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Multidimensional Analog of the Bernoulli Polynomials and its Properties

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We consider a generalization of the Bernoulli numbers and polynomials to several variables, namely, we define the Bernoulli numbers associated with a rational cone and the corresponding Bernoulli polynomials. Also, we prove some properties of the Bernoulli polynomials.

Keywords: Bernoulli numbers and polynomials, generating functions, Todd operator, rational cone.

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Introduction

The Bernoulli numbers b_μ are the coefficients of the Taylor series expansion of the function $T(\xi) = \frac{\xi}{e^\xi - 1}$:

$$T(\xi) = \sum_{\mu \geq 0} b_\mu \frac{\xi^\mu}{\mu!}.$$

The Bernoulli numbers were introduced by J. Bernoulli in connection with the problem of summation of powers of consecutive integers: $1^\mu + 2^\mu + \dots + x^\mu$.

The Bernoulli polynomials

$$B_\mu(x) = \sum_{k=0}^{\mu} C_\mu^k b_{\mu-k} x^k, \quad C_\mu^k = \frac{\mu!}{k! (\mu-k)!},$$

where $b_\mu = B_\mu(0)$ are the Bernoulli numbers were considered by J. Bernoulli [1] for natural x , and for any x these polynomials were first studied by Euler [2] who used the generating function in 1738:

$$\frac{\xi}{e^\xi - 1} e^{x\xi} = \sum_{\mu \geq 0} B_\mu(x) \frac{\xi^\mu}{\mu!}.$$

Also J. L. Raabe (1801–1859) [3] studied the Bernoulli polynomials, he found two important formulas and introduced this term (J. L. Raabe, 1851). The Bernoulli and Euler polynomials were later systematically studied by N. Nörlund [4].

The Bernoulli numbers have wide applications in computer technology [5], combinatorial analysis [6, 7] and in numerical analysis [8]. Gould [9] remarks that many sums involving binomial coefficients greatly benefit from the use of Bernoulli numbers. There is a number of papers about

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different generalizations of Bernoulli numbers and polynomials [10–15]. For example, Temme [16] uses generalized Bernoulli polynomials with complex powers.

In 1880 Appell (1855–1930) [17] characterized certain polynomial sequences $B_\mu(x)$ by the property $DB_\mu(x) = \mu B_{\mu-1}(x)$. Polynomials satisfying this condition are called Appell polynomials. For the Bernoulli polynomials this formula can be called the differentiation formula.

It was Euler who first defined a differential operator of infinite order $\frac{D}{e^D - 1} = \sum_{\mu \geq 0} b_\mu \frac{D^\mu}{\mu!}$, where D is a differential operator, now this operator is called the Todd operator and denoted $Td(D)$. It connects the solutions of the difference equation $f(x+1) - f(x) = \varphi(x)$ and the differential equation $Df(x) = Td(D)\varphi(x)$. The Bernoulli polynomials can be considered as a result of the Todd operator action on the monomial x^μ :

$$B_\mu(x) = \frac{D}{e^D - 1} x^\mu. \quad (1)$$

The following identities hold for the Bernoulli polynomials:
the argument addition formula

$$B_\mu(x+y) = \sum_{k=0}^{\mu} C_\mu^k B_k(x) y^{\mu-k}; \quad (2)$$

the differentiation formula

$$B'_\mu(x) = \mu B_{\mu-1}(x); \quad (3)$$

the complement formula

$$B_\mu(x) = (-1)^\mu B_\mu(1-x); \quad (4)$$

the multiplication formula

$$B_\mu(mx) = m^{\mu-1} \sum_{k=0}^{\mu-1} B_\mu\left(x + \frac{k}{m}\right). \quad (5)$$

In this article we consider a generalization of the Bernoulli numbers and polynomials to the case of several variables, namely, we define the Bernoulli numbers associated with a rational cone, and the corresponding Bernoulli polynomials. For functions in several variables we construct a Todd operator associated with a rational cone K , and prove (Theorem 1) that the Bernoulli polynomials are the result of the Todd operator action on monomials. Further on, we formulate and prove multidimensional analogs of formulas (2)–(5) for the Bernoulli polynomials.

1. A Todd operator and the Bernoulli polynomials in several variables

A multidimensional analog of the Euler differential operator of infinite order $\frac{D}{e^D - 1}$ is called the Todd operator (see, for example, [18]). The Todd operator [19] helps to connect volumes and a number of lattice points of convex polytopes. This formula is closely related to the Hirzebruch-Riemann-Roch theorem for smooth projective toric varieties. Note the articles [20–23] in which the operator $\frac{D_1}{e^{a_1 D_1} - 1} \dots \frac{D_n}{e^{a_n D_n} - 1}$, where a_j are some constants, is considered.

Now we define the Bernoulli numbers, polynomials and a Todd operator associated with a rational cone K . Let a^1, \dots, a^n be *linearly independent* vectors with integer coordinates $a^j =$

$(a_1^j, \dots, a_n^j), a_i^j \in \mathbb{Z}$. A *rational cone* generated by the vectors a^1, \dots, a^n , is the set $K = \{y \in \mathbb{R}^n : y = \lambda_1 a^1 + \dots + \lambda_n a^n, \lambda_j \in \mathbb{R}_+, j = 1, \dots, n\}$.

Note that such cone is *simplicial*, i.e. each element of the cone is represented by the cone generators in a *unique* way.

For points $u, v \in \mathbb{R}^n$ we define a partial order relation $\underset{K}{\geq}$ the following way:

$$u \underset{K}{\geq} v \Leftrightarrow u \in v + K,$$

where $v + K$ is a shift of the cone K by the vector v . Besides, let us write $u \not\underset{K}{\geq} v$, if $u \in K \setminus \{v + K\}$, i.e. if the condition $u \underset{K}{\geq} v$ is not fulfilled. Let $\mathbb{Z}^n = \mathbb{Z} \times \dots \times \mathbb{Z}$, and note that each element $y \in K \cap \mathbb{Z}^n$ can be expressed as a linear combination of the basis vectors $y = \lambda_1 a^1 + \dots + \lambda_n a^n$, $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$. We can write it in a matrix form $y = A\lambda$, where y and λ are column vectors, A is a matrix with the determinant $\Delta \neq 0$, and the columns are the coordinates of vectors a^j

$$A = \begin{pmatrix} a_1^1 & \dots & a_1^n \\ \dots & \dots & \dots \\ a_n^1 & \dots & a_n^n \end{pmatrix}.$$

For $j = 1, \dots, n$ consider the hyperplanes

$$L_{j,k} = \{\xi : \langle a^j, \xi \rangle = 2k\pi i\},$$

where $i = \sqrt{-1}$, $j = 1, \dots, n$, $k = 0, \pm 1, \pm 2 \dots$, and for a meromorphic function

$$T(\xi) = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1},$$

where $\langle a^j, \xi \rangle = \sum_{k=1}^n a_k^j \xi_k$, $\xi = (\xi_1, \dots, \xi_n)$, the hyperplanes $L_{j,0}$, $j = 1, \dots, n$ are ‘removable’ singular set, so the function $T(\xi)$ is holomorphic in a neighborhood of the origin, more exactly for $R = 2\pi / \max_j \|a^j\|$ it is holomorphic in a polycylinder $U_R = \{\xi : |\xi_j| < R, j = 1, \dots, n\}$, and for $k = 0, \pm 1, \pm 2 \dots$ the hyperplanes $L_{j,k}$ do not cross with the polycylinder U_R . Therefore, in this polycylinder the function $T(\xi)$ expands into a series

$$T(\xi) = \sum_{\mu \geq 0} \frac{b_\mu^A}{\mu!} \xi^\mu, \tag{6}$$

where $\mu = (\mu_1, \dots, \mu_n)$, $\mu! = \mu_1! \dots \mu_n!$, $\xi^\mu = \xi_1^{\mu_1} \dots \xi_n^{\mu_n}$, and $\mu \geq 0$ means $\mu_j \geq 0, j = 1, \dots, n$.

The coefficients b_μ^A of the series (6) we call *the Bernoulli number associated with the rational cone*.

For $\mu = (\mu_1, \dots, \mu_n)$ the Bernoulli polynomials in several variables are the polynomials

$$B_\mu^A(x) = \sum_{0 \leq k \leq \mu} \frac{\mu!}{(\mu - k)! k!} b_{\mu - k}^A x^k,$$

where b_k^A are the Bernoulli numbers, $k = (k_1, \dots, k_n)$, $x = (x_1, \dots, x_n)$, $\mu - k = (\mu_1 - k_1, \dots, \mu_n - k_n)$.

For $n = 1$ and $a = 1$ thus defined Bernoulli numbers and polynomials coincide with the classical ones.

Example 1. For the cone K generated by vectors $a^1 = (1, 0)$ and $a^2 = (1, 1)$ the first Bernoulli numbers are $b_{00} = 1$, $b_{10} = -1$, $b_{01} = -\frac{1}{2}$, $b_{11} = \frac{5}{12}$, and for $\mu = (1, 1)$ the Bernoulli polynomials are $B_{11}^A(x_1, x_2) = x_1x_2 - \frac{1}{2}x_1 - x_2 + \frac{5}{12}$.

A Todd mapping is the meromorphic function $Td^A : \mathbb{C}^n \rightarrow \mathbb{C}$ defined by $Td^A(\xi) = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1}$. It expands into a series $Td^A(\xi) = \sum_{\mu \geq 0} \frac{b_\mu^A}{\mu!} \xi^\mu$, and the Todd operator is the result of substitution of the differential operator $\partial = (\partial_1, \dots, \partial_n)$ in place of ξ :

$$Td^A(\partial) = \sum_{\mu \geq 0} \frac{b_\mu^A}{\mu!} \partial^\mu,$$

where $\mu \geq 0$ means $\mu_j \geq 0$, $j = 1, \dots, n$.

Remark 1. In general, to define the Bernoulli numbers and polynomials and the Todd operator we do not need the condition of rationality of the cone, i.e., any real numbers can be the coordinates of the vectors a^j . However, in the problem of summation of functions the rationality of the cone is a natural condition.

The Todd operator can act on a function h if the series $Td(\partial)h$ converges absolutely in the domain of the function h uniformly on compact subsets. It is obvious that polynomials can be acted on by the Todd operator and for the monomial $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}$ we have the following analog of formula (1).

Theorem 1.1. *If $Td^A(\partial)$ is the Todd operator and $B_\mu^A(x)$ are Bernoulli polynomials associated with a rational cone then we have the equality*

$$Td^A(\partial)x^\mu = B_\mu^A(x).$$

Proof. First note that the equality $\frac{\partial^\alpha x^\mu}{\alpha!} = \frac{\mu!}{\alpha!(\mu - \alpha)!} x^{\mu - \alpha}$ holds true. Indeed,

$$\begin{aligned} \frac{\partial^\alpha x^\mu}{\alpha!} &= \frac{\mu_1(\mu_1 - 1) \dots (\mu_1 - \alpha_1 + 1)}{\alpha_1!} \dots \frac{\mu_n(\mu_n - 1) \dots (\mu_n - \alpha_n + 1)}{\alpha_n!} x^{\mu_1 - \alpha_1} \dots x^{\mu_n - \alpha_n} = \\ &= \frac{\mu!}{\alpha!(\mu - \alpha)!} x^{\mu - \alpha}. \end{aligned}$$

Next, using the definition of the Todd operator and the previous equality we obtain

$$\begin{aligned} Td(\partial)x^\mu &= \prod_{j=1}^n \frac{\langle a^j, \partial \rangle}{e^{\langle a^j, \partial \rangle} - 1} x^\mu = \sum_{\alpha \geq 0} \frac{b_\alpha^A}{\alpha!} \partial^\alpha x^\mu = \\ &= \sum_{0 \leq \alpha \leq \mu} b_\alpha^A \frac{\mu!}{\alpha!(\mu - \alpha)!} x^{\mu - \alpha} = \sum_{0 \leq k \leq \mu} b_{\mu - k}^A \frac{\mu!}{k!(\mu - k)!} x^k = B_\mu^A(x). \end{aligned}$$

□

2. Complement, differentiation, addition, and multiplication formulas

To formulate an analog of the property of differentiation of Bernoulli polynomials $B_\mu^A(x)$ we need a differential operator in the direction of the vectors a^j generating the cone K :

$$D_j = \langle a^j, \partial \rangle = \sum_{k=1}^n a_k^j \partial_k,$$

where ∂_j are operators of differentiating with respect to variables, $\partial = (\partial_1, \dots, \partial_n)$, $\partial^\mu = \partial_1^{\mu_1} \dots \partial_n^{\mu_n}$, $j = 1, \dots, n$. Denote $D = D_1 \dots D_n$.

Theorem 2.1. *The following multidimensional analogs of the properties (2)–(5) of Bernoulli polynomials hold true:*

1) *the argument addition formula*

$$B_\mu^A(x+y) = \sum_{0 \leq k \leq \mu} \frac{\mu!}{k!(\mu-k)!} B_k^A(x) y^{\mu-k};$$

2) *the differentiation formula*

$$DB_\mu^A(x) = \sum_{0 \leq k \leq \mu, \|k\|=n} \frac{\mu!}{k!(\mu-k)!} B_{\mu-k}^A(x) \frac{M_k}{k!},$$

where M_k are the coefficients of the polynomial

$$\prod_{j=1}^n \langle a^j, \xi \rangle = \sum_{\|k\|=n} M_k \frac{\xi^k}{k!};$$

3) *the complement formula*

$$B_\mu^A(x) = (-1)^{\|\mu\|} B_\mu^A(a-x),$$

where $a = a^1 + \dots + a^n$, $\|\mu\| = \mu_1 + \dots + \mu_n$;

4) *the multiplication formula*

$$B_\mu^A(mx) = m^{\mu-I} \sum_{0 \leq k \leq \mu-I} B_\mu^A\left(x + \frac{1}{m} Ak\right),$$

where $I = (1, \dots, 1)$, $\frac{1}{m} = \left(\frac{1}{m_1}, \dots, \frac{1}{m_n}\right)$ is a row vector, $k = \begin{pmatrix} k_1 \\ \dots \\ k_n \end{pmatrix}$ is a column vector.

Proof. To prove this theorem it is convenient to use the definition of Bernoulli polynomials via generating function

$$\prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle x, \xi \rangle} = \sum_{\mu \geq 0} B_\mu^A(x) \frac{\xi^\mu}{\mu!}, \quad (7)$$

where $\langle x, \xi \rangle = x_1 \xi_1 + \dots + x_n \xi_n$.

1. The proof of the argument addition formula. Using the definition of Bernoulli polynomials via generating function we obtain

$$\sum_{\mu \geq 0} B_{\mu}^A(x+y) \frac{\xi^{\mu}}{\mu!} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle x+y, \xi \rangle} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle x, \xi \rangle} e^{\langle y, \xi \rangle} = \sum_{k \geq 0} B_k^A(x) \frac{\xi^k}{k!} \sum_{\alpha \geq 0} y^{\alpha} \frac{\xi^{\alpha}}{\alpha!}.$$

Multiplying power series and collecting similar terms, we get

$$\sum_{\mu \geq 0} B_{\mu}^A(x+y) \frac{\xi^{\mu}}{\mu!} = \sum_{\mu \geq 0} \left(\sum_{k+\alpha=\mu} \frac{\mu!}{k!\alpha!} B_k^A(x) y^{\alpha} \right) \frac{\xi^{\mu}}{\mu!} = \sum_{\mu \geq 0} \left(\sum_{0 \leq k \leq \mu} \frac{\mu!}{k!(\mu-k)!} B_k^A(x) y^{\mu-k} \right) \frac{\xi^{\mu}}{\mu!}.$$

The formula follows from equality of coefficients at identical powers of ξ .

2. The proof of the differentiation formula. Act by the operator D on both parts of the equality

$$\sum_{\mu \geq 0} B_{\mu}^A(x) \frac{\xi^{\mu}}{\mu!} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle x, \xi \rangle},$$

to obtain, keeping in mind that $De^{\langle x, \xi \rangle} = e^{\langle x, \xi \rangle} \prod_{j=1}^n \langle a^j, \xi \rangle$,

$$\sum_{\mu \geq 0} DB_{\mu}^A(x) \frac{\xi^{\mu}}{\mu!} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle x, \xi \rangle} \prod_{j=1}^n \langle a^j, \xi \rangle.$$

Let M_k be the coefficients of the polynomial $\prod_{j=1}^n \langle a^j, \xi \rangle$ at $\frac{\xi^k}{k!}$:

$$\prod_{j=1}^n \langle a^j, \xi \rangle = \sum_{\|k\|=n} M_k \frac{\xi^k}{k!},$$

then

$$\sum_{\mu \geq 0} DB_{\mu}^A(x) \frac{\xi^{\mu}}{\mu!} = \sum_{\alpha \geq 0} B_{\alpha}^A(x) \frac{\xi^{\alpha}}{\alpha!} \sum_{\|k\|=n} M_k \frac{\xi^k}{k!}.$$

Multiplying power series, we have

$$\sum_{\mu \geq 0} DB_{\mu}^A(x) \frac{\xi^{\mu}}{\mu!} = \sum_{\mu \geq 0} \left(\sum_{\alpha+k=\mu, \|k\|=n} \frac{\mu!}{k!\alpha!} B_{\alpha}^A(x) M_k \right) \frac{\xi^{\mu}}{\mu!}.$$

After changing indices we equal the coefficients of powers of ξ and obtain the necessary property.

3. The proof of the complement formula.

$$\begin{aligned} \sum_{\mu \geq 0} (-1)^{\|\mu\|} B_{\mu}^A(a-x) \frac{\xi^{\mu}}{\mu!} &= \sum_{\mu \geq 0} B_{\mu}^A(a-x) \frac{(-\xi)^{\mu}}{\mu!} = \prod_{j=1}^n \frac{\langle a^j, -\xi \rangle}{e^{\langle a^j, -\xi \rangle} - 1} e^{\langle a-x, -\xi \rangle} = \\ &= \prod_{j=1}^n \frac{-\langle a^j, \xi \rangle}{\frac{1}{e^{\langle a^j, \xi \rangle}} - 1} e^{\langle a-x, -\xi \rangle} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{1 - \frac{1}{e^{\langle a^j, \xi \rangle}}} e^{\langle a-x, -\xi \rangle} = \\ &= \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{\frac{e^{\langle a^j, \xi \rangle}}{e^{\langle a^j, \xi \rangle}} - \frac{1}{e^{\langle a^j, \xi \rangle}}} e^{\langle a-x, -\xi \rangle} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle a^j, \xi \rangle} e^{-\langle a, \xi \rangle} e^{\langle x, \xi \rangle}. \end{aligned}$$

Since $e^{-\langle a, \xi \rangle} = \prod_{j=1}^n e^{\langle a^j, \xi \rangle}$,

$$\sum_{\mu \geq 0} (-1)^{\|\mu\|} B_\mu^A(a-x) \frac{\xi^\mu}{\mu!} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle a^j, \xi \rangle} e^{-\langle a^j, \xi \rangle} e^{\langle x, \xi \rangle} = \sum_{\mu \geq 0} B_\mu^A(x) \frac{\xi^\mu}{\mu!}.$$

Equal the coefficients at powers of ξ to obtain the complement formula.

4. The proof of the multiplication formula. From the definition of Bernoulli polynomials we have

$$\sum_{\mu \geq 0} B_\mu^A(mx) \frac{\xi^\mu}{\mu!} = \prod_{j=1}^n \frac{\langle a^j, \xi \rangle}{e^{\langle a^j, \xi \rangle} - 1} e^{\langle mx, \xi \rangle}. \quad (8)$$

Using the formula of geometric progression with denominator $e^{\langle a^j, \xi \rangle}$, we obtain

$$\frac{1}{e^{\langle a^j, \xi \rangle} - 1} = \frac{1 + e^{\langle a^j, \xi \rangle} + \dots + e^{\langle (m_j-1)a^j, \xi \rangle}}{e^{\langle m_j a^j, \xi \rangle} - 1},$$

After substituting this equality into the right hand side of (8) and after some transformations we obtain

$$\begin{aligned} \sum_{\mu \geq 0} B_\mu^A(mx) \frac{\xi^\mu}{\mu!} &= \prod_{j=1}^n \frac{1 + e^{\langle a^j, \xi \rangle} + \dots + e^{\langle (m_j-1)a^j, \xi \rangle}}{e^{\langle m_j a^j, \xi \rangle} - 1} \langle m_j a^j, \xi \rangle m_j^{-1} e^{\langle mx, \xi \rangle} = \\ &= m^{-I} \prod_{j=1}^n \frac{\langle a^j, m_j \xi \rangle}{e^{\langle a^j, m_j \xi \rangle} - 1} e^{\langle x, m_j \xi \rangle} \prod_{j=1}^n \left(1 + e^{\langle a^j, \xi \rangle} + \dots + e^{\langle (m_j-1)a^j, \xi \rangle} \right). \end{aligned}$$

Since

$$\prod_{j=1}^n \left(1 + e^{\langle a^j, \xi \rangle} + \dots + e^{\langle (m_j-1)a^j, \xi \rangle} \right) = \sum_{0 \leq k \leq m-I} e^{\langle k_1 a^1 + \dots + k_n a^n, \xi \rangle},$$

we get

$$\begin{aligned} \sum_{\mu \geq 0} B_\mu^A(mx) \frac{\xi^\mu}{\mu!} &= m^{-I} \prod_{j=1}^n \frac{\langle a^j, m_j \xi \rangle}{e^{\langle a^j, m_j \xi \rangle} - 1} e^{\langle x, m_j \xi \rangle} \sum_{0 \leq k \leq m-I} e^{\langle k_1 a^1 + \dots + k_n a^n, \xi \rangle} = \\ &= \sum_{0 \leq k \leq m-I} m^{-I} \prod_{j=1}^n \frac{\langle a^j, m_j \xi \rangle}{e^{\langle a^j, m_j \xi \rangle} - 1} e^{\langle mx + k_1 a^1 + \dots + k_n a^n, \xi \rangle}. \end{aligned}$$

Since $k_1 a^1 + \dots + k_n a^n = Ak$, where $A = \begin{pmatrix} a_1^1 & \dots & a_1^n \\ \dots & \dots & \dots \\ a_n^1 & \dots & a_n^n \end{pmatrix}$ and $k = \begin{pmatrix} k_1 \\ \dots \\ k_n \end{pmatrix}$, $\frac{1}{m} = \left(\frac{1}{m_1}, \dots, \frac{1}{m_n} \right)$, we obtain

$$\begin{aligned} \sum_{\mu \geq 0} B_\mu^A(mx) \frac{\xi^\mu}{\mu!} &= \sum_{0 \leq k \leq m-I} m^{-I} \prod_{j=1}^n \frac{\langle a^j, m_j \xi \rangle}{e^{\langle a^j, m_j \xi \rangle} - 1} e^{\langle x + \frac{1}{m} Ak, m \xi \rangle} = \\ &= m^{-I} \sum_{0 \leq k \leq m-I} \sum_{\mu \geq 0} B_\mu^A \left(x + \frac{1}{m} Ak \right) m^\mu \frac{\xi^\mu}{\mu!} = \sum_{\mu \geq 0} \left(m^{\mu-I} \sum_{0 \leq k \leq m-I} B_\mu^A \left(x + \frac{1}{m} Ak \right) \right) \frac{\xi^\mu}{\mu!}. \end{aligned}$$

Equal the coefficients at powers of ξ to obtain the formula. \square

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Многомерный аналог многочленов Бернулли и его свойства

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В работе рассматривается некоторое обобщение чисел и многочленов Бернулли на случай нескольких переменных, а именно определяются числа Бернулли, ассоциированные с рациональным конусом, и соответствующие им многочлены Бернулли. Доказаны некоторые свойства многочленов Бернулли.

Ключевые слова: числа и многочлены Бернулли, производящие функции, оператор Тодда, рациональный конус.