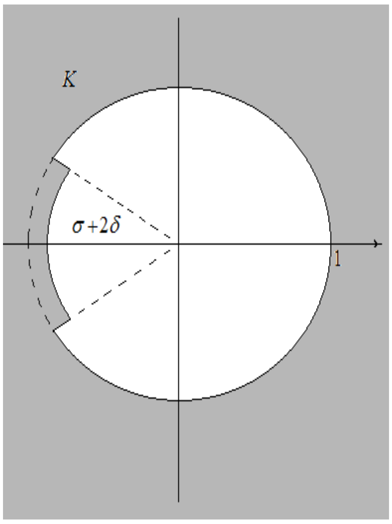

Figure 5

$$f(z)=\sum_{n=0}^{\infty}\frac{(2n-2)(2n-5)...(2n-3(n+2))}{2^{\frac{2}{3}n}n!}z^n, \tag{1.12}$$

whose domain of convergence is the unit disk. Let the coefficients f_n be given by the values of a meromorphic function of the form (1.4). Therefore the cofficients can be rewritten in the form

$$f_n=\frac{3^{n-1}(\frac{2}{3}n+\frac{1}{3}-1)...(\frac{2}{3}n+\frac{1}{3}-(n-1))}{2^{\frac{2}{3}n}n!}.$$

Using the formula

$$\Gamma(\tau+l)=(\tau)_l\Gamma(l),$$

where $(\tau)_l=\tau(\tau+1)...(\tau+l-1)$ is the Pochhammer symbol, $l\in\mathbb{N}$, we get

$$f_n=\frac{\Gamma(\frac{2}{3}n+\frac{1}{3})3^{n-1}}{\Gamma(n+1)\Gamma(-\frac{1}{3}n+\frac{4}{3})2^{\frac{2}{3}n}}.$$

Thus, the meromorphic function

$$\psi(\zeta)=\frac{3^{\zeta-1}}{2^{\frac{2}{3}\zeta}}\frac{\Gamma(\frac{2}{3}\zeta+\frac{1}{3})}{\Gamma(\zeta+1)\Gamma(-\frac{1}{3}\zeta+\frac{4}{3})}$$

interpolates the coefficients f_n.

In this case the entire function from Theorem 1.1 is

$$\varphi(\zeta) = \frac{3^{\zeta-1}}{2^{\frac{2}{3}\zeta}} \frac{(\frac{2}{3})^{\frac{2\zeta}{3}}}{(\frac{1}{3})^{\frac{-\zeta}{3}}} \equiv \frac{1}{3}.$$

Here $h_\varphi(\theta) = 0$ and $l = 1 + \frac{1}{3} + \frac{2}{3} = \frac{2}{3}$. Therefore

$$\max\{h_\varphi(-\frac{\pi}{2}) + \frac{\pi l}{2}, h_\varphi(\frac{\pi}{2}) + \frac{\pi l}{2}\} = \frac{\pi}{3}.$$

According to Theorem 1.1, the series (1.12) extends analytically to the open sector $\mathbb{C} \setminus \Delta_{\frac{\pi}{3}}$.

Note that the series (1.12) is a branch of the solution $y(z)$ to the algebraic equation $y^3 - \frac{3z}{2^{\frac{2}{3}}}y - 1 = 0$ determined by the condition $y(0) = 1$ (see [29]). The series $f(z)$ has singularities (branching points) at $e^{-i\frac{2}{3}\pi}$ and $e^{i\frac{2}{3}\pi}$ and extends to the sector $\mathbb{C} \setminus \Delta_{\frac{\pi}{3}}$ defined by the large arc of the boundary of the unit disk with endpoints $e^{-i\frac{2}{3}\pi}$ and $e^{i\frac{2}{3}\pi}$ [29].

However, an entire function interpolating the coefficients cannot be always constructed so easily, despite its existence follows from Arakelian's theorem [10].

Example 2. Consider now the series

$$f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n}{3} + \frac{1}{3})3^n}{\Gamma(n+1)\Gamma(\frac{-2n}{3} + \frac{4}{3})2^{\frac{2n}{3}}} z^n, \tag{1.13}$$

with the same domain of convergence, the unit disk. Its coefficients are

$$f_n = \frac{\Gamma(\frac{n}{3} + \frac{1}{3})3^n}{\Gamma(n+1)\Gamma(\frac{-2n}{3} + \frac{4}{3})2^{\frac{2n}{3}}},$$

They are interpolated by the following entire function

$$\varphi(z) = \frac{2\pi}{3^{\frac{1}{2}}} \frac{2^{-\frac{2}{3}z}}{\Gamma(\frac{z}{3} + \frac{2}{3})\Gamma(\frac{z}{3} + 1)\Gamma(\frac{4}{3} - \frac{2z}{3})}.$$

Indeed, in Gauss's multiplication formula [30]

$$\Gamma(w)\Gamma(w + \frac{1}{m})...\Gamma(w + \frac{m-1}{m}) = m^{\frac{1}{2}-mw}(2\pi)^{\frac{m-1}{2}}\Gamma(mw)$$

let $m=3$, $w=\frac{n}{3}+\frac{1}{3}$, then

$$\Gamma(\frac{n}{3}+\frac{1}{3})\Gamma(\frac{n}{3}+\frac{2}{3})\Gamma(\frac{n}{3}+1)=3^{-\frac{1}{2}-n}2\pi\Gamma(n+1).$$

Express $\Gamma(\frac{n}{3}+\frac{1}{3})$ through the other terms of this identity and substitute it into the expression for f_n to see that $\varphi(n)=f_n,\ n\in\mathbb{N}$.

Estimate $|\varphi(r)|$ by using Stirling's formula

$$|\varphi(r)|=\left|\frac{2}{3^{\frac{1}{2}}}\frac{2^{\frac{2}{3}r}\Gamma(\frac{2r}{3}-\frac{1}{3})sin(\pi\frac{2r-1}{3})}{\Gamma(\frac{r}{3}+\frac{2}{3})\Gamma(\frac{r}{3}+1)}\right|\sim$$

$$\sim\frac{2}{3^{\frac{1}{2}}}\frac{2^{\frac{2}{3}r}(2\pi\frac{2r-1}{3})^{\frac{1}{2}}(\frac{2r}{3}-\frac{1}{3})^{\frac{2r}{3}-\frac{1}{3}}e^{-(\frac{2r}{3}-\frac{1}{3})}}{(2\pi\frac{r+2}{3})^{\frac{1}{2}}(\frac{r}{3}+\frac{2}{3})^{\frac{r}{3}+\frac{2}{3}}e^{-(\frac{r}{3}+\frac{2}{3})}}\frac{\sin(\pi\frac{2r-1}{3})}{(2\pi(\frac{r}{3}+1))^{\frac{1}{2}}(\frac{r}{3}+1)^{\frac{r}{3}+1}e^{-(\frac{r}{3}+1)}}\leq$$

$$\leq Cr+e^{o(r)}.$$

On the one hand

$$h_\varphi(0)=\overline{\lim_{r\to\infty}}\frac{\ln|\varphi(r)|}{r}\leq\overline{\lim_{r\to\infty}}\frac{\ln(Cr+e^{o(r)})}{r}\leq 0,$$

on the other hand

$$h_\varphi(0)\geq\overline{\lim_{n\to\infty}}\frac{\ln|\varphi(n)|}{n}=\overline{\lim_{n\to\infty}}\ln|f_n|^{\frac{1}{n}}=0,$$

therefore $h_\varphi(0)=0$.

In order to estimate $|\varphi(re^{i\frac{\pi}{2}})|$ and $|\varphi(re^{-i\frac{\pi}{2}})|$ we use the double-sided estimate for the Gamma-function (see [31]):

$$c_1(|y|+1)^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|}\leq\Gamma(x+iy)\leq c_2(|y|+1)^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|},$$

where $x\in K\subset\mathbb{R}\setminus\{0,-1,-2,...\}$, K is compact. The constants c_1 and c_2 depend on the choice of $y\in\mathbb{R}$. Then

$$|\varphi(re^{\pm i\frac{\pi}{2}})| \leq C\frac{e^{\frac{\pi}{6}r}e^{\frac{\pi}{6}r}e^{\frac{2\pi}{6}r}}{c_1(\frac{r}{3}+1)^{\frac{2}{3}-\frac{1}{2}}c_1(\frac{r}{3}+1)^{1-\frac{1}{2}}c_1(\frac{2r}{3}+1)^{\frac{4}{3}-\frac{1}{2}}},$$

or

$$\ln|\varphi(re^{\pm i\frac{\pi}{2}})| \leq \frac{2\pi}{3}r + o(r).$$

Therefore

$$h_\varphi(\pm\frac{\pi}{2}) \leq \frac{2\pi}{3}.$$

It follows from Arakelian's Theorem [24] that the series (2.2) extends to the open sector $\mathbb{C} \setminus \Delta_{\frac{2\pi}{3}}$.

On the other hand, the coefficients of the series (1.13) are interpolated by the meromorphic function

$$\psi(\zeta) = \frac{3^\zeta}{2^{\frac{2}{3}\zeta}}\frac{\Gamma(\frac{1}{3}\zeta+\frac{1}{3})}{\Gamma(\zeta+1)\Gamma(-\frac{2}{3}\zeta+\frac{4}{3})}.$$

The entire function of Theorem 1.1 is

$$\varphi(\zeta) = \frac{3^\zeta}{2^{\frac{2}{3}\zeta}}\frac{3^{-\frac{1}{3\zeta}}}{(-\frac{2}{3})^{-\frac{2}{3}\zeta}} \equiv 1,$$

and $l = 1+\frac{2}{3}-\frac{1}{3} = \frac{4}{3}, \quad h_\varphi(\theta) = 0$ and

$$\max\{h_\varphi(-\frac{\pi}{2}) - \frac{2\pi}{3}, h_\varphi(\frac{\pi}{2}) + \frac{2\pi}{3}\} \leq \frac{2\pi}{3}.$$

Therefore, by Theorem 1.1 the series (1.13) extends to the open sector $\mathbb{C} \setminus \Delta_{\frac{2\pi}{3}}$.

1.3 Non extendable one-dimensional series

The problem of describing the relations between singularities of power series in one variable and their coefficients attracted mathematicians' attention already at the end of 19th century. Remarkable results were obtained in the first half of the 20th century which allowed thinking that the development in this direction was almost completed. Many obtained results touch upon the question about series non extendable analytically across the boundary of their convergence domain and these results are connected with the names of famous Hungarian mathematicians Szegö and Pólya (see, for example, articles [4] and [3], and also the list of their articles in the book by Bieberbach [6]). Examples of series that are non extendable analytically across the boundary of their convergence domain we can find in the text-books about theory functions of complex variables. These examples deal with the so-called "strongly lacunar" series, in other words, having "many" monomials with zero coefficients. Such series, for instance, are

$$\sum_{n=0}^{\infty} z^{n!}, \qquad \sum_{n=0}^{\infty} z^{2^n}, \qquad \sum_{n=0}^{\infty} z^{n^n}.$$

In 1891, Fredholm [17] gave examples of "moderate lacunar" non extendable series, moreover, these series represented infinitely differentiable function in the closure of the convergence disk. These series depend on a parameter a, and they have the following form

$$\sum_{n=0}^{\infty} a^n z^{n^2}, \qquad 0 < a < 1.$$

Here n^2 has the power order 2 respective to the summation index n, therefore we say that Fredholm's series have the lacunarity order 2.

A more general result on non extendable series in terms of lacunarity belongs to Fabry (see [18] or [6]). It claims that if the sequence of natural numbers m_n increases faster than n (i.e. $n = o(m_n)$), then there is a series

$$\sum_{n=0}^{\infty} a_n z^{m_n},$$

converging in the unit disk and not extending across its boundary.

One of the main results in this section is given by Theorem 1.4. It demonstrates that Fredholm's example may be refined to the power order of lacunarity from 2 to $1+\varepsilon$. The precise formulation is the following:

Theorem 1.4. *If the increasing sequence of natural numbers n_k satisfies the inequality $n_k \geq const \times k^{1+\varepsilon}$ with $\varepsilon > 0$, then the power series*

$$\sum_{k=0}^{\infty} a^k z^{n_k}, \qquad 0 < a < 1, \tag{1.14}$$

are not extendable across the boundary circle and represent infinitely differentiable function in the closed disk.

Proof. Consider the following series

$$\varphi(t,u) = \sum_{k=0}^{\infty} e^{n_k t + ku}, \quad \textit{where} \quad t,u \in \mathbb{C}. \tag{1.15}$$

Its terms exponentially decrease in the product $\Pi \times \bar{\Pi}$ of subspaces $\Pi = \{u : Re\ u < 0\}$ and $\bar{\Pi} = \{t : Re\ t \leq 0\}$. These series converge uniformly on compact subsets of $\Pi \times \bar{\Pi}$ and therefore $\varphi(t,u)$ is holomorphic in the product $\Pi \times \Pi$ of open subspaces. This property is preserved for all derivatives of this series with respect to the variable t. Consequently the function $\varphi(t,u)$ is infinitely differentiable in $\bar{\Pi}$ for each fixed $u \in \Pi$.

Introduce the following notation

$$F(-t) = \sum_{k=0}^{\infty} e^{ku_0} e^{-n_k(-t)} = \sum_{k=0}^{\infty} e^{n_k t + ku_0} = \varphi(t,u_0), \tag{1.16}$$

for $t \in \bar{\Pi}$ and for each fixed $u_0 \in \Pi$. Here, the function $F(-t)$ is represented by a Dirichlet series

$$\sum_{k=0}^{\infty} a_k e^{-\lambda_k t}$$

with exponential indexes $\lambda_k = n_k$ and coefficiens $a_k = e^{ku_0}$.

If the series converges in the half plane $Rez > c$ and diverges in the half plane $Rez < c$, then the line $Rez = c$ is called a line of convergence for the

Dirichlet series, and the quantity c is called an abscissa of convergence (see [32], [33]).

Compute the value

$$L = \overline{\lim_{k\to\infty}} \frac{\ln k}{\lambda_k} = \overline{\lim_{k\to\infty}} \frac{\ln k}{n_k} \le \overline{\lim_{k\to\infty}} \frac{\ln k}{k^{1+\varepsilon}} = 0.$$

Therefore, the abscissa of convergence for the series (1.16) can be found as follows

$$= \overline{\lim_{k\to\infty}} \frac{\ln |e^{ku_0}|}{n_k} = \overline{\lim_{k\to\infty}} \frac{\ln e^{kRe\ u_0}}{n_k} = \overline{\lim_{k\to\infty}} \frac{kRe\ u_0}{n_k} \le \overline{\lim_{k\to\infty}} \frac{kRe\ u_0}{k^{1+\varepsilon}} = 0.$$

Now we demonstrate that the function $F(-t)$ satisfies the conditions of Polya's theorem [34]: *If a Dirichlet series*

$$F(z) = \sum_{k=1}^{\infty} a_k e^{-n_k z}$$

has a finite abscissa of convergence c *and*

$$\lim_{k\to\infty} \frac{k}{n_k} = 0, \qquad n_{k+1} - n_k \ge h > 0,$$

then the line of convergence $Re\ z = c$ *is the natural boundary for the function* $F(z)$.

Indeed,

$$\lim_{k\to\infty} \frac{k}{n_k} \sim \lim_{k\to\infty} \frac{k}{k^{1+\varepsilon}} \to 0,$$

$$n_{k+1} - n_k \sim (k+1)^{1+\varepsilon} - k^{1+\varepsilon} = k^{1+\varepsilon}\left(\left(1+\frac{1}{k}\right)^{1+\varepsilon} - 1\right) =$$

$$= k^{1+\varepsilon}\left((1+\varepsilon)\frac{1}{k} + o(\frac{1}{k})\right) \xrightarrow[k\to\infty]{} \infty.$$

Consequently, the function $F(-t)$ is not analytically extendable. Then, denoting $a = e^u$ (fixed) and $z = e^t$, from (1.15) we get (1.14) as desired.

Theorem 1.5. *For an arbitrary pair of natural numbers* $p > q$ *the series*

$$f(z) = \sum_{\nu=0}^{\infty} a^{\nu^q} z^{\nu^p}, \quad 0 < a < 1, \tag{1.17}$$

is not extendable across the unit disk boundary and represents an infinitely differentiable function in the closed disk.

Proof. We can prove this theorem directly, without referring to the Polya's theorem. We consider the following series

$$\varphi(t,u)=\sum_{\nu=0}^{\infty} e^{\nu^p t+\nu^q u}, \qquad \text{where} \quad t,u\in\mathbb{C}. \tag{1.18}$$

Its terms are exponentially decreasing in the product $\Pi\times\bar{\Pi}$ of subspaces $\Pi=\{u: Re\ u<0\}$ and $\bar{\Pi}=\{t: Re\ t\leq 0\}$. The series converges uniformly on the compact subsets of $\Pi\times\bar{\Pi}$ therefore $\varphi(t,u)$ is holomorphic in the product of open subspaces $\Pi\times\Pi$. Besides, the function $\varphi(t,u)$ is holomorphic in $u\in\Pi$ for any fixed $t_0\in\bar{\Pi}$. We consider the Taylor expansion of φ

$$\varphi(t,u)=\sum_{k=0}^{\infty}\frac{\partial^k\varphi}{\partial u^k}(t,u_0)\frac{(u-u_0)^k}{k!}, \tag{1.19}$$

with the centre $u_0\in\Pi$, regarding $t\in\bar{\Pi}$ as a parameter. In view of (1.18) we have

$$\frac{\partial^k\varphi}{\partial u^k}(t,u)=\sum_{\nu=0}^{\infty}(\nu^q)^k e^{\nu^p t+\nu^q u}.$$

Substituting this expression in (1.19), we obtain

$$\sum_{k=0}^{\infty}\left(\sum_{\nu=0}^{\infty}(\nu^q)^k e^{\nu^p t+\nu^q u_0}\right)\frac{(u-u_0)^k}{k!}. \tag{1.20}$$

We demonstrate that the series (1.20) has a finite convergence radius for any fixed t_0 from the boundary $\partial\,\bar{\Pi}$ (i.e. $Ret_0=0$).

The series (1.18) diverges if $Reu\geq 0$ and $Ret_0=0,$ because its general term

$$|e^{\nu^p t_0+\nu^q u}|=|e^{\nu^p t_0}||e^{\nu^q u}|=(e^{Re\ u})^{\nu^q}$$

does not tend to 0. Besides, the series (1.18) can be considered as a power series in the variable $w=e^u$. Using these facts, we obtain that the function $\varphi(t_0,u)$

has a singularity point $\hat{u}$ such that $Re\hat{u} = 0$. Hence, the series (1.20) has a finite convergence disk.

By using these facts and the Cauchy-Hadamard formula, we obtain that there is a sequence k_l with the following property

$$\left|\sum_{\nu=0}^{\infty} (\nu^q)^{k_l} e^{\nu^p t_0 + \nu^q u_0}\right| \sim \frac{k_l!}{\rho_l^k} \quad with \quad k_l \to \infty, \tag{1.21}$$

where ρ is the convergence radius of the series (1.18) which depends on the choice of points $u_0 \in \Pi$ and $t_0 \in \Pi$.

Assume that the function $\varphi(t, u_0)$ extends analytically with respect to t from $\bar{\Pi}$ across some boundary point $t_0 \in \partial\bar{\Pi}$ for some fixed $u_0 \in \Pi$. We denote by $\tilde{\varphi}(t, u_0)$ the analytic continuation of the function $\varphi(t, u_0)$. Its Taylor series is the following:

$$\tilde{\varphi}(t, u_0) = \sum_{k=0}^{\infty} \frac{\partial^k \tilde{\varphi}}{\partial t^k}(t_0, u_0) \frac{(t - t_0)^k}{k!} = \sum_{k=0}^{\infty} \frac{\partial^k \varphi}{\partial t^k}(t_0, u_0) \frac{(t - t_0)^k}{k!}. \tag{1.22}$$

Taking into account (1.18) we have

$$\frac{\partial^k \varphi}{\partial t^k}(t, u) = \sum_{\nu=0}^{\infty} (\nu^p)^k e^{\nu^p t + \nu^q u}.$$

Substituting this expression in (1.22), we obtain

$$\tilde{\varphi}(t, u_0) = \sum_{k=0}^{\infty} \left(\sum_{\nu=0}^{\infty} (\nu^p)^k e^{\nu^p t_0 + \nu^q u_0} \right) \frac{(t - t_0)^k}{k!} =$$

$$= \sum_{k=0}^{\infty} \left(\sum_{\nu=0}^{\infty} (\nu^q)^{\frac{p}{q}k} e^{\nu^p t_0 + \nu^q u_0} \right) \frac{(t - t_0)^k}{k!}. \tag{1.23}$$

We investigate the convergence radius of this series by the Cauchy–Hadamard theorem. In the sequence

$$\sqrt[k]{\frac{1}{k!} \left| \sum_{\nu=0}^{\infty} (\nu^q)^{\frac{p}{q}k} e^{\nu^p t_0 + \nu^q u_0} \right|}$$

we consider the subsequence taking $k = qk_l$:

$$\sqrt[qk_l]{\frac{1}{(qk_l)!}\left|\sum_{\nu=0}^{\infty}(\nu^q)^{pk_l}e^{\nu^p t_0+\nu^q u_0}\right|}.$$

Using the estimate (1.21), we obtain

$$\sqrt[qk_l]{\frac{1}{(qk_l)!}\frac{(pk_l)!}{\rho^{pk_l}}} = \sqrt[qk_l]{\frac{(pk_l)!}{(qk_l)!}}\,\rho^{\frac{p}{q}}.$$

By Stirling's formula

$$\sqrt[qk_l]{\frac{(pk_l)^{pk_l-\frac{1}{2}}e^{-pk_l}}{(qk_l)^{qk_l-\frac{1}{2}}e^{-qk_l}}}\,\rho^{\frac{p}{q}} \sim \frac{k_l^{\frac{p}{q}}}{k_l} \xrightarrow[k_l\to\infty]{} \infty \quad \text{with} \quad p > q.$$

Thus, the series (1.22) has empty convergence domain. It follows that the series (1.18) does not continue analytically with respect to t across the point $t_0 \in \partial\,\bar{\Pi}$. The theorem is proved.

Chapter 2. Analytic continuation of multiple power series

For multiple power series there are significantly less results describing singular subsets on the boundary of the convergence domain, or, in other words, subsets on the boundary such that series analytically extend across them. In the first section we extend Arakelian's result [10] on the arc of regularity formulated in the Introduction.

Recall that this theorem establishes the size of the regularity arc (across which the series extends) on the boundary circle in terms of the indicator function of the entire function of exponential type interpolating the coefficients of the series.

In the second section we consider the problem of continuability of power series to sectorial domains of $\mathbb{C}^n$. Sectorial domains are defined by conditions on the arguments $\theta_j = \arg z_j$ of variables $(z_1, ..., z_n) \in \mathbb{C}^n$ only. In the final fourth section we construct double power series with a natural boundary. Such series are non extendable across the boundary of their convergence domains.

2.1 Criterion of continuability of multiple power series across a family of polyarcs

Consider the multiple power series

$$f(z) = \sum_{k \in \mathbb{N}^n} f_k z^k \tag{2.1}$$

with the property

$$\overline{\lim_{|k| \to \infty}} \sqrt[|k|]{|f_k| R^k} = 1, \tag{2.2}$$

where $R^k = R_1^{k_1} ... R_n^{k_n}$, and $|k| = k_1 + ... + k_n$. According to the n-dimensional Cauchy-Hadamard theorem ([27], Section 7), the property (2.2) means that R_j

constitute the family of *radii of polydisk of convergence* of the series (2.1).

A subset G on the boundary of convergence domain is said to be *the regularity set of the series* (2.1), if the sum of the series can be analytically continued across any point of this set.

In this section we describe the regularity sets G that consist of families of polyarcs (direct products of arcs) from the distinguished boundary of the polydisk of convergence of the series (2.1).

Let $D_\rho(a) := \{z \in \mathbb{C} : |z - a| < \rho\}$ be an open circle with center $a \in \mathbb{C}$ and radius $\rho > 0$. Denote $D_\rho := D_\rho(0)$, and for $\sigma \in (0, \pi]$ by $\gamma_{\sigma,\rho}$ we denote the open arc $\partial D_\rho \setminus \Delta_\sigma$.

2.1.1 Formulation of Theorem 2.1

In the multivariate case there is no universal definition of the growth indicator of an entire function. Moreover, frequently the information on the growth of an entire function is represented in geometric terms. Following Ivanov [28] ((see also [22], Section 3, §3), we introduce the following set, which implicitly contains the notion of the growth indicator of an entire function $\varphi(z) \in \mathcal{O}(\mathbb{C}^n)$:

$$T_\varphi(\theta) = \{\nu \in \mathbb{R}^n : \ln|\varphi(re^{i\theta})| \le \nu_1 r_1 + ... + \nu_n r_n + C_{\nu,\theta}\},$$

where the inequality is satisfied for any $r \in \mathbb{R}^n_+$ with some constant $C_{\nu,\theta}$. Here $re^{i\theta}$ stands for the vector $(r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n})$. Thus, $T_\varphi(\theta)$ is the set of linear majorants (up to a shift by $C_{\nu,\theta}$)

$$\nu = \nu(r) = \nu_1 r_1 + ... + \nu_n r_n$$

for the logarithm of the modulus of function φ.

Define the set

$$\mathcal{M}_\varphi(\theta) := \{\nu \in \mathbb{R}^n : \ \nu + \varepsilon \in T_\varphi(\theta), \ \ \nu - \varepsilon \notin T_\varphi(\theta) \quad \text{for any} \ \ \varepsilon \in \mathbb{R}^n_+\},$$

which can be called *a boundary set of linear majorants*.

We say that an entire function φ interpolates the coefficients of the series (2.1), if the following equality is fulfilled:

$$\varphi(k) = f_k \text{ for all } k \in \mathbb{N}^n. \tag{2.3}$$

Let $D \subset \mathbb{C}^n$ be the domain of convergence of the series (2.1). Consider a family of polyarcs $\gamma_{\sigma,R}$:

$$G = \bigcup_R \gamma_{\sigma,R} = \bigcup_R (\gamma_{\sigma_1,R_1} \times ... \times \gamma_{\sigma_n,R_n}) \subset \partial D, \tag{2.4}$$

where R runs over the surface of conjugate radii of convergence of the series (2.1), and $\sigma = \sigma(R) = (\sigma_1(R), ..., \sigma_n(R))$.

Theorem 2.1. *A family G of polyarcs (2.4) is the regularity set for the series* (2.1) *if and only if there exists an interpolating the coefficients f_k entire function $\varphi(z)$ such that the following conditions are fulfilled:*
1) $0 \in \mathcal{M}_{R^z\varphi}(0)$,
2) there exists a vector-function $\nu_R(\theta)$ with values in $\mathcal{M}_{R^z\varphi}(\theta)$ to satisfy

$$\varlimsup_{(\theta_1,..\hat{j}..,\theta_n)\to 0} \varlimsup_{\theta_j \to 0} \frac{\nu_j(\theta)}{|\theta_j|} \leq \sigma_j(R), \quad j = 1, ..., n.$$

Observe that it is enough to prove the theorem for the polyarc $\gamma_{\sigma,R}$ from the distinguished boundary of the polydisk of convergence

$$\{|z_1| < R_1, ..., |z_n| < R_n\} = D_{R_1} \times ... \times D_{R_n}.$$

Namely, we prove the following proposition for fixed $R_1, ..., R_n$.

Proposition. *The polyarc $\gamma_{\sigma_1,R_1} \times ... \times \gamma_{\sigma_n,R_n}$ is the regularity set for the series* (2.1) *if and only if there exists an interpolating the coefficients f_k entire function $\varphi(z)$ such that the following conditions are fulfilled:*

1) $0 \in \mathcal{M}_{R^z\varphi}(0)$,

2) there exists a vector-function $\nu(\theta)$ *with values in* $\mathcal{M}_{R^z\varphi}(\theta)$, *to satisfy*

$$\overline{\lim_{(\theta_1,..\hat{j}..,\theta_n)\to 0}}\ \overline{\lim_{\theta_j\to 0}} \frac{\nu_j(\theta)}{|\theta_j|} \le \sigma_j, \quad j = 1, ..., n.$$

It is worthwhile to note that for the class of hypergeometric functions (this class contains a general algebraic function, that is, a function that is determined by a polynomial equation with independent variable coefficients) the polyarc of regularity can be extended to a polytope of regularity (see [35] and [26], Chapters 4,7). By this we mean is continuation of the series across a part of boundary of the domain of convergence which in the angular coordinates $\theta_1, ..., \theta_n$ is determined by a polytope, i.e. by a bounded polyhedron.

2.1.2 Necessity of the conditions of Theorem 2.1

Assume that the sum of series (2.1) can be continued across the polyarc

$$\gamma_{\sigma,R} = \gamma_{\sigma_1,R_1} \times ... \times \gamma_{\sigma_n,R_n}.$$

We show that there exists an entire function $\varphi(\zeta)$ that interpolates the coefficients f_k and satisfies conditions 1) and 2).

According to our assumption there exists a simply connected domain Ω containing $(D_{R_1} \times ... \times D_{R_n}) \cup \gamma_{\sigma,R}$, in which the sum of series (2.1) is holomorphic. By Hartogs' theorem (see, e.g.,[27], Section 32) this sum can be holomorphically continued into a domain containing

$$(D_{R_1} \cup \gamma_{\sigma_1,R_1}) \times ... \times (D_{R_n} \cup \gamma_{\sigma_n,R_n}).$$

We fix the numbers $r_o^j \in (0, R_j|1 - e^{i\sigma_j}|), \ \ j = 1, ..., n,$ to satisfy

$$(\bar{D}_{r_o^1}(-R_1)) \times ... \times (\bar{D}_{r_o^n}(-R_n)) \subset \Omega.$$

Denote $e^{\mu_o^j} := R_j + r_o^j,\ \ j = 1, ..., n$, and for any $\delta_j \in (0, \pi - \sigma_j)$ fix the numbers $\mu_j = \mu_{\delta_j}^j \in (\ln R_j, \mu_o^j)$ to satisfy

$$(\bar{D}_{e^{\mu_1}} \setminus \Delta^o_{\sigma_1+\delta_1}) \times ... \times (\bar{D}_{e^{\mu_n}} \setminus \Delta^o_{\sigma_n+\delta_n}) \subset \Omega.$$

Then for any $\varepsilon \in \mathbb{R}_+^n$ the domain

$$\Omega_{\varepsilon,\delta} = \Omega^1_{\varepsilon_1,\delta_1} \times ... \times \Omega^n_{\varepsilon_n,\delta_n},$$

where

$$\Omega^j_{\varepsilon_j,\delta_j} := (D_{r_o^j}(-R_j)) \cup (\bar{D}_{e^{\mu_j}} \setminus \Delta_{\sigma_j+\delta_j}) \cup D_{R_j e^{-\varepsilon_j}},\ \ j = 1, ..., n,$$

satisfies the condition $\bar{\Omega}_{\varepsilon,\delta} \subset \Omega$.

Denote by $\Gamma_{\varepsilon,\delta} := \partial\Omega^1_{\varepsilon_1,\delta_1} \times ... \times \partial\Omega^n_{\varepsilon_n,\delta_n}$ the distinguished boundary of $\Omega_{\varepsilon,\delta}$. Since $f \in \mathcal{O}(\bar{\Omega}_{\varepsilon,\delta})$, we can apply the Cauchy integral formula for the coefficients of the power series (2.1) to obtain

$$f_k = (2\pi i)^{-n} \int\limits_{\Gamma_{\varepsilon,\delta}} \zeta^{-k-I} f(\zeta) d\zeta,\ \ k \in \mathbb{N}^n,$$

where $I = (1, ..., 1) \in \mathbb{N}^n$ and $d\zeta = d\zeta_1 ... d\zeta_n$. As a desired interpolating function φ we take the same integral but with a complex parameter z instead of integer k:

$$\varphi(z) = (2\pi i)^{-n} \int\limits_{\Gamma_{\varepsilon,\delta}} \zeta^{-z-I} f(\zeta) d\zeta, \quad \text{where } \zeta_j^{z_j} = e^{z_j \log \zeta_j}. \tag{2.5}$$

Observe that $\varphi(z)$ is an entire function because it is an integral over a compact set of a function continuous up to the boundary with respect to the variables $(\zeta, z) \in (\Omega \cap (\mathbb{C} \setminus \mathbb{R}_-)^n) \times \mathbb{C}^n$ and holomorphic everywhere in the variable z, where R_- stands for the negative real semiaxis (see [36]).

Now we are going to obtain an estimate for the function φ. To this end, we deform the distinguish boundary $\Gamma_{\varepsilon,\delta}$ as follows. The parts of arcs from $\partial D_{r_o^j}(-R_j)$, the dotted curves in Fig. 6, we replace by two arcs on $\partial D_{e_j^\mu}$ and a pair of segments $[-e^{\mu_o^j}, -e^{\mu_j}]$ and $[-e^{\mu_j}, -e^{\mu_o^j}]$ oriented in the opposite directions. The contour obtained for each $j = 1, ..., n$ we denote by $L^j_{\varepsilon_j,\delta_j}$. Then the entire distinguished boundary $\Gamma_{\varepsilon,\delta}$ is deformed into an n-dimensional loop

$$L_{\varepsilon,\delta} = L^1_{\varepsilon_1,\delta_1} \times ... \times L^n_{\varepsilon_n,\delta_n}.$$

Observe that for a fixed $r_0 \in \mathbb{R}^n_+$ and a chosen $\sigma \in \mathbb{R}^n_+$ the curves $L^j_{\varepsilon_j,\delta_j}$ and $L^j_{\acute{\varepsilon}_j,\acute{\delta}_j}$ bound a path, where the integrand in (2.5) is univalent and holomorphic in ζ_j, and hence the value of $\varphi(z)$ given by the integral (2.5) is independent of ε and δ.

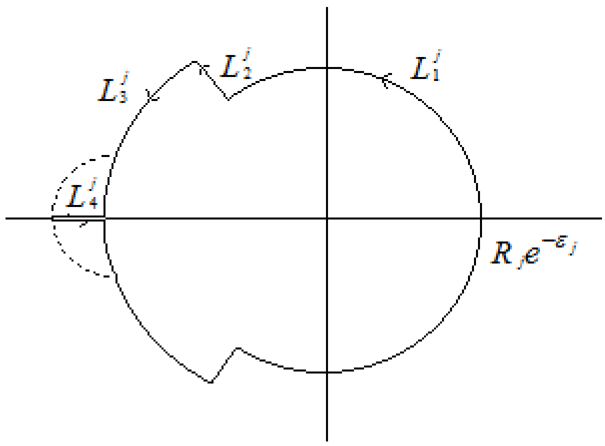

Figure 6

Denoting $z_j = \xi_j + i\eta_j, \quad j = 1, ..., n$, and

$$M_{\varepsilon,\delta} := \sup_{\zeta \in L_{\varepsilon,\delta}} |f(\zeta)|,$$

from (2.5) we obtain the estimate

$$|\varphi(z)| \le M_{\varepsilon,\delta}\mathcal{I}, \quad z \in \mathbb{C}^n, \tag{2.6}$$

where

$$\mathcal{I} = \pi^n \int\limits_{L_1^+ \times ... \times L_n^+} |\zeta_1|^{-\xi_1 - 1} e^{|\eta_1| \arg \zeta_1} ... |\zeta_n|^{-\xi_n - 1} e^{|\eta_n| \arg \zeta_n} |d\zeta_1| ... |d\zeta_n|.$$

Here L_j^+ denotes the part of $L^j_{\varepsilon_j,\delta_j}$ lying in the upper half-plane, that is,

$$L^j_+ = L^j_1 \cup L^j_2 \cup L^j_3 \cup L^j_4,$$

where

$$L^j_1 = \{R_j e^{-\varepsilon_j + i\omega_j} : \omega_j \in [0, \sigma_j + \delta_j)\},$$

$$L^j_2 = \{t_j e^{i(\sigma_j + \delta_j)} : t_j \in [R_j e^{-\varepsilon_j}, e^{\mu_j})\},$$

$$L^j_3 = \{e^{\mu_j + i\omega_j} : \omega_j \in [\sigma_j + \delta_j, \pi)\},$$

$$L^j_4 = \{t_j e^{i\pi} : t_j \in [e^{\mu_j}, e^{\mu^j_o}]\}.$$

Therefore $\mathcal{I}$ can be represented as a sum of integrals $\mathcal{I}_{p_1,...,p_n}$ over the $L^1_{p_1} \times ... \times L^n_{p_n}, \;\; p_1, ..., p_n = 1, 2, 3, 4$. Observe that each such path is a direct product of *arcs* (with centers at zero) and line segments (passing through zero). This observation allows to obtain effective estimates of the integrals $\mathcal{I}_{p_1,...,p_n}$. For instance, we have

$$\mathcal{I}_{1...1} = \frac{1}{\pi^n} \int\limits_{L^1_1 \times ... \times L^n_1} |\zeta_1|^{-\xi_1} e^{|\eta_1| \arg \zeta_1} ... |\zeta_n|^{-\xi_n} e^{|\eta_n| \arg \zeta_n} |\frac{d\zeta_1}{\zeta_1}| ... |\frac{d\zeta_n}{\zeta_n}| =$$

$$= \frac{1}{\pi^n} \int\limits_0^{\sigma_1+\delta_1} ... \int\limits_0^{\sigma_n+\delta_n} \prod_{j=1}^n (R_j^{-\xi_j} e^{\varepsilon_j \xi_j + |\eta_j| \omega_j}) d\omega_1 ... d\omega_n =$$

$$= \frac{1}{\pi^n} \prod_{j=1}^n \left(R_j^{-\xi_j} e^{\varepsilon_j \xi_j} \; \frac{e^{(\sigma_j+\delta_j)|\eta_j|} - 1}{|\eta_j|} \right).$$

Taking into account that $\sigma_j + \delta_j \le \pi$, we can use the inequality $e^{ax} - 1 \le ae^{ax}$, where $a \ge 0,\ x \ge 0$, to obtain the estimate

$$\mathcal{I}_{1\ldots 1} \le \prod_{j=1}^{n} \left(R_j^{-\xi_j} e^{\varepsilon_j \xi_j}\ e^{(\sigma_j+\delta_j)|\eta_j|} \right).$$

By similar argument we get

$$\mathcal{I}_{2\ldots 2} = \frac{1}{\pi^n} \int\limits_{L_2^1 \times \ldots \times L_2^n} |\zeta_1|^{-\xi_1} e^{|\eta_1| \arg \zeta_1} \ldots |\zeta_n|^{-\xi_n} e^{|\eta_n| \arg \zeta_n} |\frac{d\zeta_1}{\zeta_1}| \ldots |\frac{d\zeta_n}{\zeta_n}| =$$

$$= \frac{1}{\pi^n} \int\limits_{R_1 e^{-\varepsilon_1}}^{e^{\mu_1}} \ldots \int\limits_{R_j e^{-\varepsilon_n}}^{e^{\mu_n}} \prod_{j=1}^{n} \left(t_j{}^{-\xi_j - 1} e^{|\eta_j|(\sigma_j+\delta_j)} \right) dt_1 \ldots dt_n =$$

$$= \frac{1}{\pi^n} \prod_{j=1}^{n} \left(e^{|\eta_j|(\sigma_j+\delta_j)} \left(\frac{e^{-\mu_j \xi_j} - R_j^{-\xi_j} e^{\varepsilon_j \xi_j}}{-\xi_j} \right) \right).$$

Therefore for $\xi_j \ge 1$ we obtain the estimate

$$\mathcal{I}_{2\ldots 2} \le \prod_{j=1}^{n} \left(e^{|\eta_j|(\sigma_j+\delta_j)} R_j^{-\xi_j} e^{\varepsilon_j \xi_j} \right).$$

Furthermore, we have

$$\mathcal{I}_{3\ldots 3} = \frac{1}{\pi^n} \int\limits_{\sigma_1+\delta_1}^{\pi} \ldots \int\limits_{\sigma_n+\delta_n}^{\pi} \prod_{j=1}^{n} \left(e^{-\mu_j \xi_j} e^{\omega_j |\eta_j|} \right) d\omega_1 \ldots d\omega_n =$$

$$= \frac{1}{\pi^n} \prod_{j=1}^{n} \left(e^{-\mu_j \xi_j} \left(\frac{e^{|\eta_j|\pi} - e^{|\eta_j|(\sigma_j+\delta_j)}}{|\eta_j|} \right) \right)$$

$$= \frac{1}{\pi^n} \prod_{j=1}^{n} \left(e^{-\mu_j \xi_j} e^{|\eta_j|(\sigma_j+\delta_j)} \left(\frac{e^{|\eta_j|(\pi - \sigma_j - \delta_j} - 1}{|\eta_j|} \right) \right) =$$

$$= \frac{1}{\pi^n} \prod_{j=1}^{n} \left((\pi - \sigma_j - \delta_j) e^{-\mu_j \xi_j} e^{|\eta_j|(\sigma_j+\delta_j)} \left(\frac{e^{|\eta_j|(\pi - \sigma_j - \delta_j)} - 1}{|\eta_j|(\pi - \sigma_j - \delta_j)} \right) \right) \le$$

$$\leq \frac{1}{\pi^n}\prod_{j=1}^{n}\left(\pi e^{-\mu_j\xi_j}e^{|\eta_j|(\sigma_j+\delta_j)}e^{|\eta_j|(\pi-\sigma_j-\delta_j)}\right) \leq$$

$$\leq \prod_{j=1}^{n} e^{-\mu_j\xi_j+\pi|\eta_j|}.$$

Finally, we have

$$\mathcal{I}_{4\dots 4} = \frac{1}{\pi^n}\int\limits_{e^{\mu_1}}^{e^{\mu_o^1}}\dots\int\limits_{e^{\mu_n}}^{e^{\mu_o^n}} {t_1}^{-\xi_1-1}e^{\pi|\eta_1|}\dots {t_n}^{-\xi_n-1}e^{\pi|\eta_n|}dt_1\dots dt_n =$$

$$= \frac{1}{\pi^n}\prod_{j=1}^{n}\left(e^{\pi|\eta_j|}\left(\frac{e^{-\mu_j\xi_j}+e^{-\mu_o^j\xi_j}}{\xi_j}\right)\right),$$

which for $\xi_j \geq 1$ implies the estimate

$$\mathcal{I}_{4\dots 4} \leq \prod_{j=1}^{n} e^{-\mu_j\xi_j+\pi|\eta_j|}.$$

The obtained results show that in repeated calculation of the integral $\mathcal{I}_{p_1\dots p_n}$ depending on the value of p_j (indicating that integration by the variable ζ_j is over the part $L^j_{p_j}$), the contribution of this integral in the estimate is given by the following relations:

$$R_j^{-\xi_j}e^{\varepsilon_j\xi_j}\ e^{(\sigma_j+\delta_j)|\eta_j|},\ \text{if}\ \ p_j=1,$$

$$e^{|\eta_j|(\sigma_j+\delta_j)}R_j^{-\xi_j}e^{\varepsilon_j\xi_j},\ \text{if}\ p_j=2\ \text{and}\ \xi_j\geq 1,$$

$$e^{-\mu_j\xi_j+\pi|\eta_j|},\ \text{if}\ p_j=3,$$

$$e^{-\mu_j\xi_j+\pi|\eta_j|},\ \text{if}\ p_j=4\ \text{and}\ \xi_j\geq 1.$$

Each collection $p_1, ..., p_n$ we divide into 4 groups: A_1, A_2, A_3, A_4, where A_j denote the numbers of $k \in \{1, ..., n\}$, for which $p_k = j$. Then for $\xi_l \geq 1,\ j = 1, ...n$, we obtain

$$\mathcal{I}_{p_1,\dots,p_n} \leq \prod_{j\in A_1} R_j^{-\xi_j}e^{\varepsilon_j\xi_j}\ e^{(\sigma_j+\delta_j)|\eta_j|}\prod_{j\in A_2} e^{|\eta_j|(\sigma_j+\delta_j)}R_j^{-\xi_j}e^{\varepsilon_j\xi_j}\times$$

$$\times \prod_{j\in A_3} e^{-\mu_j\xi_j+\pi|\eta_j|} \prod_{j\in A_4} e^{-\mu_j\xi_j+\pi|\eta_j|}.$$

For $\pi|\eta_j| \leq \mu_j\xi_j$ we have $e^{-\mu_j\xi_j+\pi|\eta_j|} \leq 1$, Hence in view of the estimate above, we obtain the following estimate for the integral $\mathcal{I}$:

$$\mathcal{I} < C_{\varepsilon,\delta} R_1^{-\xi_1}...R_n^{-\xi_n} e^{|\eta_1|(\sigma_1+\delta_1)+\varepsilon\xi_1}...e^{|\eta_n|(\sigma_n+\delta_n)+\varepsilon\xi_n}.$$

Thus, with notation $\zeta_j = r_j e^{i\theta_j}$ and $\alpha_j = \arctan(\mu_j/\pi)$ the inequality (2.6) gives the following estimate for the function φ:

$$|\varphi(re^{i\theta})| \leq cR^{-r\cos\theta} e^{((\sigma_1+\delta_1)|\sin\theta_1|+\varepsilon_1\cos\theta_1)r_1+...+((\sigma_n+\delta_n)|\sin\theta_n|+\varepsilon_n\cos\theta_n)r_n}, \tag{2.7}$$

if $|\theta_j| \leq \alpha_j, \;\; j = 1, ..., n.$

Observe that the inequality (2.7) can be written in the following form:

$$R^{r\cos\theta}|\varphi(re^{i\theta})| \leq ce^{((\sigma_1+\delta_1)|\sin\theta_1|+\varepsilon_1\cos\theta_1)r_1+...+((\sigma_n+\delta_n)|\sin\theta_n|+\varepsilon_n\cos\theta_n)r_n}.$$

Therefore, taking its logarithm, for $|\theta_j| \leq \alpha_j, \; j = 1, ..., n$ we obtain

$$\ln\left(R^{r\cos\theta}|\varphi(re^{i\theta}|\right) \leq c + \sum_{j=1}^{n} \left(((\sigma_j+\delta_j)|\sin\theta_j| + \varepsilon_j\cos\theta_j)r_j\right). \tag{2.8}$$

Taking $\theta = 0$ in the inequality (2.8), we obtain for any $\varepsilon \in \mathbb{R}^n_+$

$$\ln\left(R^r|\varphi(r)|\right) \leq c + <\varepsilon, r> \tag{2.9}$$

implying that $0 \in T_{R^z\varphi}(0)$.

Next, in view of (2.2) and (2.3) we conclude that

$$\ln\left(R^k|\varphi(k)|\right)^{\frac{1}{|k|}} = 0 \quad \text{as} \quad |k| \to \infty, \tag{2.10}$$

implying that for any $\varepsilon \in \mathbb{R}^n_+$ we have $-\varepsilon \notin T_{R^z\varphi}(0)$. Hence, by (2.9) and (2.10) we obtain $0 \in \mathcal{M}_{R^z\varphi}(0)$.

Also, in view of the inequality (2.8) for any $\varepsilon \in \mathbb{R}^n_+$ we have

$$((\sigma_1+\delta_1)|\sin\theta_1| + \varepsilon_1\cos\theta_1, ..., (\sigma_n+\delta_n)|\sin\theta_n| + \varepsilon_n\cos\theta_n) \in T_{R^z\varphi}(\theta),$$

if $|\theta_j| \le \alpha_j,\ j = 1, ..., n.$

Therefore there exists $\nu(\theta) = (\nu_1(\theta), ..., \nu_n(\theta)) \in \mathcal{M}_{R^z\varphi}(\theta)$ with the properties

$$\nu_j(\theta) \le (\sigma_j + \delta_j)|\sin\theta_j| \text{ for } |\theta_j| \le \alpha_j,\ j = 1, ..., n.$$

For the components of $\nu(\theta)$ we obtain

$$\overline{\lim_{(\theta_1,..\hat{j}..,\theta_n)\to 0}}\ \overline{\lim_{\theta_j\to 0}} \frac{\nu_j(\theta)}{|\theta_j|} \le \sigma_j, \quad j = 1, ..., n.$$

Thus, the necessity of conditions of Proposition, and hence, of Theorem 2.1 is proved.

2.1.3 Sufficiency of the conditions of Theorem 2.1

Let φ be an entire function satisfying conditions 1) and 2) of Proposition. We show that the series (2.1) can be continued across the polyarc $\gamma_{\sigma_1,R_1} \times ... \times \gamma_{\sigma_n,R_n}$. To this end, we first observe that by condition 2) for any $\delta_j \in (0, \frac{\pi-\sigma_j}{2})$ there exists α_j such that

$$\nu_j(\theta) \le (\sigma_j + \delta_j)|\sin\theta_j|, \text{ if } |\theta_j| \le \alpha_j, \quad j = 1, ..., n.$$

Since $\nu(\theta) \in \mathcal{M}_{R^z\varphi}(\theta)$, we have

$$\ln\left(R^{r\cos\theta}|\varphi(re^{i\theta})|\right) \le ((\sigma_1 + \delta_1)|\sin\theta_1|)r_1 + ... + ((\sigma_n + \delta_n)|\sin\theta_n|)r_n + c,$$

implying the estimate

$$|\varphi(re^{i\theta})| \le e^c R_1^{-r_1\cos\theta_1} ... R_n^{-r_n\cos\theta_n} e^{((\sigma_1+\delta_1)|\sin\theta_1|)r_1+...+((\sigma_n+\delta_n)|\sin\theta_n|)r_n}. \qquad (2.11)$$

We introduce the following auxiliary function

$$g(\zeta, z) = \prod_{j=1}^{n} \frac{z_j^{\zeta_j}}{(e^{2\pi i\zeta_j} - 1)},$$

where $\zeta_j = \xi_j + i\eta_j,\ z_j = x_j + iy_j,\ j = 1, ..., n$, and observe that it is meromorphic in variables ζ from $\mathbb{C}^n$ and holomorphic in variables z from $(\mathbb{C} \setminus \mathbb{R}_+)^n$.

Denote $D^* := \cup_{m\in\mathbb{Z}} D_{1/4}(m)$, and note that there exists a constant $C > 0$ such that

$$|e^{2\pi i w} - 1| > \frac{e^{\pi(|\operatorname{Im} w| - \operatorname{Im} w)}}{C} \quad \text{for} \quad w \in D^*.$$

Therefore we have the estimate

$$g(\zeta, z) < Ce^{\langle \xi, \log|z|\rangle - \langle (\pi - |\pi - \arg z|), |\eta|\rangle} \tag{2.12}$$

for $\zeta \in (\mathbb{C} \setminus D^*)^n$ and $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$. Using (2.11) and (2.12) for $\zeta \in (\Delta_{\sigma_1} \setminus D^*) \times ... \times (\Delta_{\sigma_n} \setminus D^*)$ and $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$, we obtain

$$|\varphi(\zeta)||g(\zeta, z)| < cR^{-\xi} e^{\langle \sigma+\delta, \eta\rangle} e^{\langle \xi, \log|z|\rangle - \langle (\pi - |\pi - \arg z|), |\eta|\rangle} =$$
$$= cR^{-\xi} e^{\langle \xi, \log|z|\rangle - \langle (\pi - |\pi - \arg z| - \sigma - \delta), |\eta|\rangle}.$$

Denoting

$$d(z) = (d_1(z_1), ..., d_n(z_n)), \quad d_j(z_j) = \pi - |\pi - \arg z_j| - \sigma_j - \delta_j, \quad j = 1, ..., n,$$

we obtain

$$|\varphi(\zeta)||g(\zeta, z)| < cR^{-\xi} e^{\langle \xi, \log|z|\rangle - \langle d(z), |\eta|\rangle}. \tag{2.13}$$

Consider the sets

$$K_j = \bar{D}_{R_j e^{\varepsilon_j}} \setminus (\Delta^o_{\sigma_j + 2\delta_j} \cup D_{R_j/2}), \ \ \varepsilon_j > 0, \ \ j = 1, ..., n.$$

We show that

$$d_j(z_j) \geq \delta_j \quad \text{for} \ \ z_j \in K_j \quad j = 1, ..., n.$$

Indeed, we have

$$d_j(z_j) = \pi - \sigma_j - \delta_j - |\pi - \arg z_j|, \quad j = 1, ..., n.$$

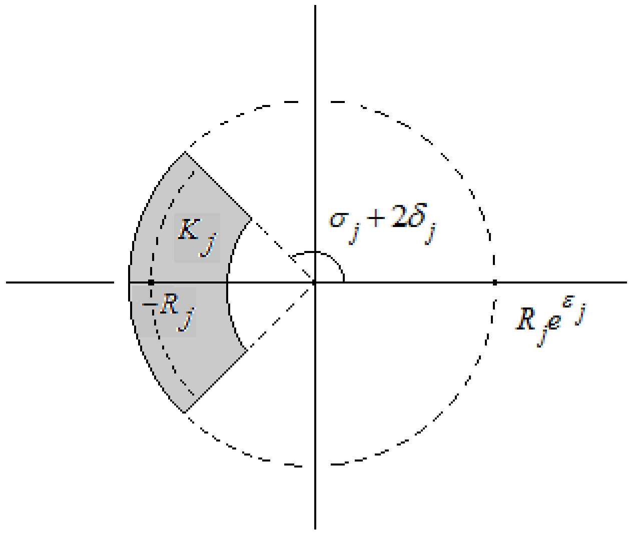

Figure 7

Taking into account that $z_j \in K_j$, we can write

$$1)\sigma_j + 2\delta_j < \arg z_j < \pi, \;\; \Rightarrow \;\; d_j(z_j) = \pi - \sigma_j - \delta_j - \pi + \sigma_j + 2\delta_j \geq \delta_j,$$

$$2)\pi < \arg z_j < 2\pi - \sigma_j - 2\delta_j, \;\; \Rightarrow \;\; d_j(z_j) = 2\pi - \sigma_j - \delta_j - 2\pi + \sigma_j + 2\delta_j \geq \delta_j.$$

Thus, for $(z_1, ..., z_n) \in (K_1 \times ... \times K_n)$ and $(\zeta_1, ..., \zeta_n) \in (\partial\Delta_{\alpha_1} \setminus D^*) \times ... \times (\partial\Delta_{\alpha_n} \setminus D^*)$ we obtain

$$|g(\zeta, z)||\varphi(\zeta)| < cR^{-\xi}e^{\langle \xi, \log|z|\rangle - \langle \delta, |\eta|\rangle} \leq$$
$$\leq cR^{-\xi}e^{\langle \xi, \log(Re^{\varepsilon})\rangle - \langle \delta, |\eta|\rangle} = ce^{\langle \xi, \varepsilon\rangle - \langle \delta, |\eta|\rangle}.$$

Taking $2\varepsilon_j = \delta_j \sin\alpha_j, \;\; j = 1, ..., n$, we get

$$|g(\zeta, z)||\varphi(\zeta)| < ce^{\langle -\varepsilon, |\zeta|\rangle}. \tag{2.14}$$

For each $j \in \{1, ..., n\}$ consider the domain

$$G_j := D_{R_j} \cup \Delta^o_{\alpha_j},$$

and let $\Gamma_j = \partial G_j$ be the boundary of this domain positively oriented with respect to the origin. Then for each natural m_j consider the following part of Γ_j :

$$\Gamma^j_{m_j} := \{\zeta_j = \xi_j + i\eta_j \in \Gamma_j : \xi_j \leq m_j + \frac{1}{2}\}.$$

Denote by L^j_m the vertical segment with vertices (see Fig. 8)

$$(m_j + \frac{1}{2})(1 \pm i \tan \alpha_j) \ \text{ for } \ m_j \in \mathbb{N},$$

oriented by movement upward. The domain bounded by the union $\Gamma^j_{m_j} \cup \ L^j_{m_j}$ we denote by $G^j_{m_j}$ and hence

$$\partial G^j_{m_j} = \Gamma^j_{m_j} \cup L^j_{m_j}.$$

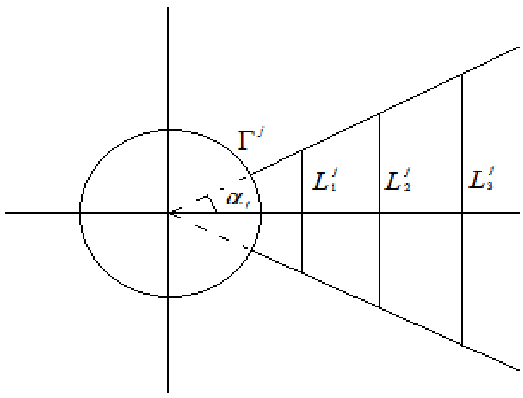

Figure 8

Consider the following integral

$$I_m = \int\limits_{\partial G_1^{m_1} \times \ldots \times \partial G_n^{m_n}} g(\zeta, z)\varphi(\zeta) d\zeta = \tag{2.15}$$

$$= \int\limits_{\partial G_1^{m_1} \times \ldots \times \partial G_n^{m_n}} \prod_{j=1}^{n} \frac{z^{\zeta_j}}{(e^{2\pi i \zeta_j} - 1)} \varphi(\zeta) d\zeta.$$

Now we compute the integral in (2.15) by means of multidimensional residues. To this end, observe first that the integrand in (2.15) defines the differential form

$$\omega = \prod_{j=1}^{n} \frac{z^{\zeta_j}}{(e^{2\pi i \zeta_j} - 1)} \varphi(\zeta) d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

with poles on divisors

$$Q_1 = \{(\zeta_1, ..., \zeta_n) : f_1 = e^{2\pi i\zeta_1} - 1 = 0\} = \mathbb{Z} \times \mathbb{C}^{n-1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Q_n = \{(\zeta_1, ..., \zeta_n) : f_n = e^{2\pi i\zeta_n} - 1 = 0\} = \mathbb{C}^{n-1} \times \mathbb{Z}.$$

Next, since the intersection $Z = Q_1 \cap ... \cap Q_n = \mathbb{Z}^n$ is discrete and the Jacobian is different from zero, that is, $\partial(f)/\partial(\zeta) = (2\pi i)^n \neq 0$ at points $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, then for any point $k \in \mathbb{Z}^n$ we can define the local residue (see [37], [31]) :

$$res_k\omega = \frac{z^k\varphi(k)}{\frac{\partial(f)}{\partial(\zeta)}(k)} = \varphi(k)z^k. \tag{2.16}$$

The position of the distinguished boundary and the polar divisors $Q_1, ..., Q_n$ is such that

$$Q_1 \cap (\partial G_1^{m_1} \times ... \times G_n^{m_n}) = (\mathbb{Z} \times \mathbb{C}^{n-1}) \cap (\partial G_1^{m_1} \times ... \times G_n^{m_n}) = \emptyset,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Q_n \cap (G_1^{m_1} \times ... \times \partial G_n^{m_n}) = (\mathbb{C}^{n-1} \times \mathbb{Z}) \cap (G_1^{m_1} \times ... \times \partial G_n^{m_n}) = \emptyset.$$

According to the terminology of [37], this means that the polyhedron $G_1^{m_1} \times ... \times G_n^{m_n}$ is consistent with the divisors $Q_1, ..., Q_n$. Therefore, according to the principle of separating cycles the integral (2.15) after multiplication by $(2\pi i)^{-n}$ is equal to the sum of residues over all points

$$k \in (G_1^{m_1} \times ... \times G_n^{m_n}) \cap (\mathbb{Z} \times ... \times \mathbb{Z}).$$

Hence, in view of (2.16) we obtain

$$I_m = \sum_{k_1=0}^{m_1} ... \sum_{k_n=0}^{m_n} \varphi(k_1, ..., k_n) z_1^{k_1} ... z_n^{k_n}.$$

Now we represent the integral (2.15) as a sum of 2^n integrals over the paths

$$\Gamma^1_{m_1} \times ... \times \Gamma^n_{m_n}, ..., L^1_{m_1} \times ... \times L^n_{m_n}.$$

For each of such path we split the integration variables $\zeta_1, ..., \zeta_n$ into two groups: B_1 and B_2, where B_1 stands for the numbers $j \in \{1, ..., n\}$ for which $\zeta_j \in L^j_{m_j}$, while B_2 stands for the numbers $j \in \{1, ..., n\}$, for which $\zeta_j \in \Gamma^j_{m_j}$. Then using (2.13), we obtain the following estimate

$$|g(\zeta, z)||\varphi(\zeta)| < C \prod_{j \in B_1} e^{m_j \log \frac{|z_j|}{R_j}} \prod_{j \in B_2} e^{-\varepsilon_j |\zeta_j|}, \tag{2.17}$$

where $z \in (D_{R_1} \cap K^o_1) \times ... \times (D_{R_n} \cap K^o_n)$.

It follows from (2.17) that if $B_1 \neq \emptyset$, then the integral over the corresponding path tends to zero as $m_j \to \infty, \ j \in B_1$.

Finally, we consider the integral

$$\mathcal{I} = \int\limits_{\Gamma_1 \times ... \times \Gamma_n} g(\zeta, z)\varphi(\zeta) d\zeta.$$

It follows from (2.14) that the integral I converges uniformly on z from the compact set $(K_1 \times ... \times K_n)$, and defines a holomorphic function in its interior.

Taking into account that $I_m \to \mathcal{I}$ for $m_j \to \infty, \quad j = 1, ..., n$, we obtain $\mathcal{I}(z) = f(z)$ for $z \in (D_{R_1} \cap K^o_1) \times ... \times (D_{R_n} \cap K^o_n)$. This means that $\gamma_{\sigma+2\delta, R}$ is a polyarc of regularity for f, provided that δ is sufficiently close to zero. Thus, $\gamma_{\sigma, R}$ is a polyarc of regularity for f, and the result follows.

2.2 Conditions of continuability of multiple power series into a sectorial domain

Here we give sufficient conditions for analytic continuability of a multiple power series to a sectorial domain. A domain $G \subset \mathbb{C}^n$ is called sectorial if it is defined by the conditions on the arguments $\theta = (argz_1, ..., argz_n)$ of elements $z \in \mathbb{C}^n$ only. As in Theorem 2.1, the conditions of continuability are expressed in terms of a vector-function $\nu(\theta)$ with the values in $\mathcal{M}_\varphi(\theta)$, however, more precisely. Recall that φ is an entire function interpolating the coefficients.

Denote

$$T_\varphi := \bigcap_{\theta_j = \pm\frac{\pi}{2}} T_\varphi(\theta_1, ..., \theta_n).$$

$$\mathcal{M}_\varphi := \{\nu \in [0,\pi)^n : \ \nu + \varepsilon \in T_\varphi, \ \ \nu - \varepsilon \notin T_\varphi \quad \text{for any} \ \ \varepsilon \in \mathbb{R}^n_+\}.$$

Let G be a sectorial set

$$G = \bigcup_{\nu \in \mathcal{M}_\varphi} G_\nu \tag{2.18}$$

where

$$G_\nu = (\mathbb{C} \setminus \Delta_{\nu_1}) \times ... \times (\mathbb{C} \setminus \Delta_{\nu_n}).$$

This set is a domain: it is open and connected because every polysector G_ν is connected and contains the point $(-1, ..., -1)$.

Theorem 2.2. *The sum of the series* (2.1) *extends analytically to a sectorial domain G of the form* (2.18) *if there is an entire function* $\varphi(\zeta)$ *of exponential type interpolating the coefficients* f_k *and a vector-function* $\nu(\theta)$ *on* $[-\frac{\pi}{2}, \frac{\pi}{2}]^n$ *with values in* $\mathcal{M}_\varphi(\theta)$ *to satisfy*

$$\nu_j(\theta) \le a|\sin\theta_j| + b\cos\theta_j, \ \ j = 1, ..., n, \tag{2.19}$$

with some constants $a \in [0,\pi), \ \ b \in [0,\infty)$.

We need the following proposition regarding the properties of the set T_φ. Let $I = (1, ..., 1) \in \mathbb{Z}^n$.

Proposition 1. *If* $\nu \in T_\varphi(\theta) \cap T_\varphi(\theta + \pi I)$ *then* $\nu \in \mathbb{R}^n_{\ge 0}$.
It follows that $T_\varphi \subset \mathbb{R}^n_{\ge 0}$.

Proof. For a given entire function φ in $\mathbb{C}^n$ consider the function

$$L_r(z,\varphi)=\overline{\lim_{t\to\infty}}\frac{\ln|\varphi(tz)|}{t},\quad z\in\mathbb{C}^n$$

called the *radial indicator* of φ. The function $L_r(z,\varphi)$ is plurisubharmonic, positively homogenous in $\mathbb{C}^n$ and has the following property $([22],\mathrm{Ch.3},\S5)$:

$$L_r(e^{i\pi}z,\varphi)+L_r(z,\varphi)\geq 0. \tag{2.20}$$

From the definition of $T_\varphi(\theta)$ it follows that

$$L_r(re^{i\theta},\varphi)=\overline{\lim_{t\to\infty}}\frac{\ln|\varphi(tre^{i\theta})|}{t}\leq \nu_1r_1+...+\nu_nr_n$$

for all $\nu\in T_\varphi(\theta)$. Analogously

$$L_r(e^{i\pi}re^{i\theta},\varphi)=\overline{\lim_{t\to\infty}}\frac{\ln|\varphi(tre^{i(\theta+\pi})|}{t}\leq \tilde{\nu}_1r_1+...+\tilde{\nu}_nr_n$$

for all $\tilde{\nu}\in T_\varphi(\theta+\pi I)$.

According to (2.20) we see that for all $\nu\in T_\varphi(\theta)$, $\tilde{\nu}\in T_\varphi(\theta+\pi I)$ the inequality

$$(\nu_1+\tilde{\nu}_1)r_1+...+(\nu_n+\tilde{\nu}_n)r_n\geq 0$$

hold for all $r\in\mathbb{R}^n_{\geq 0}$. In particular, if $\nu\in T_\varphi(\theta)\cap T_\varphi(\theta+\pi I)$ then

$$2(\nu_1r_1+...+\nu_nr_n)\geq 0,\quad \text{for any}\ \ r\in\mathbb{R}^n_{\geq 0},$$

hence $\nu\in\mathbb{R}^n_{\geq 0}$.

Let now $\nu\in T_\varphi$. Then

$$\nu\in T_\varphi(-\frac{\pi}{2}I)\cap T_\varphi(\frac{\pi}{2}I)=T_\varphi(-\frac{\pi}{2}I)\cap T_\varphi(-\frac{\pi}{2}I+\pi I).$$

According to proved above, one has $\nu\in\mathbb{R}^n_{\geq 0}$.

Proof of Theorem 2.2. Let φ be an entire function satisfying the conditions of Theorem 2.2. For all $\nu\in T_\varphi(\theta)$ this function satisfies the inequality

$$|\varphi(re^{i\theta})|\leq A_{\nu,\theta}e^{\langle\nu,r\rangle}\ \ \forall r\in{\mathbb{R}_+}^n.$$

Hence for $\nu \in \mathcal{M}_\varphi(\theta)$ it satisfies the inequality

$$|\varphi(re^{i\theta})| \le A_{\nu,\theta} e^{\langle \nu, r\rangle + o(r)} \quad \forall r \in {\mathbb{R}_+}^n.$$

From (2.19) we get that for $\theta_j \le \frac{\pi}{2}, j = 1, ..., n$.

$$|\varphi(re^{i\theta})| \le A_{\nu,\theta} e^{a\sum_{j=1}^n r_j |\sin\theta_j| + b\sum_{j=1}^n r_j \cos\theta_j + o(r)}.$$

Write $\zeta_j = \xi_j + i\eta_j = r_j e^{i\theta_j}$, then

$$|\varphi(\zeta)| \le A_{\nu,\theta} e^{a\sum |\eta_j| + b\sum \xi_j + o(|\zeta|)} \tag{2.21}$$

for $\zeta_j \in \Delta_{\pi/2}, \;\; j = 1, ..., n$. Consider the following function

$$g(\zeta, z) = \prod_{j=1}^n \frac{z_j^{\zeta_j}}{(e^{2\pi i \zeta_j} - 1)},$$

where $z_j = x_j + iy_j, \; j = 1, ..., n$. It is meromorphic in $\zeta \in \mathbb{C}^n$ and holomorphic in $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$.

Using (2.21) and (2.12) for $\zeta \in (\Delta_{\pi/2} \setminus D^*)^n$ and $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$, we obtain

$$|\varphi(\zeta)||g(\zeta, z)| \le c e^{b\sum \xi_j + a\sum |\eta_j|} e^{\sum \xi_j \ln |z_j| - \sum (\pi - |\pi - \arg z_j|)|\eta_j| + o(|\zeta|)} =$$

$$= c e^{\sum \xi_j (\ln |z_j| + b) - \sum (\pi - a - |\pi - \arg z_j|)|\eta_j| + o(|\zeta|)}.$$

Denoting $d(z_j) = \pi - a - |\pi - \arg z_j|$ we get

$$|\varphi(\zeta)||g(\zeta, z)| \le c e^{\sum \xi_j (\ln |z_j| + b) - \sum d(z_j)|\eta_j| + o(|\zeta|)}.$$

Let

$$K = \bar{D}_{e^{-b-\delta}} \setminus (\Delta^o_{a+\delta} \cup D_{e^{-2b}}).$$

Note that

$$d(z_j) \ge \delta \quad \text{as} \quad z_j \in K \quad j = 1, ..., n.$$

Thus, for $z \in K^n$ and $\zeta \in (\Delta_{\frac{\pi}{2}} \setminus D^*)^n$ we get

$$|g(\zeta, z)||\varphi(\zeta)| < ce^{\sum \xi_j(\ln|z_j|+b)-\delta\sum|\eta_j|+o(|\zeta|)}. \tag{2.22}$$

For any $m = (m_1, ..., m_n) \in \mathbb{N}^n$ we consider the integral

$$I_m = \int\limits_{\partial G_m} \varphi(\zeta)g(\zeta, z)d\zeta, \tag{2.23}$$

where $\partial G_m = \partial G_{m_1} \times ... \times \partial G_{m_n}$. Each of the plane domains G_{m_j} is bounded by segments: $\partial G^j_{m_j} = \Gamma^1_{m_j} \cup \Gamma^2_{m_j} \cup \Gamma^3_{m_j} \cup \Gamma^4_{m_j}$,

$$\Gamma^1_{m_j} = [-im_j, im_j],$$

$$\Gamma^2_{m_j} = [im_j, a_j + m_j + im_j],$$

$$\Gamma^3_{m_j} = [a_j + m_j + im_j, a_j + m_j - im_j],$$

$$\Gamma^4_{m_j} = [a_j + m_j - im_j, -im_j]$$

where $\frac{1}{4} < a_j < \frac{3}{4}$.

The integral I_m can be represented as a sum of 4^n integrals over paths

$$\Gamma^1_{m_1} \times ... \times \Gamma^1_{m_n}, ..., \Gamma^{i_1}_{m_1} \times ... \times \Gamma^{i_n}_{m_n}, ..., \Gamma^4_{m_1} \times ... \times \Gamma^4_{m_n}.$$

where $i_1, ..., i_n$ are random collections of numbers 1,2,3,4.

For each of such path we split the integration variables $\zeta_1, ..., \zeta_n$ into 2 groups B_1 and B_2 where B_1 stands for the indexes $j \in \{1, ..., n\}$, for which $\zeta_j \in \Gamma^1_{m_j}$, and B_2 stands for the indexes $j \in \{1, ..., n\}$, for which $\zeta_j \in \Gamma^2_{m_j} \cup \Gamma^3_{m_j} \cup \Gamma^4_{m_j}$.

Using inequalities (2.22) for K^n, we obtain

$$|g(\zeta, z)||\varphi(\zeta)| < c \prod_{j\in B_1} e^{-\delta|\eta_j|} \prod_{j\in B_2} e^{-\delta m_j} e^{o(|\zeta|)}.$$

Thus, if $B_2 \neq \emptyset$, the integral over the corresponding contour vanishes as $m_j \to \infty, \ \ j = 1, ..., n$. Therefore, for K_n we get

$$I_m = \int\limits_{\partial G_m} \varphi(\zeta) g(\zeta, z) d\zeta = \int\limits_{\Gamma_m^1} \varphi(\zeta) g(\zeta, z) d\zeta = I_m^1$$

as $m_j \to \infty$.

On the other hand, the integral I_m can be computed by means of multidimentional residues, as was done in section 2.1.3.

The integrand in (2.23) defines the differential form

$$\omega = \prod_{j=1}^{n} \frac{z^{\zeta_j}}{(e^{2\pi i \zeta_j} - 1)} \varphi(\zeta) d\zeta_1 \wedge ... \wedge d\zeta_n$$

with poles on divisors

$$Q_1 = \{(\zeta_1, ..., \zeta_n) : f_1 = e^{2\pi i \zeta_1} - 1 = 0\} = \mathbb{Z} \times \mathbb{C}^{n-1},$$

$$. \ .$$

$$Q_n = \{(\zeta_1, ..., \zeta_n) : f_n = e^{2\pi i \zeta_n} - 1 = 0\} = \mathbb{C}^{n-1} \times \mathbb{Z}.$$

Since the intersection $Z = Q_1 \cap ... \cap Q_n = \mathbb{Z}^n$ is discrete and the Jacobian is different from zero, that is, $\partial(f)/\partial(\zeta) = (2\pi i)^n \neq 0$ at $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, then for any point $k \in \mathbb{Z}^n$ the local residue can be defined (see Appendix A.2) :

$$res_k \omega = \frac{z^k \varphi(k)}{\frac{\partial(f)}{\partial(\zeta)}(k)} = \varphi(k) z^k. \tag{2.24}$$

The integration set in (2.23) is related with the polar dicisor $Q_1, ..., Q_n$ by the relations:

$$Q_1 \cap (\partial G_{m_1}^1 \times ... \times G_{m_n}^n) = (\mathbb{Z} \times \mathbb{C}^{n-1}) \cap (\partial G_{m_1}^1 \times ... \times G_{m_n}^n) = \emptyset,$$

$$. \ .$$

$$Q_n \cap (G_{m_1}^1 \times ... \times \partial G_{m_n}^n) = (\mathbb{C}^{n-1} \times \mathbb{Z}) \cap (G_{m_1}^1 \times ... \times \partial G_{m_n}^n) = \emptyset.$$

This means that the polyhedron $G_1^{m_1} \times ... \times G_n^{m_n}$ is compatible with the divisors $Q_1, ..., Q_n$. Therefore, by the Theorem from A.2 the integral (2.23) multiplied by $(2\pi i)^{-n}$ is equal to the sum of residues at

$$k \in (G_{m_1}^1 \times ... \times G_{m_n}^n) \cap (\mathbb{Z} \times ... \times \mathbb{Z}).$$

Taking into account (2.24) we see that

$$I_m = \sum_{k_1=0}^{m_1} ... \sum_{k_n=0}^{m_n} \varphi(k_1, ..., k_n) z_1^{k_1} ... z_n^{k_n}.$$

Consider the integral

$$\mathbb{I} = \int_{\Gamma^1} g(\zeta, z)\varphi(\zeta) d\zeta.$$

where $\Gamma^1 = \{\zeta \in \mathbb{C}^n : \xi_j = 0, j = 1, ..., n\}$ is the imaginary subspace $i\mathbb{R}^n$.

Let us show that for $\zeta \in \Gamma^1$ $(\zeta_j = i\eta_j)$ and $z \in G$ the absolute value of the integrand $|\varphi(\zeta)||g(\zeta, z)|$ is estimated by

$$|g(\zeta, z)||\varphi(\zeta)| < e^{-\delta \sum |\eta_j|}.$$

Indeed, it follows from the definition of the set T_φ (T_φ describes the growth of the function φ along the imaginary subspace) that

$$|\varphi(\zeta)| \leq e^{C_{\nu,\theta}} e^{\sum \nu_j |\eta_j|} \quad \text{as} \quad \zeta \in \Gamma^1,$$

where ν_j is a jth component of the vector ν which run over the set T_φ.
From the previous inequality and (2.12), for $\zeta \in \Gamma^1$ and $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$ we get

$$|\varphi(\zeta)||g(\zeta, z)| \leq e^{-\sum d(z_j)|\eta_j|},$$

where $d(z_j) = \pi - \nu_j - |\pi - \arg z_j|$.

Note that

$$d(z_j) \geq \delta$$

for $z \in (\mathbb{C} \setminus \Delta^o_{\nu_1+\delta}) \times ... \times (\mathbb{C} \setminus \Delta^o_{\nu_n+\delta})$, for any $\nu_j < \pi$.

From two previous inequalities and Proposition 1 we have

$$|\varphi(\zeta)||g(\zeta, z)| \le e^{-\delta \sum |\eta_j|}$$

for $\zeta \in \Gamma^1$ and

$$z \in \bigcup_{\{\nu \in T_\varphi,\ \nu_j < \pi\}} G_{\nu+\delta}. \tag{2.25}$$

From the definition of $\mathcal{M}_\varphi$ it follows that (2.25) is equivalent to

$$z \in \bigcup_{\nu \in \mathcal{M}_\varphi} G_\nu.$$

Thus, the integral $\mathbb{I}$ converges for $z \in G$.

Since $I_m \to \mathbb{I}$ as $m_j \to \infty$, $j = 1, ..., n$, we get $\mathbb{I}(z) = f(z)$ as $z \in (K^o)^n$. Therefore the sum of the series (2.1) extends analytically to the sectorial set G, which was to be proved.

To conclude the section let us note that along with Theorem 2.2 on extendability to a sectorial domain by means of entire interpolation, one can ask about extendability be means of meromorphic interpolation. Denote by Ψ the class of meromorphic functions $\psi(\zeta)$ that does not have poles in the set

$$\{\zeta : Re\zeta_j \ge 0, j = 1, ..., n\}.$$

For the function $\psi(\zeta) \in \Psi$ we can correctly define the sets $T_\psi(\theta)$ and $\mathcal{M}_\psi(\theta)$ for $|\theta_j| \le \frac{\pi}{2}$ $j = 1, ..., n$. The same reasoning as in the proof of Theorem 2.2 leads us to the following statement.

Theorem 2.3. *The sum of the series* (2.1) *extends analytically to a sectorial domain* G *of the form* (2.18) *if there is a meromorphic function* $\psi(\zeta)$ *of the class* Ψ *interpolating the coefficients* f_k *and a vector-function* $\nu(\theta)$ *on* $[-\frac{\pi}{2}, \frac{\pi}{2}]^n$ *with values in* $\mathcal{M}_\varphi(\theta)$ *to satisfy*

$$\nu_j(\theta) \le a|\sin\theta_j| + b\cos\theta_j, \quad j = 1, ..., n$$

with some constants $a \in (0, \pi)$, $b \in (0, \infty)$.

2.3 Example

Consider a power series

$$f(z_1, z_2) = \sum_{k_1, k_2 \in \mathbb{N}^2} \cos \sqrt{k_1 k_2}\; z_1^{k_1} z_2^{k_2}. \tag{2.26}$$

Obviously the function $\varphi(\zeta_1, \zeta_2) = \cos(\zeta_1 \zeta_2)^{\frac{1}{2}}$ is entire and interpolates the coefficients of the series (2.26). Write $\zeta_j = r_j e^{i\theta_j}$, the absolute value of the function admits an asymptotic expansion

$$|\varphi(\zeta_1, \zeta_2)| = |\cos\left((r_1 r_2)^{\frac{1}{2}} e^{i\frac{\theta_1+\theta_2}{2}}\right)| = \frac{1}{2} e^{(r_1 r_2)^{\frac{1}{2}} |\sin(\frac{\theta_1+\theta_2}{2})|} + o(1)$$

as $r_1 r_2 \to \infty$. Therefore, the set $T_\varphi(\theta)$ is

$$T_\varphi(\theta) = \{\nu \in \mathbb{R}^2 : (r_1 r_2)^{\frac{1}{2}} |\sin(\frac{\theta_1 + \theta_2}{2})| \le \nu_1 r_1 + \nu_2 r_2 + C_{\nu,\theta}\},$$

and, consequently, consist of solutions $\nu = \nu(\theta)$ of the inequality

$$(r_1 r_2)^{\frac{1}{2}} |\sin(\frac{\theta_1 + \theta_2}{2})| \le \nu_1 r_1 + \nu_2 r_2, \quad r_1, r_2 \ge 0.$$

Taking $r_j = 0$ we see that $\nu_j \ge 0,\ j = 1, 2.$

To study this inequality for $r_1 r_2 \ne 0$ we take into account that it is homogeneous with respect to r_1 and r_2. Namely, divide it by r_2, then

$$|\sin(\frac{\theta_1 + \theta_2}{2})| \le \nu_1 \left(\frac{r_1}{r_2}\right)^{\frac{1}{2}} + \nu_2 \left(\frac{r_2}{r_1}\right)^{\frac{1}{2}},$$

and denote

$$t = \left(\frac{r_1}{r_2}\right)^{\frac{1}{2}}.$$

Thus, the inequality reduces to the following (not homogeneous) inequality

$$\nu_1 t^2 - |\sin(\frac{\theta_1 + \theta_2}{2})| t + \nu_2 \ge 0, \quad t \ge 0. \tag{2.27}$$

As stated above, we are interested only in solutions ν with non negative coordinates. It follows from Viéte's formulas that for $\nu_1 \geq 0, \nu_2 \geq 0$ the quadratic trinomial (2.27) in t does not have negative roots, therefore we may consider the inequality for all $t \in \mathbb{R}$. Thus, the solutions ν of the inequality (2.27) are defined by the condition that its discriminant is non-positive:

$$|\sin(\frac{\theta_1 + \theta_2}{2})|^2 - 4\nu_1\nu_2 \leq 0,$$

i.e.

$$\nu_1\nu_2 \geq \frac{1}{4}|\sin(\frac{\theta_1 + \theta_2}{2})|^2.$$

In the end we get

$$T_\varphi(\theta_1, \theta_2) = \{\nu \in \mathbb{R}^2 : \nu_1\nu_2 \geq \frac{1}{4}|\sin(\frac{\theta_1 + \theta_2}{2})|^2, \ \ \nu_1 \geq 0, \ \ \nu_2 \geq 0\}.$$

The set $\mathcal{M}_\varphi(\theta_1, \theta_2)$ coincides with the topological boundary of $T_\varphi(\theta_1, \theta_2)$, that is the positive part of the hyperbola

$$\mathcal{M}_\varphi(\theta_1, \theta_2) = \{\nu \in \mathbb{R}^2 : \nu_1\nu_2 = \frac{1}{4}|\sin(\frac{\theta_1 + \theta_2}{2})|^2, \ \ \nu_1 \geq 0, \ \ \nu_2 \geq 0\}. \qquad (2.28)$$

Obviously, $T_\varphi(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ consists of the quadrant $\nu_1 \geq 0, \nu_2 \geq 0$, if the signs are the same, and $\{\nu \in \mathbb{R}^2_+ : \nu_1\nu_2 \geq \frac{1}{4}\}$, if the signs are different.

As a result, the intersection $T_\varphi = \bigcap T_\varphi(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ is

$$T_\varphi = \{\nu \in \mathbb{R}^2 : \nu_1\nu_2 \geq \frac{1}{4}, \ \ \nu_1 \geq 0, \ \ \nu_2 \geq 0\}.$$

Thereby, the sets $\mathcal{M}_\varphi$ are of the form (see Fig. 9)

$$\mathcal{M}_\varphi = \{\nu \in [0, \pi)^2 : \nu_1\nu_2 = \frac{1}{4}\}.$$

Now, make sure that the condition (2.19) of Theorem 2.2 is fulfilled. Taking into account (2.28), it is enough to find constants $a \in [0, \pi), \ \ b \in [0, \infty)$ such that

$$(a|\sin\theta_1| + b\cos\theta_1)(a|\sin\theta_2| + b\cos\theta_2) \geq \frac{1}{4}|\sin(\frac{\theta_1 + \theta_2}{2})|^2.$$

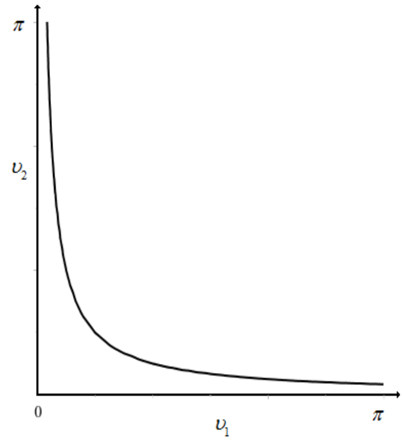

Figure 9

For $a = 1,\ b = 1$ the left hand side

$$(|\sin\theta_1| + \cos\theta_1)(|\sin\theta_2| + \cos\theta_2) \geq 1,$$

and the right hand side does not exceed $\frac{1}{4}$, i.e. the inequality holds.

Thus according to Theorem 2.2 the sum of the series extends into the union G of polysectors $G_\nu = (\mathbb{C} \setminus \Delta_{\nu_1}) \times \ldots \times (\mathbb{C} \setminus \Delta_{\nu_n})$ over all $\nu \in \mathcal{M}_\varphi$. Fig. 10 depicts the set of arguments $\theta = (\theta_1, \theta_2)$ defining the sectorial set G. It is a union of rectangles $(\nu_1, 2\pi - \nu_1) \times (\nu_2, 2\pi - \nu_2)$ over all $(\nu - 1, \nu_2) \in \mathcal{M}_\varphi$.

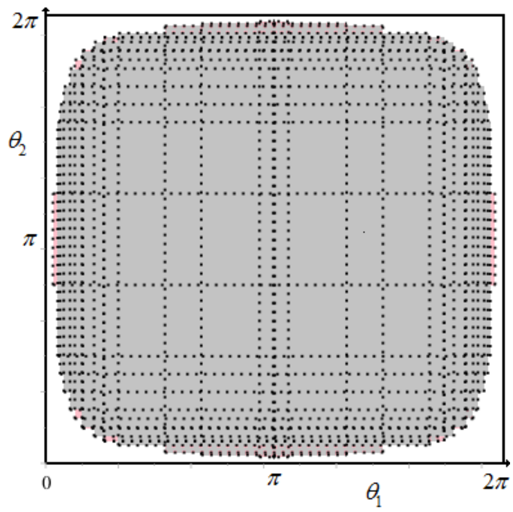

Figure 10

2.4 Non extendable multiple power series

Here we consider two examples of double power series. The first one can be viewed as a two-dimensional analog of the Fredholm example. Its coefficients assume only two values: zero or one. The second series is of the same type, it is a restriction of a geometric series to a cone. In the case the cone is rational such a series represents a rational function of a certain type. We may conjecture that a restriction to an irrational cone is a series with natural boundary.

Theorem 2.4. *If the support A of a double power series*

$$\sum_{(k_1,k_2)\in A} {z_1}^{k_1}{z_2}^{k_2} \tag{2.29}$$

is ofthe type

$$A=\{(k_1,k_2)\in {Z_+}^2 : k_2\geq {k_1}^{1+\varepsilon}\}\cup\{(k_1,k_2)\in {Z_+}^2 : k_1\geq {k_2,}^{1+\varepsilon}\}\ \ \varepsilon>0,$$

then the double series (2.29) *is not extendable across the boundary of the bidisk*

$$U^2=\{(z_1,z_2):\ |z_1|<1,\ |z_2|<1\}$$

and represents an infinitely differentiable function in $\bar{U}^2\setminus T^2$, where

$$T^2=\{(z_1,z_2):|z_1|=1,|z_2|=1\}.$$

Proof. We can represent the power series (2.29) by the sum of two series:

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}{z_1}^{k_1}{z_2}^{k_2+[{k_1}^{1+\varepsilon}]}+\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}{z_1}^{k_1+[{k_2}^{1+\varepsilon}]}{z_2}^{k_2}=$$

$$\sum_{k_2=0}^{\infty}{z_2}^{k_2}\sum_{k_1=0}^{\infty}{z_1}^{k_1}{z_2}^{[{k_1}^{1+\varepsilon}]}+\sum_{k_1=0}^{\infty}{z_1}^{k_1}\sum_{k_2=0}^{\infty}{z_2}^{k_2}{z_1}^{[{k_2}^{1+\varepsilon}]}=$$

$$=\frac{1}{1-z_2}\sum_{k_1=0}^{\infty}{z_1}^{k_1}{z_2}^{n_{k_1}}+\frac{1}{1-z_1}\sum_{k_2=0}^{\infty}{z_2}^{k_2}{z_1}^{n_{k_2}}.$$

Here $[k_j{}^{1+\varepsilon}]$ denotes the integer part of the number $k_j{}^{1+\varepsilon}$.

According to Theorem 1.4 the series

$$\sum_{k_1=0}^{\infty} z_1{}^{k_1} z_2{}^{n_{k_1}}, \tag{2.30}$$

if considered in the variable $|z_2| < 1$, converges in the unit disk and does not extend across the boundary circle when $0 < |z_1| < 1$. Using the change of variables $e^u = z_1$ and $e^t = z_2$, we rewrite (2.30) as an exponential series

$$\sum_{(k_1,k_2)\in A_1} e^{k_1 u} e^{k_2 t},$$

where $A_1 = \{(k_1, k_2) \in \mathbb{Z}^2 : k_2 \geq k_1^{1+\varepsilon}\}$. This represents an infinitely differentiable function in

$$\{(u,t) : Re\ u \leq 0,\ Re\ t \leq 0\} \setminus \{(u,t) : Re\ u = 0,\ Re\ t = 0\}.$$

Consequently, the series (2.30) represent an infinitely differentiable function in $\bar{U}^2 \setminus T^2$.

The similar holds for the series

$$\sum_{k_2=0}^{\infty} z_2{}^{k_2} z_1{}^{n_{k_2}},$$

it converges in the unit disk, it does not extend with respect to the variable z_1, if $0 < z_2 < 1$, and represents an infinitely differentiable function in $\bar{U}^2 \setminus T^2$.

Therefore, we obtain the desired statement for the series (2.29).

Proposition 2. *Let K be a sector with integer generating vectors $m_1 = (m_{11}, m_{12})$ and $m_2 = (m_{21}, m_{22})$, then the series*

$$f(z) = \sum_{k\in N^2\cap K} z_1{}^{k_1} z_2{}^{k_2}$$

represents a rational function of the form

$$f(z) = \frac{P(z)}{(1 - z_1{}^{m_{11}} z_2{}^{m_{12}})(1 - z_1{}^{m_{21}} z_2{}^{m_{22}})},$$

with

$$P(z) = 1 + \sum_{\alpha \in (\mathbb{N}^2 \cap intD)} z^{\alpha},$$

where $intD$ is the interior of the parallelogram D with vertices $(0,0), m_1, m_2$ and $m_1 + m_2$.

Proof. We can cover all the integer points of K by the semigroup

$$L = \{(l_1 m_{11} + l_2 m_{21}, l_1 m_{12} + l_2 m_{22}), \quad l_i \in Z_{\geq 0}, \quad i = 1, 2\}$$

and its shifts $L_j = a_j + L$, where a_j runs over $\mathbb{N}^2 \cup intD$. Thus, we have

$$\sum_{k \in N^2 \cap K} {z_1}^{k_1} {z_2}^{k_2} = \sum_{k \in L} {z_1}^{k_1} {z_2}^{k_2} + \sum_{k \in L_1} {z_1}^{k_1} {z_2}^{k_2} + \ldots + \sum_{k \in L_p} {z_1}^{k_1} {z_2}^{k_2},$$

where p is the cardinality of $N^2 \cap intD$. Summing up the geometric series we get

$$\sum_{k \in L} {z_1}^{k_1} {z_2}^{k_2} = \sum_{l_1, l_2 \geq 0} z^{l_1 m_1 + l_2 m_2} = \sum_{l_1, l_2 \geq 0} (z^{m_1})^{l_1} (z^{m_2})^{l_2} =$$

$$= \frac{1}{(1 - z^{m_1})(1 - z^{m_1})} = \frac{1}{(1 - {z_1}^{m_{11}} {z_2}^{m_{12}})(1 - {z_1}^{m_{21}} {z_2}^{m_{22}})}.$$

Obviously

$$\sum_{k \in L_j} {z_1}^{k_1} {z_2}^{k_2} = {z_1}^{a_{j1}} {z_2}^{a_{j2}} \sum_{k \in L} {z_1}^{k_1} {z_2}^{k_2},$$

therefore we obtain

$$\sum_{k \in N^2 \cap K} {z_1}^{k_1} {z_2}^{k_2} = \frac{1 + {z_1}^{a_{11}} {z_2}^{a_{12}} + \ldots + {z_1}^{a_{p1}} {z_2}^{a_{p2}}}{(1 - {z_1}^{m_{11}} {z_2}^{m_{12}})(1 - {z_1}^{m_{21}} {z_2}^{m_{22}})}$$

as desired.

In the end let us conjecture the following: The series (2.4) for an arbitrary cone K (not necessary whit rational m_1 and m_2) is either non-extendable across the boundary of convergence domain (2.4) or represent a rational function of the form

$$f(z) = \frac{P(z)}{(1 - {z_1}^{m_{11}} {z_2}^{m_{12}})(1 - {z_1}^{m_{21}} {z_2}^{m_{22}})},$$

where $P(z)$ is a polynomial. Such a statement can be seen as a two-dimensional analog of Szegö's theorem ([4], [6]) on series whose coefficients take a finite number of different values.

APPENDIX

A.1 Growth indicator of entire functions

An entire function $\varphi(z)$ of a complex variable $z \in \mathbb{C}$ is said to be of exponential type if

$$\overline{\lim_{z\to\infty}} \frac{\ln|\varphi(z)|}{|z|} < +\infty.$$

The indicator function of an entire function $\varphi(z)$ of exponential type is defined as the upper limit [6]

$$h_\varphi(\theta) := \overline{\lim_{r\to\infty}} \frac{\ln|\varphi(re^{i\theta})|}{r}, \quad \theta \in \mathbb{R}.$$

The indicator function describes the growth of the function φ on rays $z = re^{i\theta}$ (here $r \in \mathbb{R}_+$ and θ is fixed). It follows from the definition that $h_\varphi(\theta)$ is a real valued function with the period 2π. One of the basic properties of the indicator function $h_\varphi(\theta)$ is the *trigonometric convexity* [38],[6]:

If $\theta_1 < \theta < \theta_2$ and $\theta_2 - \theta_1 < \pi$ then

$$h_\varphi(\theta)\sin(\theta_2 - \theta_1) \le h_\varphi(\theta_1)\sin(\theta_2 - \theta) + h_\varphi(\theta_2)\sin(\theta - \theta_1).$$

If an entire function $\varphi(z)$ is represented by a power series

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then the Laurent series

$$\hat{\varphi} = \sum_{k=0}^{\infty} a_k k! z^{-k-1} \tag{3.1}$$

is called the *Borel transform* of φ.

The connection between the set of singularities of $\hat{\varphi}$ and the indicator function φ is described by a theorem of Pólya [34], [22]. Recall that the *support function* $K(\theta)$ for a convex set K is defined as

$$k(\theta) = \sup_{z \in K} Re(ze^{-i\theta}).$$

Note that if $z = x + iy$ then

$$Re(ze^{-i\theta}) = x\cos\theta + y\sin\theta.$$

Theorem (Pólya [34]) *The indicator function $h_\varphi(\theta)$ for an entire function φ of exponential type and the support function $k(\theta)$ of the minimal convex compact set K, outside of which h_φ extends analytically, are related through*

$$h_\varphi(\theta) = k(-\theta).$$

Note that K is convex implies that it is an intersection of half-planes

$$K = \bigcap_{\theta \in [0, 2\pi]} \{z : Re(ze^{-i\theta}) < \nu\}.$$

This fact has been taken as a basis of multidimensional formulation of Pólya's theorem.

In n variables by an entire function of exponential type we understand a function $\varphi(z) = \varphi(z_1, ..., z_n)$, for which there exist positive $A, \sigma_1, ..., \sigma_n$ such that $\forall$z$\in \mathbb{C}^n$ there holds an inequality

$$|\varphi(z)| \leq Ae^{\sigma_1|z_1| + ... + \sigma_n|z_n|}.$$

As in one-dimensional case, to an entire function

$$\varphi(z) = \sum_{k \in \mathbb{N}^n} a_k z^k, \tag{3.2}$$

where $k = (k_1, ..., k_n), \;\; z^k = z_1^{k_1} ... z_n^{k_n}$, there is an associated Borel transform

$$\hat{\varphi}(z) = \sum_{|k| \geq 0}^{\infty} a_k k! z^{-k-1},$$

where $k! = k_1!...k_n!$.

For an entire function φ of exponential type we define the set

$$T_{\varphi}(\theta) = \{\nu \in \mathbb{R}^n : \ln|\varphi(re^{i\theta})| \leq \nu_1 r_1 + ... + \nu_n r_n + C_{\nu,\theta}\},$$

where the inequality is satisfied for any $r \in \mathbb{R}^n_+$ with some constant $C_{\nu.\theta}$.

Let $C_{\varphi}(\theta)$ be a set of vectors $\nu \in \mathbb{R}^n$ such that the function $\hat{\varphi}(z)$ extends into the domain

$$G_{\nu,\theta} = \{z : Re(z_j e^{-i\theta_j}) > \nu_j, \;\; j = 1, .., n\},$$

from a neighborhood of $(\infty, ..., \infty)$ $G_{\nu,\theta}$ is a direct product of half-planes.

Theorem (Ivanov-Stavski [28], [22]) Let $\varphi(z)$ be an entire function of exponential type, then

$$T_{\varphi}(\theta) = C_f(-\theta).$$

A.2 Multidimensional residues and an analog of the Jordan Lemma

Let ω be a meromorphic differential form in $\mathbb{C}^n$ of the form

$$\omega = \frac{h(z)dz_1 \wedge ... \wedge dz_n}{f_1(z)...f_n(z)} \tag{3.3}$$

with poles on the divisors $D_j = \{z : f_j(z) = 0\}, \quad j = 1, ..., n$. Assuming the intersection $Z = D_1 \cap ... \cap D_n$ is discrete, for every $a \in Z$ we define the local (Grothendieck) residue with respect to the system of divisors $\{D_j\}$ to be the following integral (see [39], Chapter 5 or [37], §5)

$$res_a \omega = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_a} \omega, \tag{3.4}$$

where $\Gamma_a = \{z \in U_a : |f_j(z)| = \varepsilon, \ j = 1, ..., n\}$ is a cycle in some small neighborhood U_a about a with orientation determined by the inequality

$$d(arg f_1) \wedge ... \wedge d(arg f_n) \geq 0.$$

If $f_1, ..., f_n$ are such that the Jacobian $\partial(f)/\partial(z)$ at a differs from zero then (by Cauchy's formula) the local residue equals

$$res_a \omega = \frac{h(a)}{\frac{\partial(f)}{\partial(z)}(a)}. \tag{3.5}$$

Consider the question of when the integral

$$\frac{1}{(2\pi i)^n} \int\limits_{\Gamma_a} \omega \tag{3.6}$$

of a meromorphic form (3.3) over the skeleton σ of some polyhedron Π equals the sum of the residues (3.4) at points $a \in \Pi$. A polyhedron is the inverse image $g^{-1}(\partial G)$ of the domain $G = G_1 \times ... \times G_n$ under a proper mapping $g : \mathbb{C}^n \to \mathbb{C}^n$

with each G_j being a domain with piecewise smooth boundary ∂G_j in the complex plane. The skeleton of such polyhedron is the set $g^{-1}(\partial G_1 \times ... \times \partial G_n)$, with the orientation determined by the order of parameters $\tau_1, ..., \tau_n$ in parametrizations of the boundaries $\partial G_1, ..., \partial G_n$.

Given a multi-index $K = \{k_1, ..., k_s\} \subset \{1, ..., n\}$ we associate with K the face

$$\sigma_K = \{z : g_k(z) \in \partial G_k, k \in K, g_j(z) \in G_j, j \notin K\}.$$

We say that a family of divisors $\{D_j\}$ is *compatible* with Π if

$$D_j \cap \sigma_j =, \quad j = 1, ..., n. \tag{3.7}$$

If Π is a bounded polyhedron and $\{D_j\}$ is a family of divisors compatible with Π then the integral (3.6) equals the sum of the residues (3.4) over all points $a \in \Pi$ [36]. For an unbounded polyhedron we must additionally require that the integrand vanishes at infinity in accordance with the classical one-dimensional Jordan lemma [36]. Using the functions f_j determining the divisors D_j, we introduce the functions

$$\rho_j = \frac{|f_j|^2}{||f||^2}, \quad \text{where} \quad ||f||^2 = |f_1|^2 + ... + |f_n|^2.$$

Given a multi-index $J = \{j_1, ..., j_s\} \subset \{1, ..., n\}$ with $1 \leq s \leq n$, we associate with J the $(n, s-1)$-differential form

$$\xi_J = \sum_{j \in J} (-1)^{(j,J)-1} \rho_j \bar{\partial} \rho_J[j] \wedge \omega,$$

where (j, J) indicates the position of j in J and $\bar{\partial}\rho_J[j] = \bar{\partial}\rho_1 \wedge ...[j]... \wedge \bar{\partial}\rho_s$.

We say that a differential form ξ_j *satisfies the Jordan condition on the face* σ_{J^o} where $J^o = \{1, ..., n\} \setminus J$, if there is a sequence of reals R_k converging to $+\infty$ as $k \to \infty$ and such that

$$\lim_{k \to \infty} \int\limits_{S_{R_k} \cap \, \sigma_{J^o}} \xi_j = 0, \tag{3.8}$$

where S_R is the sphere of radius R with center at some point of the skeleton $\sigma = \sigma_{1\ldots n}$ of Π.

Theorem (the multidimensional abstract Jordan lemma [31], [48]). *If the family of divisors $\{D_j\}$ is compatible with the polyhedron Π and for every multi-index J the form ξ_J satisfies the Jordan condition on the face σ_{J^o} then*

$$\int\limits_{\sigma} \omega = (2\pi i)^n \sum_{a \in \Pi} res_a \omega.$$

Observe that the sequence of spheres S_{R_k} in the lemma may be replaced with an arbitrary sequence of piecewise smooth surfaces such that the domains bounded by the faces of the polyhedron and the surfaces of the sequence exhaust the whole polyhedron as $R \to \infty$.

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