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# Polynomials, $\alpha$-ideals, and the Principal Lattice

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*Let $R$ be a commutative ring with an identity, $\mathfrak{R}$ be an almost distributive lattice and $I_\alpha(\mathfrak{R})$ be the set of all $\alpha$-ideals of $\mathfrak{R}$. If $L(R)$ is the principal lattice of $R$, then $R[I_\alpha(\mathfrak{R})]$ is Cohen-Macaulay. In particular, $R[I_\alpha(\mathfrak{R})][X_1, X_2, \cdots]$ is WB-height-unmixed.*

*Keywords: Almost distributive lattice, principal lattice, $\alpha$-ideals, multiplicative lattice; complete lattice, WB-height-unmixedness, Cohen-Macaulay rings, unmixedness.*

## Introduction

Lattices, in general, (specially multiplicative lattices), are natural abstractions of the set of ideals of a ring. However, Wihout a good notion of principal lattice, it is impossible to get very deep results, see [3]. Dilworth overcame this in [4], with a new notion of a principal element. Recall, that a multiplicative lattice is a complete lattice $L$ with a commutative, associative multiplication which distributes over arbitrary joins and its largest element $I$, is the identity for the multiplication, see [4]. Basically, an element $E$ of a multiplicative lattice $L$, is said to be meet-(join-)principal if $(A \wedge (B : E))E = (AE) \wedge B$ (if $(BE \vee A) : E = B \vee (A : E)$) for all $A$ and $B$ in $L$. A principal element is an element that is both meet-principal and join-principal or $A \wedge E = (A : E)E$ and $AE : E = A \vee (0 : E)$, for all $A \in L$. A lattice $L$, is called a principal lattice, when each of its elements is principal. Here, the residual quotient of two elements $A$ and $B$ is denoted by $A : B$, so $A : B = \vee\{X \in L | XB \leqslant A\}$. An almost distributive lattice (ADL), was introduced by U. M. Swamy and G. C. Rao in [11], as an algebra $(R, \vee, \wedge)$ of type $(2, 2)$, which satisfies almost all the properties of a distributive lattice, except possibly the commutativity of $\vee$, the commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$. W. H. Cornish studied in to [2], the properties of $\alpha$-ideals in a distributive lattice, (see the next section for the definition). In this paper, the concept of an $\alpha$-ideal in an ADL is introduced, analogous to the case of distributive lattices. In section 2, it is shown that, if $R$ is a commutative ring with an identity and $L(R)$ is the principal lattice, then $R$ is a Cohen-Macaulay ring. In section 3, we prove that, if $R$ is Cohen-Macaulay ring and if $P$ is a distributive lattice, then $R[P]$ is Cohen-Macaulay ring, where $R[P]$ is the polynomial ring over $P$. Finally, in section 4, we conclude some properties of $L(R)$.

## 1. The Principal Lattice and $\alpha$-ideals

A Noether lattice is a modular multiplicative lattice satisfying the ascending chain condition in which every element is a join of elements called principal elements. The multiplication, meet,

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and join in a Noether lattice are supposed to mirror the multiplication, intersection, and sum of ideals. Because of this, a multiplicative lattice is defined to be a complete lattice $L$, containing a unite element $I$ and a null element 0, and provided with a commutative, associative, join-distributive multiplication, for which $I$ is an identity element. We will use $\wedge$ and $\vee$ to denote meet and join, respectively, and $\leqslant$ to denote lattice partial ordering, with $<$ reserved for strict inequality.

**Remark 1.1.** *Let $R$ be a commutative ring with an identity, $I$ and $J$ be ideals of $R$, $I+J$ and $IJ$, be the ordinary sum and product of ideals. With these two operations as join and meet, the set of all ideals of a given ring, forms a complete modular lattice. Remember that, a principal lattice, is a lattice in which every element is principal. The following theorem is proved in [6].*

**Theorem 1.1.** *Let $R$ be a commutative ring with identity. Then $L(R)$ is a principal lattice, if and only if, $R$ is a Noetherian multiplication ring.*

Note that, a ring is called a multiplication ring, if every ideal of $R$ is product of two ideals. Let $M$ be a finitely generated module over a Noetherian ring $R$. We say that $x \in R$ is an $M$-regular element, if $xg = 0$ for $g \in M$ implies $g = 0$, in the other words, if $x$ is not a zero-divisor on $M$. A sequence $x_1, \cdots x_r$ of elements of the ring $R$, is called an $M$-regular sequence or simply an $M$-sequence if the following conditions are satisfied:

(1) $x_i$ is an $M/(x_1, \cdots, x_{i-1})M$-regular element for $i = 1, \ldots, r$;

(2) $M/(x_1, \ldots, x_r)M \neq 0$.

Suppose $I \subseteq R$ is an ideal with $IM \neq M$. The *depth* of $I$ on $M$ is maximal length of an $M$-regular sequence in $I$, denoted by *depth*$(I, M)$. If $R$ is a local ring with a unique maximal ideal $\mathfrak{m}$, we write *depth*$(\mathfrak{m})$, for *depth*$(\mathfrak{m}, M)$.

Let $R$ be a Noetherian local ring. A finitely generated $R$-module $M$, is a Cohen-Macaulay module, if *depth*$(M)$ = *dim*$(M)$. If $R$ itself is a Cohen-Macaulay module, then it is called a Cohen-Macaulay ring. For the proof of the following theorem, see [9].

**Theorem 1.2.** *Suppose $R$ is a Noetherian multiplication ring. Then $R$ is a Cohen-Macaulay ring.*

## 2. Cohen-Macaulay and Unmixedness

We begin this section by a definition from Bourbaki.

**Definition 2.1.** *A prime ideal $P$ is an associated prime of $I$, if $P = I : x$ for some $x \in R$.*

Remember that the height of a prime ideal $P$ is the maximum length of the chains of prime ideals of the following form,

$$P_1 \subset P_2 \subset \cdots \subset P_k = P.$$

We will denote the height of $P$ by $ht(P)$. An ideal $I$ of $R$ is said to be height-unmixed, if all the associated primes of $I$ have equal height. That is $ht(P) = ht(Q)$, for all $P, Q \in Ass(I)$, where $Ass(I)$ denotes the set of associated primes of $I$. An ideal $I$ is said to be unmixed if there are no embedded primes among the associated primes of $I$. That is, $P \subseteq Q \Rightarrow P = Q$, for all $P, Q \in Ass(I)$.1 We will say that an ideal is WB-height-unmixed, if it is height-unmixed with respect to the set of weak Bourbaki associated primes and an ideal is WB-unmixed if it is unmixed with respect to the set of weak Bourbaki associated primes. The set of weak Bourbaki associated primes of an ideal $I$ is denoted by $Ass_f(I)$. A prime ideal $P$ is a weak Bourbaki associated prime of the ideal $I$ if it is a minimal ideal of the form $I : a$, for some $a \in R$.

**Theorem 2.1.** *If $R$ satisfies GPIT (generalized principal ideal theorem), then $R$ is WB-height-unmixed if and only if $R$ is WB-unmixed.*

*Proof.* Suppose $R$ is a ring which satisfies GPIT.

($\Rightarrow$) Suppose $R$ is WB-height-unmixed and let $I$ be a height-generated ideal in $R$. In a ring which satisfies GPIT every ideal $I$ satisfies $ht(I) \leqslant \ell(I)$ where $\ell(I)$ denotes the minimal number of generators of $I$. Thus, in a ring with GPIT, $I$ is height-generated if and only if $ht(I) = \ell(I) < \infty$. To show that $I$ is WB-unmixed, suppose $P, Q \in Ass_f(I)$, with $P \subseteq Q$. Since $R$ is WB-height-unmixed so $I$ is WB-height-unmixed. Thus, $ht(P) = ht(Q) = ht(I) < \infty$. So, $P$ and $Q$ are prime ideals with $P \subseteq Q$ and $ht(P) = ht(Q) < \infty$. Thus, $P = Q$ and so $I$ is WB-unmixed. Therefore, $R$ is WB-unmixed.

($\Leftarrow$) Suppose $R$ is WB-unmixed and let $I$ be a height-generated ideal in $R$. As in the first part of this proof, we have $ht(I) = \ell(I) < \infty$. Let $n = ht(I) = \ell(I)$. Note that, since $R$ satisfies GPIT, we have $ht(P) \leqslant n$ for all $P \in Min(I)$. However, since $ht(I) \leqslant ht(P)$ for all $P \in Min(I)$ and $ht(I) = n$, we have $ht(P) = n$ for all minimal associated prime $P$. Since $I$ is WB-unmixed, we have $Ass_f(I)$ is the set of all minimal associated primes of $I$. Therefore, $ht(P) = n$ for all $P \in Ass_f(I)$ and thus, $I$ is WB-height-unmixed. Therefore, $R$ is WB-height-unmixed. □

**Theorem 2.2.** *Let $R$ be a Noetherian ring and let $S = R[X_1, X_2, \ldots]$, the ring of polynomials in the variables $X_1, X_2, \ldots$. For any prime ideal $P$ in $R$ we have $ht(P) = ht(PS)$ where $ht(PS)$ refers to the height of the ideal $PS$ in $S$.*

*Proof.* Note that the proof of this theorem depends only on the weaker condition that $R$ is a strong $S$-ring ( see [8] for more information on strong $S$-rings). It is not necessary for the ring to be Noetherian. First, note that, we have trivially $ht(P) \leqslant ht(PS)$, since the extensions of a chain of distinct prime ideals in $R$, is a chain of distinct prime ideals in $S$. For $i \geqslant 1$, let $R_i = R[X_1, X_2, \ldots, X_i]$. So, $S = \varinjlim R_i$. Let $P_i = PR_i$. Since $R$ is Noetherian (and thus a strong $S$-ring), we have $ht(P) = ht(P_i)$, see [8](Theorem 149, page 108). Now, suppose $h(PS) > n$, where $n = ht(P)$. Then there is a chain of prime ideals

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{n+1} = PS,$$

in $S$. For $1 \leqslant i \leqslant n+1$, choose $x_i \in Q_i \setminus Q_{i-1}$. Since $S = \varinjlim R_i$, there is a positive integer $j$ such that $\{x_1, \cdots, x_{n+1}\} \in R_j$. For $0 \leqslant i \leqslant n+1$, let $T_i = Q_i \bigcap R_i$. Then

$$T_0 \subset T_1 \subset \cdots \subset T_{n+1}$$

is a chain of prime ideals in $R_j$. So, $ht(T_{n+1}) \geqslant n+1$. However, $T_{n+1} = Q_{n+1} \bigcap R_j = PS \bigcap R_j = P_j$ and we have already noted that $ht(P_j) = n$, a contradiction. Therefore, $ht(PS) = ht(P)$. □

In [1], it was shown that $R[X_1, X_2, \ldots]$ satisfies GPIT (if $R$ is a Noetherian ring). The statement of this fact in [1] actually makes the assumption that $R$ is a domain, however, the fact that $R$ is a domain, is not necessary in the proof given in [1], so we will use the more general result. By applying 2.1 to this result, we get the following theorem

**Theorem 2.3.** *Let $R$ be a Cohen-Macaulay ring. Then $R[X_1, X_2, \ldots]$ is WB-height-unmixed.*

**Theorem 2.4.** *Let $L$ be a Noetherian multiplicative lattice. Every element of $L$ is principal element, if and only if, for all $a \leqslant b$, there is an element $c \in L$, such that $a = bc$.*

*Proof.* Suppose that elements of $L$ are principal and let $a, b \in L$ and $a \leqslant b$. Then $a = a \cap b = (a : b)b$, and so $c = (a : b)$. Conversely, it follows from (ACC), that each element of $L$ is a join of a finite number of principal elements. Therefore, to prove the theorem it is sufficient to show that if $m$ and $n$ are principal elements of $L$, then $m \cup n$ is principal. Let $m$ be principal and let $m \leqslant d$, where $d \in L$. Then $m = cd$, for some $c \in L$, and since $m$ is join principal ,$(a \cup bcd) : cd = a : m \cup b$, for all $a, b \in L$. Hence $((a \cup bcd) : c) : d = a : m \cup b$. However, $(a \cup bd)c = ac \cup bdc \leqslant a \cup bcd$, and so $a \cup bd \leqslant (a \cup bcd) : c$. Therefore, $(a \cup bd) : d \leqslant a : m \cup b$ for all $a, b \in L$. Thus, if $m$ and $n$ are principal elements of $L$, we have for all $a, b \in L$,

$$\begin{aligned} (a \cup b(m \cup n)) : (m \cup n) &\leqslant (a : m \cup b) \cap (a : n \cup b) \\ &= (a : m \cap a : n) \cup b \\ &= a : (m \cup n) \cup b. \end{aligned}$$

□

**Corollary 2.1.** *Let $R$ be a commutative ring with an identity and $L(R)$ be a Noether lattice. Every ideal of $R$ is an principal element in $L(R)$, if and only if, $R$ is a multiplication ring.*

For the proof of the following theorem, see [5].

**Theorem 2.5.** *If $R$ is Cohen-Macaulay ring, and if $P$ is a distributive lattice, then $R[P]$ is Cohen-Macaulay.*

## 3. $\alpha$-ideals and Cohen-Macaulay Rings

In this section we introduce the concept of an $\alpha$-ideal in an ADL with zero, analogous to that in a distributive lattice [2]. An Almost Distributive Lattice (ADL) is an algebra $(\mathfrak{R}, \vee, \wedge)$ of type (1.2) satisfying:

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$ for any $x, y, z \in \mathfrak{R}$

If $\mathfrak{R}$ has an element 0, and satisfies $0 \wedge x = 0$ and $0 \vee x = 0$ alogn with the above properties, then $\mathfrak{R}$ is called an ADL with 0.

**Definition 3.1.** *For any non-empty subset $A$ of an ADL, $\mathfrak{R}$ with 0, define $A^* = \{x \in \mathfrak{R} \mid a \wedge x = 0$, for all $a \in A\}$. Then $A^*$ is called the annihilator of $A$. For any $a \in \mathfrak{R}$, we have $\{a\}^* = (a]^*$, where $(a]$ is the principal ideal generated by $a$. For any $\emptyset \neq A \subseteq \mathfrak{R}$, we have clearly $A \cap A^* = (0]$.*

For the proof of the next lemmas, see [10].

**Lemma.** *For any non-empty subset $A$ of $\mathfrak{R}$, $A^*$ is an ideal of $\mathfrak{R}$.*

**Lemma.** *For any non-empty subsets $I$, $J$ of $\mathfrak{R}$, we have the following:*

1. *If $I \subseteq J$, then $J^* \subseteq I^*$*
2. *$I \subseteq I^{**}$*
3. *$I^{***} = I^*$*
4. *$(I \vee J)^* = I^* \cap J^*$.*

**Definition 3.2.** *Let $\mathfrak{R}$ be a ADL with 0. An ideal $I$ of $\mathfrak{R}$ is called an $\alpha$-ideal if $(x]^{**} \subseteq I$ for all $x \in I$.*

We now denote the set of all $\alpha$-ideal of an ADL $\mathfrak{R}$ by $I_\alpha(\mathfrak{R})$. If $\mathfrak{R}$ is an ADL, then we know that $(I(\mathfrak{R}), \vee, \wedge)$ is a distributive lattice. But the set $I_\alpha(\mathfrak{R})$ is not a sublattice of $I(\mathfrak{R})$.

**Definition 3.3.** *A Noether lattice is said to be complete if it is complete in the topology of the Jacobson radical.*

In [7], the following theorem is proved.

**Theorem 3.1.** *Let $(L, \mathsf{m})$ be a distributive local Noether lattice of dimension $d$. Then $L$ is complete in the $\mathsf{m}$-adic topology.*

Let $R$ be a local Noetherian ring with the maximal ideal $M$. Then $L(R)$, the lattice of ideals of $R$, is a local Noether lattice and also $L(R)$ is a complete modular lattice. A ring $R$ is called an arithmetical ring, if $L(R)$ is distributive.

**Corollary 3.1.** *If $(R, m)$ is a local Noetherian ring and is arithmetical ring, then $L(R)$ is a complete in the m-adic topology.*

*Proof.* This is immediate from Remark 1.1 and Theorem 3.1 and if $L$ is an ADL, then $L(R)$ is a distributive lattice. □

**Corollary 3.2.** *If $(\mathfrak{R}, \boldsymbol{m})$ is local Noether lattice and is an ADL, then $I(\mathfrak{R})$ is a complete in the $\boldsymbol{m}$-adic topology.*

In [10], it is proved that, if $\mathfrak{R}$ is an ADL with 0, then $I_\alpha(\mathfrak{R})$ forms a distributive lattice. So we have

**Theorem 3.2.** *Let $R$ be a commutative ring with an identity and let $I_\alpha(\mathfrak{R})$ be the set of all $\alpha$-ideal of an ADL $\mathfrak{R}$. If $L(R)$ is a principal lattice, then $R[I_\alpha(\mathfrak{R})][X_1, X_2, \ldots]$ is WB-height-unmixed.*

*Proof.* Since $L(R)$ is a principal lattice, $R$ is Cohen-Macaulay . By assumption, $I_\alpha(\mathfrak{R})$ is a distributive lattice. Thus the $R[I_\alpha(\mathfrak{R})]$ is Cohen-Macaulay and $R[I_\alpha(\mathfrak{R})][X_1, X_2, \ldots]$ is WB-height-unmixed. □

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## Полиномы, $\alpha$-идеалы и главные решетки

**Али Молхаси**

*Пусть $R$ — коммутативное кольцо с единицей, $\mathfrak{R}$ — почти дистрибутивная решетка и $I_\alpha(\mathfrak{R})$ — множество всех $\alpha$-идеалов в $\mathfrak{R}$. Если $L(R)$ — главная решетка $R$, то $R[I_\alpha(\mathfrak{R})]$ — кольцо Коэна-Маколея. В частности, $R[I_\alpha(\mathfrak{R})][X_1, X_2, \cdots]$ — $WB$-высота несмешанности.*

*Ключевые слова: почти дистрибутивные решетки, главные решетки, $\alpha$-идеалы, $WB$-высота несмешанности, полные решетки, Коэна-Маколея кольца, несмешанность.*