Let $R$ be a commutative ring with an identity, $\mathcal{R}$ be an almost distributive lattice and $I_\alpha(\mathcal{R})$ be the set of all $\alpha$-ideals of $\mathcal{R}$. If $L(R)$ is the principal lattice of $R$, then $R[I_\alpha(\mathcal{R})]$ is Cohen-Macaulay. In particular, $R[I_\alpha(\mathcal{R})][X_1, X_2, \cdots]$ is WB-height-unmixed.

Keywords: Almost distributive lattice, principal lattice, $\alpha$-ideals, multiplicative lattice; complete lattice, WB-height-unmixedness, Cohen-Macaulay rings, unmixedness.

Introduction

Lattices, in general, (specially multiplicative lattices), are natural abstractions of the set of ideals of a ring. However, Without a good notion of principal lattice, it is impossible to get very deep results, see [3]. Dilworth overcame this in [4], with a new notion of a principal element. Recall, that a multiplicative lattice is a complete lattice $L$ with a commutative, associative multiplication which distributes over arbitrary joins and its largest element $I$, is the identity for the multiplication, see [4]. Basically, an element $E$ of a multiplicative lattice $L$, is said to be meet-(join-)principal if $(A \land (B : E))E = (AE) \land B$ (if $(BE \lor A) : E = B \lor (A : E)$) for all $A$ and $B$ in $L$. A principal element is an element that is both meet-principal and join-principal or $A \land E = (A : E)E$ and $AE : E = A \lor (0 : E)$, for all $A \in L$. A lattice $L$, is called a principal lattice, when each of its elements is principal. Here, the residual quotient of two elements $A$ and $B$ is denoted by $A : B$, so $A : B = \lor \{X \in L | XB \leq A\}$. An almost distributive lattice (ADL), was introduced by U. M. Swamy and G. C. Rao in [11], as an algebra $(R, \lor, \land)$ of type $(2, 2)$, which satisfies almost all the properties of a distributive lattice, except possibly the commutativity of $\lor$, the commutativity of $\land$ and the right distributivity of $\lor$ over $\land$. W. H. Cornish studied in to [2], the properties of $\alpha$-ideals in a distributive lattice, (see the next section for the definition). In this paper, the concept of an $\alpha$-ideal in an ADL is introduced, analogous to the case of distributive lattices. In section 2, it is shown that, if $R$ is a commutative ring with an identity and $L(R)$ is the principal lattice, then $R$ is a Cohen-Macaulay ring. In section 3, we prove that, if $R$ is Cohen-Macaulay ring and if $P$ is a distributive lattice, then $R[P]$ is Cohen-Macaulay ring, where $R[P]$ is the polynomial ring over $P$. Finally, in section 4, we conclude some properties of $L(R)$.

1. The Principal Lattice and $\alpha$-ideals

A Noether lattice is a modular multiplicative lattice satisfying the ascending chain condition in which every element is a join of elements called principal elements. The multiplication, meet,
and join in a Noether lattice are supposed to mirror the multiplication, intersection, and sum of ideals. Because of this, a multiplicative lattice is defined to be a complete lattice \( L \), containing a unite element \( I \) and a null element \( 0 \), and provided with a commutative, associative, join-distributive multiplication, for which \( I \) is an identity element. We will use \( \wedge \) and \( \vee \) to denote meet and join, respectively, and \( \leq \) to denote lattice partial ordering, with \( < \) reserved for strict inequality.

**Remark 1.1.** Let \( R \) be a commutative ring with an identity, \( I \) and \( J \) be ideals of \( R \), \( I + J \) and \( IJ \), be the ordinary sum and product of ideals. With these two operations as join and meet, the set of all ideals of a given ring, forms a complete modular lattice. Remember that, a principal lattice, is a lattice in which every element is principal. The following theorem is proved in [6].

**Theorem 1.1.** Let \( R \) be a commutative ring with identity. Then \( L(R) \) is a principal lattice, if and only if, \( R \) is a Noetherian multiplication ring.

Note that, a ring is called a multiplication ring, if every ideal of \( R \) is product of two ideals. Let \( M \) be a finitely generated module over a Noetherian ring \( R \). We say that \( x \in R \) is an \( M \)-regular element, if \( xg = 0 \) for \( g \in M \) implies \( g = 0 \), in the other words, if \( x \) is not a zero-divisor on \( M \).

A sequence \( x_1, \ldots, x_r \) of elements of the ring \( R \), is called an \( M \)-regular sequence or simply an \( M \)-sequence if the following conditions are satisfied:

1. \( x_i \) is an \( M/(x_1, \ldots, x_{i-1})M \) regular element for \( i = 1, \ldots, r \);
2. \( M/(x_1, \ldots, x_r)M \neq 0 \).

Suppose \( I \subseteq R \) is an ideal with \( IM \neq M \). The depth of \( I \) on \( M \) is maximal length of an \( M \)-regular sequence in \( I \), denoted by \( \text{depth}(I, M) \). If \( R \) is a local ring with a unique maximal ideal \( m \), we write \( \text{depth}(m) \), for \( \text{depth}(m, M) \).

Let \( R \) be a Noetherian local ring. A finitely generated \( R \)-module \( M \), is a Cohen-Macaulay module, if \( \text{depth}(M) = \dim(M) \). If \( R \) itself is a Cohen-Macaulay module, then it is called a Cohen-Macaulay ring. For the proof of the following theorem, see [9].

**Theorem 1.2.** Suppose \( R \) is a Noetherian multiplication ring. Then \( R \) is a Cohen-Macaulay ring.

## 2. Cohen-Macaulay and Unmixedness

We begin this section by a definition from Bourbaki.

**Definition 2.1.** A prime ideal \( P \) is an associated prime of \( I \), if \( P = I : x \) for some \( x \in R \).

Remember that the height of a prime ideal \( P \) is the maximum length of the chains of prime ideals of the following form,

\[
P_1 \subset P_2 \subset \cdots \subset P_k = P.
\]

We will denote the height of \( P \) by \( ht(P) \). An ideal \( I \) of \( R \) is said to be height-unmixed, if all the associated primes of \( I \) have equal height. That is \( ht(P) = ht(Q) \), for all \( P, Q \in \text{Ass}(I) \), where \( \text{Ass}(I) \) denotes the set of associated primes of \( I \). An ideal \( I \) is said to be unmixed if there are no embedded primes among the associated primes of \( I \). That is, \( P \subseteq Q \Rightarrow P = Q \), for all \( P, Q \in \text{Ass}(I) \).

We will say that an ideal is WB-height-unmixed, if it is height-unmixed with respect to the set of weak Bourbaki associated primes and an ideal is WB-unmixed if it is unmixed with respect to the set of weak Bourbaki associated primes. The set of weak Bourbaki
associated primes of an ideal \( I \) is denoted by \( \text{Ass}_R(I) \). A prime ideal \( P \) is a weak Bourbaki associated prime of the ideal \( I \) if it is a minimal ideal of the form \( I : a \), for some \( a \in R \).

**Theorem 2.1.** If \( R \) satisfies GPIT (generalized principal ideal theorem), then \( R \) is WB-height-unmixed if and only if \( R \) is WB-unmixed.

**Proof.** Suppose \( R \) is a ring which satisfies GPIT.

\( (\Rightarrow) \) Suppose \( R \) is WB-height-unmixed and let \( I \) be a height-generated ideal in \( R \). In a ring which satisfies GPIT every ideal \( I \) satisfies \( \text{ht}(I) \leq \ell(I) \), where \( \ell(I) \) denotes the minimal number of generators of \( I \). Thus, in a ring with GPIT, \( I \) is height-generated if and only if \( \ell(I) = \text{ht}(I) < \infty \). To show that \( I \) is WB-height-unmixed, suppose \( P, Q \in \text{Ass}_R(I) \), with \( P \subseteq Q \). Since \( R \) is WB-height-unmixed so \( I \) is WB-height-unmixed. Thus, \( \text{ht}(P) = \text{ht}(Q) = \text{ht}(I) < \infty \). So, \( P \) and \( Q \) are prime ideals with \( P \subseteq Q \) and \( \text{ht}(P) = \text{ht}(Q) < \infty \). Thus, \( P = Q \) and so \( I \) is WB-unmixed. Therefore, \( R \) is WB-unmixed.

\( (\Leftarrow) \) Suppose \( R \) is WB-unmixed and let \( I \) be a height-generated ideal in \( R \). In the first part of this proof, we have \( \text{ht}(I) = \ell(I) < \infty \). Let \( n = \text{ht}(I) = \ell(I) \). Note that, since \( R \) satisfies GPIT, we have \( \text{ht}(P) \leq n \) for all \( P \in \text{Min}(I) \). However, since \( \ell(I) = n \), we have \( \text{ht}(P) = n \) for all maximal associated prime \( P \). Since \( I \) is WB-unmixed, we have \( \text{Ass}_R(I) \) is the set of all minimal associated primes of \( I \). Therefore, \( \text{ht}(P) = n \) for all \( P \in \text{Ass}_R(I) \) and thus, \( I \) is WB-height-unmixed. Therefore, \( R \) is WB-height-unmixed. \( \square \)

**Theorem 2.2.** Let \( R \) be a Noetherian ring and let \( S = R[X_1, X_2, \ldots] \), the ring of polynomials in the variables \( X_1, X_2, \ldots \). For any prime ideal \( P \) in \( R \) we have \( \text{ht}(P) = \text{ht}(PS) \) where \( \text{ht}(PS) \) refers to the height of the ideal \( PS \) in \( S \).

**Proof.** Note that the proof of this theorem depends only on the weaker condition that \( R \) is a strong \( S \)-ring (see [8] for more information on strong \( S \)-rings). It is not necessary for the ring to be Noetherian. First, note that, we have trivially \( \text{ht}(P) \leq \text{ht}(PS) \), since the extensions of a chain of distinct prime ideals in \( R \) is a chain of distinct prime ideals in \( S \). For \( i \geq 1 \), let \( R_i = R[X_1, X_2, \ldots, X_i] \). Since \( R \) is Noetherian (and thus a strong \( S \)-ring), we have \( \text{ht}(P) = \text{ht}(P_i) \), see [8](Theorem 149, page 108). Now, suppose \( \text{ht}(PS) > n \), where \( n = \text{ht}(P) \). Then there is a chain of prime ideals

\[
Q_0 \subset Q_1 \subset \cdots \subset Q_{n+1} = PS,
\]

in \( S \). For \( 1 \leq i \leq n+1 \), choose \( x_i \in Q_i \setminus Q_{i-1} \). Since \( S = \lim R_i \), there is a positive integer \( j \) such that \( \{x_1, \cdots, x_{n+1}\} \subseteq R_j \). For \( 0 \leq i \leq n+1 \), let \( T_i = Q_i \cap R_i \). Then

\[
T_0 \subset T_1 \subset \cdots \subset T_{n+1}
\]

is a chain of prime ideals in \( R_j \). So, \( \text{ht}(T_{n+1}) \geq n+1 \). However, \( T_{n+1} = Q_{n+1} \cap R_j = PS \cap R_j = P_j \) and we have already noted that \( \text{ht}(P_j) = n \), a contradiction. Therefore, \( \text{ht}(PS) = \text{ht}(P) \). \( \square \)

In [1], it was shown that \( R[X_1, X_2, \ldots] \) satisfies GPIT (if \( R \) is a Noetherian ring). The statement of this fact in [1] actually makes the assumption that \( R \) is a domain, however, the fact that \( R \) is a domain, is not necessary in the proof given in [1], so we will use the more general result. By applying 2.1 to this result, we get the following theorem

**Theorem 2.3.** Let \( R \) be a Cohen-Macaulay ring. Then \( R[X_1, X_2, \ldots] \) is WB-height-unmixed.

**Theorem 2.4.** Let \( L \) be a Noetherian multiplicative lattice. Every element of \( L \) is principal element, if and only if, for all \( a \leq b \), there is an element \( c \in L \), such that \( a = bc \).

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Proof. Suppose that elements of \( L \) are principal and let \( a, b \in L \) and \( a \leq b \). Then \( a = a \cap b = (a : b)b \), and so \( c = (a : b) \). Conversely, it follows from (ACC), that each element of \( L \) is a join of a finite number of principal elements. Therefore, to prove the theorem it is sufficient to show that if \( m \) and \( n \) are principal elements of \( L \), then \( m \cup n \) is principal. Let \( m \) be principal and let \( m \leq d \), where \( d \in L \). Then \( m = cd, \) for some \( c \in L \), and since \( m \) is join principal \((a \cup b)\) : \( cd = a : m \cup b \), for all \( a, b \in L \). Hence \((a \cup b) : c \) : \( d = a : m \cup b \). However, \((a \cup b)c = ac \cup bdc \leq a \cup bdc \), and so \( a \cup b \leq (a \cup b) : c \). Therefore, \((a \cup b) : d \leq a : m \cup b \) for all \( a, b \in L \). Thus, if \( m \) and \( n \) are principal elements of \( L \), we have for all \( a, b \in L \),

\[
(a \cup b(m \cup n)) : (m \cup n) & \leq (a : m \cup b) \cap (a : n \cup b) \\
& = (a : m \cap a : n) \cup b \\
& = a : (m \cap n) \cup b.
\]

\[\square\]

Corollary 2.1. Let \( R \) be a commutative ring with an identity and \( L(R) \) be a Noether lattice. Every ideal of \( R \) is an principal element in \( L(R) \), if and only if, \( R \) is a multiplication ring.

For the proof of the following theorem, see [5].

Theorem 2.5. If \( R \) is Cohen-Macaulay ring, and if \( P \) is a distributive lattice, then \( R[P] \) is Cohen-Macaulay.

3. \( \alpha \)-ideals and Cohen-Macaulay Rings

In this section we introduce the concept of an \( \alpha \)-ideal in an ADL with zero, analogous to that in a distributive lattice [2]. An Almost Distributive Lattice (ADL) is an algebra \((\mathcal{R}, \lor, \land)\) of type \((1.2)\) satisfying:

1. \((x \lor y) \land z = (x \land z) \lor (y \land z)\)
2. \(x \land (y \lor z) = (x \land y) \lor (x \land z)\)
3. \((x \lor y) \land y = y\)
4. \((x \lor y) \land x = x\)
5. \(x \lor (x \land y) = x \) for any \(x, y, z \in \mathcal{R}\)

If \(\mathcal{R}\) has an element \(0\), and satisfies \(0 \land x = 0\) and \(0 \lor x = 0\) along with the above properties, then \(\mathcal{R}\) is called an ADL with 0.

Definition 3.1. For any non-empty subset \(A\) of an ADL, \(\mathcal{R}\) with 0, define \(A^* = \{x \in \mathcal{R} \mid a \land x = 0, \text{for all } a \in A\}\). Then \(A^*\) is called the annihilator of \(A\). For any \(a \in \mathcal{R}\), we have \(\{a\}^* = [a]^*\), where \([a]\) is the principal ideal generated by \(a\). For any \(\emptyset \neq A \subseteq \mathcal{R}\), we have clearly \(A \land A^* = \{0\}\).

For the proof of the next lemmas, see [10].

Lemma. For any non-empty subset \(A\) of \(\mathcal{R}\), \(A^*\) is an ideal of \(\mathcal{R}\).

Lemma. For any non-empty subsets \(I, J\) of \(\mathcal{R}\), we have the following:

1. If \(I \subseteq J\), then \(J^* \subseteq I^*\)
2. \(I \subseteq I^{**}\)
3. \(I^{***} = I^*\)
4. \((I \lor J)^* = I^* \cap J^*\)
Definition 3.2. Let $\mathfrak{R}$ be a ADL with $0$. An ideal $I$ of $\mathfrak{R}$ is called an $\alpha$-ideal if $(x)^{**} \subseteq I$ for all $x \in I$.

We now denote the set of all $\alpha$-ideal of an ADL $\mathfrak{R}$ by $I_\alpha(\mathfrak{R})$. If $\mathfrak{R}$ is an ADL, then we know that $(I(\mathfrak{R}), \lor, \land)$ is a distributive lattice. But the set $I_\alpha(\mathfrak{R})$ is not a sublattice of $I(\mathfrak{R})$.

Definition 3.3. A Noether lattice is said to be complete if it is complete in the topology of the Jacobson radical.

In [7], the following theorem is proved.

Theorem 3.1. Let $(L, m)$ be a distributive local Noether lattice of dimension $d$. Then $L$ is complete in the $m$-adic topology.

Let $R$ be a local Noetherian ring with the maximal ideal $M$. Then $L(R)$, the lattice of ideals of $R$, is a local Noether lattice and also $L(R)$ is a complete modular lattice. A ring $R$ is called an arithmetical ring, if $L(R)$ is distributive.

Corollary 3.1. If $(R, m)$ is a local Noetherian ring and is arithmetical ring, then $L(R)$ is a complete in the $m$-adic topology.

Proof. This is immediate from Remark 1.1 and Theorem 3.1 and if $L$ is an ADL, then $L(R)$ is a distributive lattice.

Corollary 3.2. If $(\mathfrak{R}, \mathfrak{m})$ is local Noether lattice and is an ADL, then $I(\mathfrak{R})$ is a complete in the $m$-adic topology.

In [10], it is proved that, if $\mathfrak{R}$ is an ADL with $0$, then $I_\alpha(\mathfrak{R})$ forms a distributive lattice. So we have

Theorem 3.2. Let $R$ be a commutative ring with an identity and let $I_\alpha(\mathfrak{R})$ be the set of all $\alpha$-ideal of an ADL $\mathfrak{R}$. If $L(R)$ is a principal lattice, then $R[I_\alpha(\mathfrak{R})][X_1, X_2, \ldots]$ is WB-height-unmixed.

Proof. Since $L(R)$ is a principal lattice, $R$ is Cohen-Macaulay. By assumption, $I_\alpha(\mathfrak{R})$ is a distributive lattice. Thus the $R[I_\alpha(\mathfrak{R})][X_1, X_2, \ldots]$ is WB-height-unmixed. 

References


Полиномы, α-идеалы и главные решетки

Али Молхаси

Пусть $R$ — коммутативное кольцо с единицей, $R$ — почти дистрибутивная решетка и $I_\alpha(R)$ — множество всех α-идеалов в $R$. Если $L(R)$ — главная решетка $R$, то $R[I_\alpha(R)]$ — кольцо Коэна-Маколея. В частности, $R[I_\alpha(R)][X_1, X_2, \cdots]$ — $WB$-высота несмешанности.

Ключевые слова: почти дистрибутивные решетки, главные решетки, α-идеалы, WB-высота несмешанности, полные решетки, Коэна-Маколея кольца, несмешанность.