Journal of Siberian Federal University. Mathematics & Physics 2016, 9(3), 347–352

УДК 519.21

The Trigonometry of Harnack Curves

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Received 20.06.2016, received in revised form 20.06.2016, accepted 20.06.2016

Derive an explicit integral formula for the amoeba-to-coamoeba mapping in the case of polynomials that define Harnack curves. As a consequence obtain an exact description of the coamoebas of such polynomials. This formula can be viewed as a generalization of the familiar law of cosines that is used for solving triangles.

Keywords: Harnack curves, amoeba of polynomial, coamoeba of polynomial, Newton polygon, Ronkin function, law of cosines.
DOI: 10.17516/1997-1397-2016-9-3-347-352.

Introduction

When studying Laurent expansions and Mellin transforms of multivariate rational functions, or equivalently Fourier series and Fourier transforms of rational functions in exponentials, one is naturally led to the concepts of amoebas and coamoebas. We recall that the amoeba $\mathscr{A}_f$ of a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is by definition the image of the zero set $f^{-1}(0) \subset (\mathbb{C} \setminus \{0\})^n$ under the logarithmic projection given by $\mathrm{Log}(z) = (\log|z_1|, \ldots, \log|z_n|)$. The coamoeba $\mathscr{A}'_f$ is defined similarly by means of the argument projection $\mathrm{Arg}(z) = (\arg z_1, \ldots, \arg z_n)$.

In this paper we shall only be interested in the bivariate case $n = 2$ and we denote the coordinates by (z, w) rather than (z_1, z_2). We also write $z = \exp(x + i\theta)$ and $w = \exp(y + i\omega)$, so that our amoebas and coamoebas live in real (x, y)-space and (θ, ω)-space respectively.

Suppose we want to determine all Laurent series expansions (centered at the origin) of a rational function $1/f$, where f is a polynomial. It is then not hard to realize that there is one such expansion for each connected component of the amoeba complement $\mathbb{R}^n \setminus \mathscr{A}_f$. Similarly, there is a different Mellin transform of $1/f$ associated with each connected component of the complement of the coamoeba $\mathscr{A}'_f$, see [6]. To each component of the amoeba complement there is associated an integer vector in the Newton polygon Δ_f, called the order of the component, see [1].

Let us look at the simple example $f(z, w) = 1 - z - w$. The amoeba of this polynomial, depicted on the upper left in Fig. 1, has three complement components determined by the explicit inequalities $e^x + e^y < 1$, $1 + e^y < e^x$, and $1 + e^x < e^y$. Their respective orders are $(0, 0)$, $(1, 0)$, and $(0, 1)$. The three corresponding Laurent series expansions of $1/(1 - z - w)$ are given by

$$\sum_{j,k \geqslant 0} a_{jk} z^j w^k, \quad \sum_{j,k \geqslant 0} b_{jk} z^{-1-j-k} w^k, \quad \text{and} \quad \sum_{j,k \geqslant 0} c_{jk} z^j w^{-1-j-k},$$

where $a_{jk} = \binom{j+k}{j}$, $b_{jk} = (-1)^{1+k}\binom{j+k}{j}$, and $c_{jk} = (-1)^{1+j}\binom{j+k}{j}$. The Mellin transform of the same function has the explicit form $\Gamma(s)\Gamma(t)\Gamma(1 - s - t)$, see [6].

The condition $f(z, w) = 1 - z - w = 0$ may be rephrased as the requirement that the three complex numbers 1, $-z$, and $-w$ should form the sides of a (possibly degenerate) triangle. For

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each point (x, y) in the interior of the amoeba there are two points, conjugate to each other, on the complex line $1 - z - w = 0$ that get mapped to (x, y) by the mapping Log. These two points are mapped by the coamoeba shown on the lower left in Fig. 1. The boundary points of $\mathscr{A}_f$ correspond to real values of z and $w = 1 - z$, and such points (z, w) are mapped by Arg to the vertices of the coamoeba $\mathscr{A}'_f$.

The composed mapping $\operatorname{Arg} \circ \operatorname{Log}^{-1}$ from the amoeba to the coamoeba, which is thus well defined up to sign, can easily be written down in this case. It just a question of finding the angles in a triangle with given side lengths 1, e^x, and e^y. Using the classical law of cosines we readily arrive at the formula

$$\operatorname{Arg} \circ \operatorname{Log}^{-1}(x, y) = \pm\Big(\arccos \frac{1 + e^{2x} - e^{2y}}{2e^x}, -\arccos \frac{1 - e^{2x} + e^{2y}}{2e^y}\Big). \tag{1}$$

One can of course also go in the reverse direction, by means of the law of sines, and write down the formula

$$\operatorname{Log} \circ \operatorname{Arg}^{-1}(\theta, \omega) = \Big(\log \frac{\sin \omega}{\sin(\omega - \theta)}, \log \frac{\sin \omega}{\sin(\theta - \omega)}\Big)$$

for the inverse mapping.

It is the purpose of the present note to extend the formula (1) to more general polynomials than affine linear ones. The class of polynomials which we shall be considering consists of those f for which there are at most two points on the complex algebraic curve $f^{-1}(0)$ above each point in the amoeba $\mathscr{A}_f$. For reasons that will presently be explained, we will refer to them as Harnack polynomials.

1. Harnack curves and Ronkin functions

In [8] an optimal upper bound was obtained for the area of a planar amoeba $\mathscr{A}_f$ in terms of the area of the Newton polygon Δ_f. In fact, one always has the inequality

$$\operatorname{Area} \mathscr{A}_f \leqslant \pi^2 \operatorname{Area} \Delta_f,$$

and for any given lattice polygon Δ one can find a polynomial f, with $\Delta_f = \Delta$, such that its amoeba has the maximal area. Polynomials with this property are special in several ways, and the following beautiful result was proved in [5].

Theorem (Mikhalkin-Rullgård). *Provided that Δ_f has non-zero area, the following three conditions are equivalent:*

(i) The amoeba $\mathscr{A}_f$ is of maximal area.

(ii) The mapping $\operatorname{Log} : f^{-1}(0) \to \mathbb{R}^2$ is at most two-to-one, and there are constants $a, b, c \in \mathbb{C}^*$ such that $af(bz, cw)$ has real coefficients.

(iii) There are constants $a, b, c \in \mathbb{C}^*$ such that $af(bz, cw)$ has real coefficients, and the corresponds real algebraic curve is a Harnack curve for the polygon Δ_f.

Polynomials having the above properties will be called Harnack polynomials. Condition (ii) can be thought of as saying that topologically the amoeba is obtained from the complex curve $f^{-1}(0)$ by flat-ironing it so well that there are no wrinkles at all, except for the fold along the boundary. The exact definition of a Harnack curve for a given polygon is somewhat involved, see [5] for the details. Essentially it is a real algebraic curve in $(\mathbb{R}^*)^2$ with one bounded component (possibly reduced to a point) associated with each interior lattice point of the polygon, plus some unbounded curve pieces coming from a single curve component in the corresponding toric compactification that cyclically intersects the toric divisors associated with the edges of the polygon. Under the mapping Log the Harnack curve gets sent to the boundary $\partial\mathscr{A}_f$ of the amoeba of the corresponding complex curve $f^{-1}(0)$. The bounded components of the Harnack curve are mapped to the boundaries of the bounded holes of the amoeba, whereas the unbounded pieces are mapped to the outer boundary pieces of the amoeba.

Examples. *The following polynomials are special instances of Harnack polynomials of degree one, two, and three respectively:*

$$f(z,w) = 1 - z - w; \quad f(z,w) = 1 - z - w - zw; \quad f(z,w) = 1 - 4z - w + z^2 - 6zw - zw^2.$$

The amoebas and the coamoebas of these polynomials are shown in Fig. 1 below. It may be instructive for the reader to draw the corresponding Newton polygons and to visually examine what the order of each component of the amoebas should be. The appearance of the Newton polygons will also allow the reader to anticipate our result below, Corollary 1, on the structure of the coamoebas of Harnack polynomials.

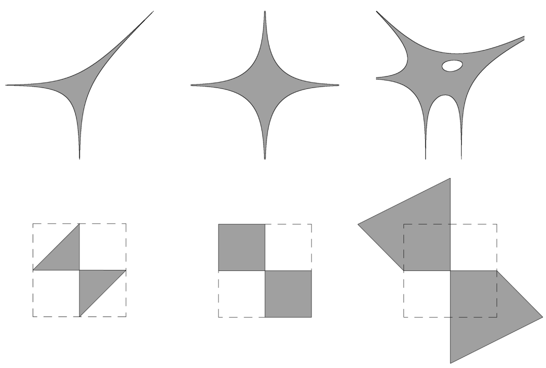

Fig. 1. The amoebas (top row) and the coamoebas (bottom row) of the three example polynomials. Each half of the coamoeba has the same area as the corresponding amoeba: $\pi^2/2$, π^2, and $5\pi^2/2$. The dashed square is centered at the origin and has side length 2π

Definition 1. *We say that a Harnack polynomial f is normalized if it has real coefficients and if the following two conditions are satisfied:*

- $(0,0)$ *is a vertex of the Newton polygon* Δ_f*;*
- *the piece of $\partial\mathscr{A}_f$ that bounds the complement component of order (0,0) is included in the image* $\mathrm{Log}(f^{-1}(0) \cap \mathbb{R}^2_+)$.

The first of these two conditions will be fulfilled after multiplying f by a monomial, which does not affect the (co)amoeba, but adds a fixed integer vector to the order of each complement component. The second property may then be achieved by changing the sign of z and/or w, which does not affect the amoeba, but translates the coamoeba by π in the direction of θ and/or ω.

Given a Harnack polynomial f we shall be dealing with the amoeba-to-coamoeba mapping $\mathrm{Arg} \circ \mathrm{Log}^{-1}$ that sends each point $(x,y) \in \mathscr{A}_f$ to the coordinatewise arguments $(\arg z, \arg w)$ of two points on the complex curve $f^{-1}(0)$ whose absolute values $(|z|,|w|)$ are (e^x, e^y). Since we may, and will, assume that f has real coefficients, the two points on $f^{-1}(0)$ will be conjugate to each other and hance the two values of the amoeba-to-coamoeba mapping will just differ by a sign.

Our aim is to generalize the explicit formula (1) for the mapping $\mathrm{Arg} \circ \mathrm{Log}^{-1}$ and to this end it is beneficial to first express it as an integral formula. This may be done with the help of the so-called Ronkin function, which we now define.

Definition 2. *The Ronkin function $N_f : \mathbb{R}^2 \to \mathbb{R}$ associated with a polynomial p is given by the mean value integral*

$$N_p(x,y) = \frac{1}{(2\pi i)^2} \int\limits_{T_{x,y}} \log\big|(z,w)\big| \frac{dzdw}{dz} = \frac{1}{(2\pi)^2} \int\limits_{[-\pi,\pi]^2} \log\big|(e^{x+i\theta}, e^{y+i\omega})\big| d\theta d\omega,$$

where $T_{x,y}$ denotes the real torus defined by $(|z|,|w|) = (e^x, e^y)$.

The Ronkin function is convex and it is affine linear precisely outside the amoeba $\mathscr{A}_f$. In any component of $\mathbb{R}^2 \setminus \mathscr{A}_f$ its gradient $(\partial N_f/\partial x, \partial N_f/\partial y)$ is thus constant. It is actually integer-valued and equal to the order of the complement component in question, see [8]. The derivatives of the Ronkin function can also be express as integrals

$$\frac{\partial N_f}{\partial x}(x,y) = \frac{1}{(2\pi i)^2} \int\limits_{T_{x,y}} \frac{f'_z(z,w)dzdw}{f(z,w)w} = \frac{1}{(2\pi)^2} \int\limits_{[-\pi,\pi]^2} \frac{e^{x+i\theta} f'_z(e^{x+i\theta}, e^{y+i\omega})d\theta d\omega}{f(e^{x+i\theta}, e^{y+i\omega})}, \tag{2}$$

and similarly for $\partial N_f/\partial y$.

2. The generalized law of cosines

From our discussion in the previous section we know that for a Harnack polynomial f with real coefficients the amoeba-to-coamoeba mapping is well-defined up to sing. In other words, as soon as the absolute values of the monomials of f are given, the condition $f(z,w) = 0$ that their sum should vanish, automatically determines their arguments. This generalizes the familiar fact that the side lengths of a triangle determine its angles. Our main result provides an explicit formula for computing the arguments, when the absolute values are given. For convenience we formulate it for normalized Harnack polynomials, but this is not essential.

Theorem 1. *If f is a normalized Harnack polynomial, then the amoeba-to-coamoeba mapping is given by*

$$\mathrm{Arg} \circ \mathrm{Log}^{-1} = \pm\Big(\pi \frac{\partial N_f}{\partial y}(x,y), -\pi \frac{\partial N_f}{\partial x}(x,y)\Big).$$

Proof. Since both *Log* and *Arg* are real-analytic mapping, it is clear that so is the amoeba-to-coamoeba mapping $\mathrm{Arg} \circ \mathrm{Log}^{-1}$. Moreover, the derivatives of the Ronkin function also depend analytically on x and y. It will therefore be enough to prove the identity locally, that is, for (x,y) in some small open subset of $\mathscr{A}_f$. To be specific, we choose a point (x_1,y_1) on the piece of $\partial \mathscr{A}_f$ that bounds the complement component of order $(0,0)$, and we assume that the boundary curve has a finite non-zero slope at (x_1,y_1). Let $(z_1,w_1) = \mathrm{Log}^{-1}(x,y)$ be the corresponding unique point on the complex curve $f^{-1}(0)$. In fact, the assumed normalization of f implies that $\mathrm{Arg}(z_1,z_2) = 0$, which means that $z_1 = e^{x_1}$ and $w_2 = e^{y_1}$. since the normal direction to the boundary at (x_1,y_1) is given by $[z_1 f'_z(z_1,w_1) : w_1 f'_w(z_1,w_1)]$, we see that neither f'_z nor f'_w vanishes at z_1, w_1, so the implicit function theorem provides us with locally defined complex analytic functions $z(w)$ and $w(z)$ that are inverse to each other and satisfy $f(z(w),w) = 0 = f(z,w(z))$.

Now let (x_0, y_0) be an interior point of the amoeba $\mathscr{A}_f$ very close to the boundary point (x_1, y_1). We shall show that the claimed identity holds at (x_0, y_0), and we begin by computing $\partial N_f/\partial y$. Consider the auxiliary function

$$\theta \mapsto \frac{1}{2\pi i} \int\limits_{|w|=e^{y_0}} \frac{f'_w(e^{x_0+i\theta}, w)dw}{f(e^{x_0+i\theta}, w)}, \quad \theta \in [-\pi, \pi]. \tag{3}$$

Since f is a Harnack polynomial, and (x_0, y_0) is an interior point of $\mathscr{A}_f$, there are precisely two values of θ for which the denominator vanishes, and since f has real coefficients these points are located symmetrically around the origin. Let us denote them by $\pm\theta_0$ with $\theta_0 > 0$. In fact, since (x_0, y_0) is close to the boundary point (x_1, y_1), the value θ_0 is close to zero. Since we are close to the $(0, 0)$ component of the amoeba complement the polynomial $w \mapsto f(e^{x_0+i\theta}, w)$ has one root inside the circle $|w| = e^{y_0}$ for $|\theta| < \theta_0$, and no such root for $|\theta| > \theta_0$. Elementary residue calculus then shows that the function (3) is a simple step function equal to 1 for $|\theta| < \theta_0$ and equal 0 for $|\theta| > \theta_0$. Recalling the formula (2), we see that the derivative $\partial N_f/\partial y$ is obtained as the mean value of the function (3), that is, we arrive at the formula $\partial N_f/\partial y(x_0, y_0) = 2\theta_0/(2\pi) = \theta_0/\pi$. By analogous reasoning one obtains $\partial N_f/\partial x(x_0, y_0) = \omega_0/\pi$, where $\omega_0 > 0$ and $\pm\omega_0$ are the two values of ω for which the polynomial $z \mapsto f(z, e^{y_0+i\omega})$ has a root on the circle $|z| = e^{x_0}$.

We thus know that the coordinates of $\operatorname{Arg} \circ \operatorname{Log}^{-1}(x_0, y_0)$ are equal to

$$\pm\theta_0 = \pm\pi\partial N_f/\partial y(x_0, y_0), \quad \textit{and} \quad \pm\omega_0 = \pm\pi\partial N_f/\partial x(x_0, y_0),$$

so what remains is to combine the sings properly. More precisely, we must show that the two values of $\operatorname{Arg} \circ \operatorname{Log}^{-1}(x_0, y_0)$ are $\pm(\theta_0, -\omega_0)$. The fact that the argument θ_0 goes together with $-\omega_0$ is a consequence of the conformality of the local holomorphic mapping $z(w)$. Indeed, the positively oriented circle $|z| = e^{x_0}$ and the oriented curve $w \mapsto z(e^{y_0+i\omega})$ intersect at the points $z = e^{x_0 \pm i\theta_0}$, see Fig. 2 below. Their biholomorphic image curves in the w-plane have the intersection points $w = e^{y_0 \pm i\omega_0}$. Since holomorphic functions preserve orientation, we find by inspection in Fig. 2 that $e^{x_0 \pm i\theta_0}$ must be mapped to $e^{y_0 \mp i\omega_0}$ as claimed. The theorem follows. $\square$

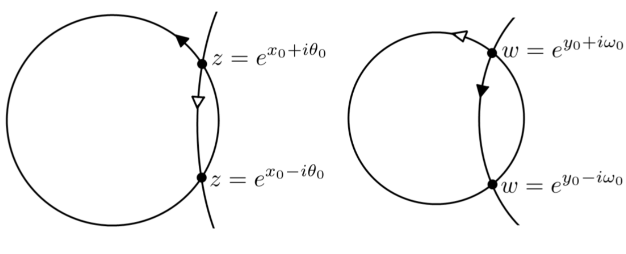

Fig. 2. *The z-plane (left) with the oriented circle $|z| = e^{x_0}$ and part of the curve $\omega \mapsto z(e^{y_0+i\omega})$, and the w-plane (right) with the oriented circle $|w| = e^{y_0}$ and part of the curve $\theta \mapsto w(e^{x_0+i\theta})$.*

Having established the connection between the gradient of the Ronkin function and the amoeba-to-coamoeba mapping, we can now give a precise description of the coamoeba of a normalized Harnack polynomial f. It turns out that it consists of dilated and rotated images of the Newton polygon Δ_f.

Corollary 1. *The coamoeba of a normalized Harnack polynomial f is given by*

$$\mathscr{A}'_f = i\pi\Delta_f \cup -i\pi\Delta_f,$$

where the factor $\pm i$ indicates a rotation around the origin through the angle $\pm\pi/2$. The vertices of the polygons, but not the remaining parts of their edges, belong to the coamoeba.

Remark. The fact that (each half of) the coamoeba has the same area $\pi^2 \operatorname{Area} \Delta_f$ as the amoeba can also be seen by considering the Jacobian of the amoeba-to-coamoeba mapping. It is not hard to prove that this Jacobian is in fact $\equiv 1$, so that the mapping is even locally area-preserving. See [7] for a proof in the case of a linear polynomial. It can easily be modified to give the general statement.

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Тригонометрия гарнаковских кривых

Микаэл Пассаре

Выводится явная интегральная формула отображения амёбы в коамёбу для случая полиномов, определяющих кривые Гарнака. Как следствие, получается точное описание коамёб таких полиномов. Эта формула может быть рассмотрена как обобщение известной теоремы косинусов, которая используется для решения треугольников.

Ключевые слова: гарнаковские кривые, амёба полинома, коамёба полинома, многоугольник Ньютона, функция Ронкина, теорема косинусов.