Rigidity Conditions for the Boundaries of Submanifolds in a Riemannian Manifold

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Developing A.D. Aleksandrov's ideas, the first author proposed the following approach to study of rigidity problems for the boundary of a C^0-submanifold in a smooth Riemannian manifold. Let Y_1 be a two-dimensional compact connected C^0-submanifold with non-empty boundary in some smooth two-dimensional Riemannian manifold (X, g) without boundary. Let us consider the intrinsic metric (the infimum of the lengths of paths, connecting a pair of points".) of the interior $\operatorname{Int} Y_1$ of Y_1, and extend it by continuity (operation $\underline{\lim}$) to the boundary points of ∂Y_1. In this paper the rigidity conditions are studied, i.e., when the constructed limiting metric defines ∂Y_1 up to isometry of ambient space (X, g). We also consider the case $\dim Y_j = \dim X = n$, $n > 2$.

1. Introduction: unique determination of surfaces by their relative metrics on boundaries

A classical theorem says (see [3]): *If two bounded closed convex surfaces in the three-dimensional Euclidean space are isometric in their intrinsic metrics then they are equal, i.e., they can be matched by a motion.*

The problems of unique determination of closed convex surfaces by their intrinsic metrics goes back to the result of Cauchy, obtained already in 1813, that *any closed convex polyhedrons P_1 and P_2 (in the three-dimensional Euclidean space) that are equally composed of congruent faces are equal.* Since then this problem has been studied by many people for about 140 years (for example, by Minkowski, Hilbert, Weyl, Blaschke, Cohn-Vossen, Aleksandrov, Pogorelov and other prominent mathematicians (see, for instance, the historical overview in [3], Chapter 3); finally, its complete solution, which is just the theorem we have cited at the beginning, was obtained by A. V. Pogorelov. For generalizations of Pogorelov's result to higher dimensions, see [4].

In [5], we proposed a new approach to the problem of unique determination of surfaces, which enabled us to substantially enlarge the framework of the problem. The following model situation illustrates the essence of this approach fairly well:

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Let U_1 and U_2 be two domains (i.e., open connected sets) in the real n-dimensional Euclidean space $\mathbb{R}^n$ whose closures $\operatorname{cl} U_j$, where $j = 1, 2$, are Lipschitz manifolds (such that $\partial(\operatorname{cl} U_j) = \partial U_j \neq \varnothing$, where ∂E is the boundary of E in $\mathbb{R}^n$). Assume also that the boundaries ∂U_1 and ∂U_2 of these domains, which coincide with the boundaries of the manifolds $\operatorname{cl} U_1$ and $\operatorname{cl} U_2$, are isometric with respect to their relative metrics $\rho_{\partial U_j, U_j}$ $(j = 1, 2)$, i.e., with respect to the metrics that are the restrictions to the boundaries ∂U_j of the extensions $\rho_{\operatorname{cl} U_j}$ (by continuity) of the intrinsic metrics ρ_{U_j} of the domains U_j to $\operatorname{cl} U_j$. The following natural question arises: *Under which additional conditions are the domains U_1 and U_2 themselves isometric (in the Euclidean metric)?* In particular, the natural character of this problem is determined by the circumstance that the problem of unique determination of closed convex surfaces mentioned at the beginning of the article is its most important particular case. Indeed, assume that S_1 and S_2 are two closed convex surfaces in $\mathbb{R}^3$, i.e., they are the boundaries of two bounded convex domains $G_1 \subset \mathbb{R}^3$ and $G_2 \subset \mathbb{R}^3$. Let $U_j = \mathbb{R}^3 \setminus \operatorname{cl} G_j$ be the complement of the closure $\operatorname{cl} G_j$ of the domain G_j, $j = 1, 2$. Then the intrinsic metrics on the surfaces $S_1 = \partial U_1$ and $S_2 = \partial U_2$ coincide with the relative metrics $\rho_{\partial U_1, U_1}$ and $\rho_{\partial U_2, U_2}$ on the boundaries of the domains U_1 and U_2, and thus the problem of unique determination of closed convex surfaces by their intrinsic metrics is indeed a particular case of the problem of unique determination of domains by the relative metrics on their boundaries.

The generalization of the problem of the unique determination of surfaces ensuing from a new approach suggested in [5] manifests itself in the fact that the unique determination of domains by the relative metrics on their boundaries holds not only when their complements are bounded convex sets but, for example, also in the following cases.

The domain U_1 is bounded and convex and the domain U_2 is arbitrary (A. P. Kopylov [5]).

The domain U_1 is strictly convex and the domain U_2 is arbitrary (A. D. Aleksandrov (see [6])).

The domains U_1, U_2 are bounded and their boundaries are smooth (V. A. Aleksandrov [6]).

The domains U_1 and U_2 have nonempty bounded complements, while their boundaries are $(n-1)$-dimensional connected C^1-manifolds without boundary, $n > 2$ (V. A. Aleksandrov [7]).

In the papers [8–10], M. V. Korobkov (in particular) obtained a complete solution to the problem of unique determination of a plane (space) domain in the class of all plane (space) domains by the relative metric on its boundary.

In this connection, there appears the following question: Is it possible to construct an analog of the theory of rigidity of surfaces in Euclidean spaces in the general case of the boundaries of submanifolds in Riemannian manifolds?

Our article is devoted to a detailed discussion of this question. In it, we in particular obtain new results concerning rigidity problems for the boundaries of n-dimensional connected submanifolds with boundary in n-dimensional smooth connected Riemannian manifolds without boundary $(n \geqslant 2)$.

In what follows, all paths $\gamma : [\alpha, \beta] \to \mathbb{R}^n$, where $\alpha, \beta \in \mathbb{R}$, are assumed continuous and non-constant, and $l(\gamma)$ means the length of a path γ.

2. Rigidity problems and intrinsic geometry of submanifolds in riemannian manifolds

Let (X, g) be an n-dimensional smooth connected Riemannian manifold without boundary and let Y be an n-dimensional compact connected C^0-submanifold in X with nonempty boundary ∂Y $(n \geqslant 2)$.

A classical object of investigations (see, for example, [11]) is given by the intrinsic metric $\rho_{\partial Y}$ on the hypersurface ∂Y defined for $x, y \in \partial Y$ as the infimum of the lengths of curves $\nu \subset \partial Y$ joining x and y. In the recent decades, an alternative approach arose in the rigidity theory for submanifolds of Riemannian manifolds (see, for instance, the recent articles [1, 10], and [2], which

also contain a historical survey of works on the topic). In accordance with this approach, the metric on ∂Y is induced by the intrinsic metric of the interior $\operatorname{Int} Y$ of the submanifold Y.

Namely, suppose that Y satisfies the following condition:

(i) if $x, y \in Y$, then

$$\rho_Y(x, y) = \liminf_{x' \to x, y' \to y; x', y' \in \operatorname{Int} Y} \{\inf[l(\gamma_{x', y', \operatorname{Int} Y})]\} < \infty, \tag{2.1}$$

where $\inf[l(\gamma_{x', y', \operatorname{Int} Y})]$ is the infimum of the lengths $l(\gamma_{x', y', \operatorname{Int} Y})$ of smooth paths $\gamma_{x', y', \operatorname{Int} Y} : [0, 1] \to \operatorname{Int} Y$ joining x' and y' in the interior $\operatorname{Int} Y$ of Y.

Remark 2.1. Easy examples show that if X is an n-dimensional connected smooth Riemannian manifold without boundary then an n-dimensional compact connected C^0-submanifold in X with nonempty boundary may fail to satisfy condition (i). For $n = 2$, we have the following counterexample:

Let (X, g) be the space $\mathbb{R}^2$ equipped with the Euclidean metric and let Y be a closed Jordan domain in $\mathbb{R}^2$ whose boundary is the union of the singleton $\{0\}$ consisting of the origin 0, the segment $\{(1 - t)(e_1 + 2e_2) + t(e_1 + e_2) : 0 \leqslant t \leqslant 1\}$, and of the segments of the following four types:

$$\left\{ \frac{(1 - t)(e_1 + e_2)}{n} + \frac{te_1}{n + 1} : 0 \leqslant t \leqslant 1 \right\} \quad (n = 1, 2, \dots);$$

$$\left\{ \frac{e_1 + (1 - t)e_2}{n} : 0 \leqslant t \leqslant 1 \right\} \quad (n = 2, 3, \dots);$$

$$\left\{ \frac{(1 - t)(e_1 + 2e_2)}{n} + \frac{2t(2e_1 + e_2)}{4n + 3} : 0 \leqslant t \leqslant 1 \right\} \quad (n = 1, 2, \dots);$$

$$\left\{ \frac{(1 - t)(e_1 + 2e_2)}{n + 1} + \frac{2t(2e_1 + e_2)}{4n + 3} : 0 \leqslant t \leqslant 1 \right\} \quad (n = 1, 2, \dots).$$

Here e_1, e_2 is the canonical basis in $\mathbb{R}^2$. By the construction of Y, we have $\rho_Y(0, E) = \infty$ for every $E \in Y \setminus \{0\}$ (see Fig. 1).

Remark 2.2. Note that if $X = \mathbb{R}^n$ and U is a domain in $\mathbb{R}^n$ whose closure $Y = \operatorname{cl} U$ is a Lipschitz manifold (such that $\partial(\operatorname{cl} U) = \partial U \neq \varnothing$), then $\rho_{\partial U, U}(x, y) = \rho_Y(x, y)$ $(x, y \in \partial U)$ and Y satisfies (i). Hence, this example is an important particular case of submanifolds Y in a Riemannian manifold X satisfying (i).

To prove our rigidity results for boundaries of submanifolds in a Riemannian manifold (see Sec. 3.), we first need to study the properties of the intrinsic geometry of these submanifolds.

One of the main results of this section is as follows:

Theorem 2.1. *Let $n = 2$. Then, under condition* (i)*, the function ρ_Y defined by* (2.1) *is a metric on Y.*

Proof. It suffices to prove that ρ_Y satisfies the triangle inequality. Let A, O, and D be three points on the boundary of Y (note that this case is basic because the other cases are simpler). Consider $\varepsilon > 0$ and assume that $\gamma_{A_\varepsilon O_\varepsilon^1} : [0, 1] \to \operatorname{Int} Y$ and $\gamma_{O_\varepsilon^2 D_\varepsilon} : [2, 3] \to \operatorname{Int} Y$ are smooth paths with the endpoints $A_\varepsilon = \gamma_{A_\varepsilon O_\varepsilon^1}(0)$, $O_\varepsilon^1 = \gamma_{A_\varepsilon O_\varepsilon^1}(1)$ and $D_\varepsilon = \gamma_{O_\varepsilon^2 D_\varepsilon}(3)$, $O_\varepsilon^2 = \gamma_{O_\varepsilon^2 D_\varepsilon}(2)$ satisfying the conditions $\rho_X(A_\varepsilon, A) \leqslant \varepsilon$, $\rho_X(D_\varepsilon, D) \leqslant \varepsilon$, $\rho_X(O_\varepsilon^j, O) \leqslant \varepsilon$ $(j = 1; 2)$, $|l(\gamma_{A_\varepsilon O_\varepsilon^1}) - \rho_Y(A, O)| \leqslant \varepsilon$, and $|l(\gamma_{O_\varepsilon^2 D_\varepsilon}) - \rho_Y(O, D)| \leqslant \varepsilon$. Let (U, h) be a chart of the manifold X such that U is an open neighborhood of the point O in X, $h(U)$ is the unit disk $B(0, 1) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ in $\mathbb{R}^2$, and $h(O) = 0$ $(0 = (0, 0)$ is the origin in $\mathbb{R}^2)$; moreover $h : U \to h(U)$ is a diffeomorphism having the following property: there exists a chart (Z, ψ) of Y with $\psi(O) = 0$, $A, D \in U \setminus \operatorname{cl}_X Z$ $(\operatorname{cl}_X Z$ is the closure of Z in the space $(X, g))$ and $Z = \widetilde{U} \cap Y$ is the intersection of an open neighborhood $\widetilde{U}$ $(\subset U)$ of O in X and Y whose image $\psi(Z)$ under ψ is the half-disk $B_+(0, 1) = \{(x_1, x_2) \in B(0, 1) : x_1 \geqslant 0\}$. Suppose that σ_r is an arc of the circle $\partial B(0, r)$ which is a connected component of the set $V \cap \partial B(0, r)$, where
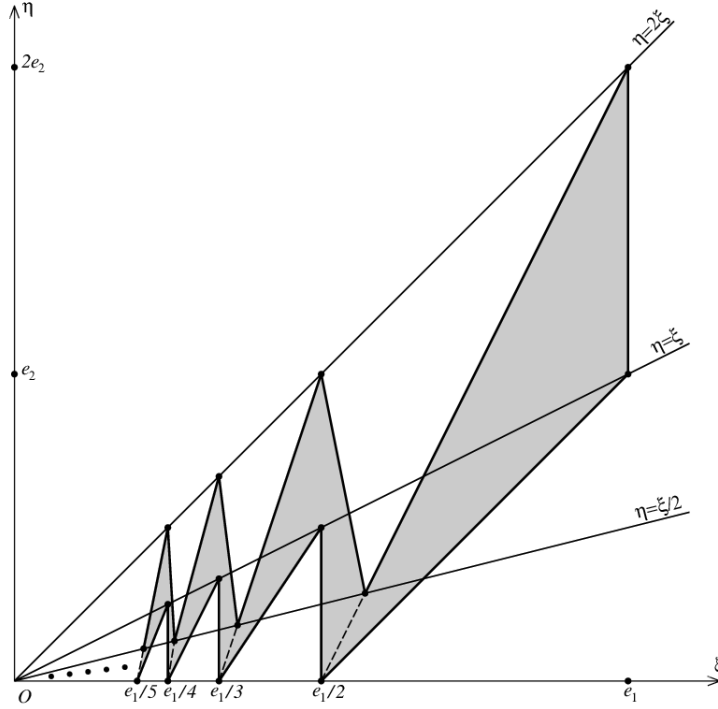
Fig. 1. An example of 2-dimensional compact connected C^0-submanifold with nonempty boundary which does not satisfy condition (i)

$V = h(Z)$ and $0 < r < r^* = \min\{|h(\psi^{-1}(x_1, x_2))| : x_1^2 + x_2^2 = 1/4, \, x_1 \geqslant 0\}$. Among these components, there is at least one (preserve the notation σ_r for it) whose ends belong to the sets $h(\psi^{-1}(\{-te_2 : 0 < t < 1\}))$ and $h(\psi^{-1}(\{te_2 : 0 < t < 1\}))$ respectively. Otherwise, the closure of the connected component of the set $V \cap B(0, r)$ whose boundary contains the origin would contain a point belonging to the arc $\{e^{i\theta}/2 : |\theta| \leqslant \pi/2\}$ (here we make use of the complex notation $z = re^{i\theta}$ for points $z \in \mathbb{R}^2 \, (= \mathbb{C})$). But this is impossible. Therefore, the above-mentioned arc σ_r exists.

It is easy to check that if ε is sufficiently small then the images of the paths $h \circ \gamma_{A_\varepsilon O_\varepsilon^1}$ and $h \circ \gamma_{O_\varepsilon^2 D_\varepsilon}$ also intersect the arc σ_r, i.e., there are $t_1 \in]0, 1[$, $t_2 \in]2, 3[$ such that $\gamma_{A_\varepsilon O_\varepsilon^1}(t_1) = x^1 \in Z$, $\gamma_{O_\varepsilon^2 D_\varepsilon}(t_2) = x^2 \in Z$ and $h(x^j) \in \sigma_r$, $j = 1, 2$. Let $\gamma_r : [t_1, t_2] \to \sigma_r$ be a smooth parametrization of the corresponding subarc of σ_r, i.e., $\gamma_r(t_j) = h(x^j)$, $j = 1, 2$. Now we can define a mapping $\widetilde{\gamma}_\varepsilon : [0, 3] \to \operatorname{Int} Y$ by setting

$$\widetilde{\gamma}_\varepsilon(t) = \begin{cases} \gamma_{A_\varepsilon O_\varepsilon^1}(t), & t \in [0, t_1]; \\ h^{-1}(\gamma_r(t)), & t \in]t_1, t_2[; \\ \gamma_{O_\varepsilon^2 D_\varepsilon}(t), & t \in [t_2, 3]. \end{cases}$$

By construction, $\widetilde{\gamma}_\varepsilon$ is a piecewise smooth path joining the points $A_\varepsilon = \widetilde{\gamma}_\varepsilon(0)$, $D_\varepsilon = \widetilde{\gamma}_\varepsilon(3)$ in $\operatorname{Int} Y$; moreover,

$$l(\widetilde{\gamma}_\varepsilon) \leqslant l(\gamma_{A_\varepsilon O_\varepsilon^1}) + l(\gamma_{O_\varepsilon^2 D_\varepsilon}) + l(h^{-1}(\sigma_r)).$$

By an appropriate choice of $\varepsilon > 0$, we can make $r > 0$ arbitrarily small, and since a piecewise smooth path can be approximated by smooth paths, we have $\rho_Y(A, D) \leqslant \rho_Y(A, O) + \rho_Y(O, D)$.

$\square$

In connection with Theorem 2.1, there appears a natural question: Are there analogs of this theorem for $n \geqslant 3$? The following Theorem 2.2 answers this question in the negative:

Theorem 2.2. *If $n \geqslant 3$ then there exists an n-dimensional compact connected C^0-manifold $Y \subset \mathbb{R}^n$ with nonempty boundary ∂Y such that condition* (i) *(where now $X = \mathbb{R}^n$) is fulfilled for Y but the function ρ_Y in this condition is not a metric on Y.*

Proof. It suffices to consider the case of $n = 3$. Suppose that A, O, D are points in $\mathbb{R}^3$, O is the origin in $\mathbb{R}^3$, $|A| = |D| = 1$, and the angle between the segments OA and OD is equal to $\frac{\pi}{6}$.

The manifold Y will be constructed so that $O \in \partial Y$, and $]O, A] \subset \operatorname{Int} Y$, $]O, D] \subset \operatorname{Int} Y$. Under these conditions, $\rho_Y(O, A) = \rho_Y(O, D) = 1$. However, the boundary of Y will create "obstacles" between A and D such that the length of any curve joining A and D in $\operatorname{Int} Y$ will be greater than $\frac{12}{5}$ (this means the violation of the triangle inequality for ρ_Y).

Consider a countable collection of mutually disjoint segments $\{I_j^k\}_{j \in \mathbb{N},\, k=1,\ldots,k_j}$ lying in the interior of the triangle $6\Delta AOD$ (which is obtained from the original triangle ΔAOD by dilation with coefficient 6) with the following properties:

($*$) every segment $I_j^k = [x_j^k, y_j^k]$ lies on a ray starting at the origin, $y_j^k = 11 x_j^k$, and $|x_j^k| = 2^{-j}$;

($**$) any curve γ with ends A, D whose interior points lie in the interior of the triangle $4\Delta AOD$ and belong to no segment I_j^k, satisfies the estimate $l(\gamma) \geqslant 6$.

The existence of such a family of segments is certain: the segments of the family must be situated chequerwise so that any curve disjoint from them be sawtooth, with the total length of its "teeth" greater than 6 (it can clearly be made greater than any prescribed positive number). However, below we exactly describe the construction.

It is easy to include the above-indicated family of segments in the boundary ∂Y of Y. Thus, it creates a desired "obstacle" to joining A and D in the plane of ΔAOD. But it makes no obstacle to joining A and D in the space. The simplest way to create such a space obstacle is as follows: Rotate each segment I_j^k along a spiral around the axis OA. Make the number of coils so large that the length of this spiral be large and its pitch (i.e., the distance between the origin and the end of a coil) be sufficiently small. Then the set S_j^k obtained as the result of the rotation of the segment I_j^k is diffeomorphic to a plane rectangle, and it lies in a small neighborhood of the cone of revolution with axis AO containing the segment I_j^k. The last circumstance guarantees that the sets S_j^k are disjoint as before, and so (as above) it is easy to include them in the boundary ∂Y but, due to the properties of the I_j^k's and a large number of coils of the spirals S_j^k, any curve joining A, D and disjoint from each S_j^k has length $\geqslant \frac{12}{5}$.

We turn to an exact description of the constructions used. First describe the construction of the family of segments I_j^k. They are chosen on the basis of the following observation:

Let $\gamma : [0, 1] \to 4\Delta AOD$ be any curve with ends $\gamma(0) = A$, $\gamma(1) = D$ whose interior points lie in the interior of the triangle $4\Delta AOD$. For $j \in \mathbb{N}$, put $R_j = \{x \in 4\Delta AOD : |x| \in [8 \cdot 2^{-j}, 4 \cdot 2^{-j}]\}$. It is clear that

$$4\Delta AOD \setminus \{O\} = \cup_{j \in \mathbb{N}} R_j.$$

Introduce the polar system of coordinates on the plane of the triangle ΔAOD with center O such that the coordinates of the points A, D are $r = 1$, $\varphi = 0$ and $r = 1$, $\varphi = \frac{\pi}{6}$, respectively. Given a point $x \in 6\Delta AOD$, let φ_x be the angular coordinate of x in $\left[0, \frac{\pi}{6}\right]$. Let $\Phi_j = \{\varphi_{\gamma(t)} : \gamma(t) \in R_j\}$. Obviously, there is $j_0 \in \mathbb{N}$ such that

$$\mathcal{H}^1(\Phi_{j_0}) \geqslant 2^{-j_0} \frac{\pi}{6}, \tag{2.2}$$

where $\mathcal{H}^1$ is the Hausdorff 1-measure. This means that, while in the layer R_{j_0}, the curve γ covers the angular distance $\geqslant 2^{-j_0} \frac{\pi}{6}$. The segments I_j^k must be chosen such that (2.2) together with

the condition
$$\gamma(t) \cap I_j^k = \varnothing \quad \forall t \in [0,1], \ \forall j \in \mathbb{N}, \ \forall k \in \{1, \ldots, k_j\}$$

give the desired estimate $l(\gamma) \geqslant 6$. To this end, it suffices to take $k_j = [(2\pi)^j]$ (the integral part of $(2\pi)^j$) and
$$I_j^k = \left\{ x \in 6\Delta AOD : \varphi_x = k(2\pi)^{-j} \frac{\pi}{6}, \ |x| \in [11 \cdot 2^{-j}, 2^{-j}] \right\},$$

$k = 1, \ldots, k_j$. Indeed, under this choice of the I_j^k's, estimate (2.2) implies that γ must intersect at least $(2\pi)^{j_0} 2^{-j_0} = \pi^{j_0} > 3^{j_0}$ of the figures
$$U_k = \left\{ x \in R_{j_0} : \varphi_x \in \left(k(2\pi)^{-j_0} \frac{\pi}{6}, (k+1)(2\pi)^{-j_0} \frac{\pi}{6} \right) \right\}.$$

Since these figures are separated by the segments $I_{j_0}^k$ in the layer R_{j_0}, the curve γ must be disjoint from them each time in passing from one figure to another. The number of these passages must be at least $3^{j_0} - 1$, and a fragment of γ of length at least $2 \cdot 3 \cdot 2^{-j_0}$ is required for each passage (because the ends of the segments $I_{j_0}^k$ go beyond the boundary of the layer R_{j_0} containing the figures U_k at distance $3 \cdot 2^{-j_0}$). Thus, for all these passages, a section of γ is spent of length at least
$$6 \cdot 2^{-j_0}(3^{j_0} - 1) \geqslant 6.$$

Hence, the construction of the segments I_j^k satisfying $(*)$–$(**)$ is finished.

Let us now describe the construction of the above-mentioned space spirals.

For $x \in \mathbb{R}^3$, denote by Π_x the plane that passes through x and is perpendicular to the segment OA. On $\Pi_{x_j^k}$, consider the polar coordinates (ρ, ψ) with origin at the point of intersection of $\Pi_{x_j^k}$ and $[O, A]$ (in this system, the point x_j^k has coordinates $\rho = \rho_j^k$, $\psi = 0$). Suppose that a point $x(\psi) \in \Pi_{x_j^k}$ moves along an Archimedean spiral, namely, the polar coordinates of the point $x(\psi)$ are $\rho(\psi) = \rho_j^k - \varepsilon_j \psi$, $\psi \in [0, 2\pi M_j]$, where ε_j is a small parameter to be specified below, and $M_j \in \mathbb{N}$ is chosen so large that the length of any curve passing between all coils of the spiral is at least 10.

Describe the choice of M_j more exactly. To this end, consider the points $x(2\pi)$, $x(2\pi(M_j-1))$, $x(2\pi M_j)$, which are the ends of the first, penultimate, and last coils of the spiral respectively (with $x(0) = x_j^k$ taken as the starting point of the spiral). Then M_j is chosen so large that the following condition hold:

$(*_1)$ *The length of any curve on the plane $\Pi_{x_j^k}$, joining the segments $[x_j^k, x(2\pi)]$ and $[x(2\pi(M_j - 1)), x(2\pi M_j)]$ and disjoint from the spiral $\{x(\psi) : \psi \in [0, 2\pi M_j]\}$, is at least 10.*

Figuratively speaking, the constructed spiral bounds a "labyrinth", the mentioned segments are the entrance to and the exit from this labyrinth, and thus any path through the labyrinth has length $\geqslant 10$.

Now, start rotating the entire segment I_j^k in space along the above-mentioned spiral, i.e., assume that $I_j^k(\psi) = \{y = \lambda x(\psi) : \lambda \in [1, 11]\}$. Thus, the segment $I_j^k(\psi)$ lies on the ray joining O with $x(\psi)$ and has the same length as the original segment $I_j^k = I_j^k(0)$. Define the surface $S_j^k = \cup_{\psi \in [0, 2\pi M_j]} I_j^k(\psi)$. This surface is diffeomorphic to a plane rectangle (strip). Taking $\varepsilon_j > 0$ sufficiently small, we may assume without loss of generality that $2\pi M_j \varepsilon_j$ is substantially less than ρ_j^k; moreover, that the surfaces S_j^k are mutually disjoint (obviously, the smallness of ε_j does not affect property $(*_1)$ which in fact depends on M_j).

Denote by $y(\psi) = 11x(\psi)$ the second end of the segment $I_j^k(\psi)$. Consider the trapezium P_j^k with vertices y_j^k, x_j^k, $x(2\pi M_j)$, $y(2\pi M_j)$ and sides I_j^k, $I_j^k(2\pi M_j)$, $[x_j^k, x(2\pi M_j)]$, and $[y_j^k, y(2\pi M_j)]$ (the last two sides are parallel since they are perpendicular to the segment AO). By construction, P_j^k lies on the plane AOD; moreover, taking ε_j sufficiently small, we can obtain the situation where the trapeziums P_j^k are mutually disjoint (since $P_j^k \to I_j^k$ under fixed M_j and $\varepsilon_j \to 0$).

Take an arbitrary triangle whose vertices lie on P_j^k and such that one of these vertices is also a vertex at an acute angle in P_j^k. By construction, this acute angle is at least $\frac{\pi}{2} - \angle AOD = \frac{\pi}{3}$. Therefore, the ratio of the side of the triangle lying inside the trapezium P_j^k to the sum of the other two sides (lying on the corresponding sides of P_j^k) is at least $\frac{1}{2}\sin\frac{\pi}{3} > \frac{2}{5}$. If we consider the same ratio for the case of a triangle with a vertex at an obtuse angle of P_j^k then it is greater than $\frac{1}{2}$. Thus, we have the following property:

($*_2$) *For arbitrary triangle whose vertices lie on the trapezium P_j^k and one of these vertices is also a vertex in P_j^k, the sum of lengths of the sides situated on the corresponding sides of P_j^k is less than $\frac{5}{2}$ of the length of the third side (lying inside P_j^k).*

Let a point x lie inside the cone K formed by the rotation of the angle $\angle AOD$ around the ray OA. Denote by $\mathrm{Proj_{rot}}x$ the point of the angle $\angle AOD$ which is the image of x under this rotation. Finally, let $K_{4\Delta AOD}$ stand for the corresponding truncated cone obtained by the rotation of the triangle $4\Delta AOD$, i.e., $K_{4\Delta AOD} = \{x \in K : \mathrm{Proj_{rot}}x \in 4\Delta AOD\}$.

The key ingredient in the proof of our theorem is the following assertion:

($*_3$) *For arbitrary space curve γ of length less than 10 joining the points A and D, contained in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$, and disjoint from each strip S_j^k, there exists a plane curve $\tilde{\gamma}$ contained in the triangle $4\Delta AOD \setminus \{O\}$, that joins A and D, is disjoint from all segments I_j^k and such that the length of $\tilde{\gamma}$ is less than $\frac{5}{2}$ of the length of $\mathrm{Proj_{rot}}\gamma$.*

Prove ($*_3$). Suppose that its hypotheses are fulfilled. In particular, assume that the inclusion $\mathrm{Proj_{rot}}\gamma \subset 4\Delta AOD \setminus \{O\}$ holds. We need to modify $\mathrm{Proj_{rot}}\gamma$ so that the new curve be contained in the same set but be disjoint from each of the I_j^k's. The construction splits into several steps.

Step 1. If $\mathrm{Proj_{rot}}\gamma$ intersects a segment I_j^k then it necessarily intersects also at least one of the shorter sides of P_j^k.

Recall that, by construction, $P_j^k = \mathrm{Proj_{rot}}S_j^k$; moreover, γ intersects no spiral strip S_j^k. If $\mathrm{Proj_{rot}}\gamma$ intersected P_j^k without intersecting its shorter sides then γ would pass through all coils of the corresponding spiral. Then, by ($*_1$), the length of the corresponding fragment of γ would be $\geqslant 10$ in contradiction to our assumptions. Thus, the assertion of step 1 is proved.

Step 2. Denote by $\gamma_{P_j^k}$ the fragment of the plane curve $\mathrm{Proj_{rot}}\gamma$ beginning at the first point of its entrance into the trapezium P_j^k to the point of its exit from P_j^k (i.e., to its last intersection point with P_j^k). Then this fragment $\gamma_{P_j^k}$ can be deformed without changing the first and the last points so that the corresponding fragment of the new curve lie entirely on the union of the sides of P_j^k; moreover, its length is less than $\frac{5}{2}$ of the length of $\gamma_{P_j^k}$.

The assertion of step 2 immediately follows from the assertions of step 1 and ($*_2$).

The assertion of step 2 in turn directly implies the desired assertion ($*_3$). The proof of ($*_3$) is finished.

Now, we are ready to pass to the final part of the proof of Theorem 2.2.

($*_4$) *The length of any space curve $\gamma \subset \mathbb{R}^3 \setminus \{O\}$ joining A and D and disjoint from each strip S_j^k is at least $\frac{12}{5}$.*

Prove the last assertion. Without loss of generality, we may also assume that all interior points of γ are inside the cone K (otherwise the initial curve can be modified without any increase of its length so that assumptions of ($*_4$) are still fulfilled and the modified curve lies in K). If γ is not included in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$ then $\mathrm{Proj_{rot}}\gamma$ intersects the segment $[4A, 4D]$; consequently, the length of γ is at least $2(4\sin\angle OAD - 1) = 2\left(4\sin\frac{\pi}{3} - 1\right) = 2(2\sqrt{3} - 1) > 4$, and the desired estimate is fulfilled. Similarly, if the length of γ is at least 10 then the desired

estimate is fulfilled automatically, and there is nothing to prove. Hence, we may further assume without loss of generality that γ is included in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$ and its length is less than 10. Then, by $(*_3)$, there is a plane curve $\tilde{\gamma}$ contained in the triangle $4\Delta AOD \setminus \{O\}$, joining the points A and D, disjoint from each segment I_j^k, and such that the length of $\tilde{\gamma}$ is at most $\dfrac{5}{2}$ of the length of $\mathrm{Proj}_{\mathrm{rot}}\gamma$. By property $(**)$ of the family of segments I_j^k, the length of $\tilde{\gamma}$ is at least 6. Consequently, the length of $\mathrm{Proj}_{\mathrm{rot}}\gamma$ is at least $\dfrac{12}{5}$, which implies the desired estimate. Assertion $(*_4)$ is proved.

The just-proven property $(*_4)$ of the constructed objects implies Theorem 2.2. Indeed, since the strips S_j^k are mutually disjoint and, outside every neighborhood of the origin O, there are only finitely many of these strips, it is easy to construct a C^0-manifold $Y \subset \mathbb{R}^3$ that is homeomorphic to a closed ball (i.e., ∂Y is homeomorphic to a two-dimensional sphere) and has the following properties:

(I) $O \in \partial Y$, $[A, O[\cup[D, O[\subset \mathrm{Int}\, Y$;

(II) for every point $y \in (\partial Y) \setminus \{O\}$, there exists a neighborhood $U(y)$ such that $U(y) \cap \partial Y$ is C^1-diffeomorphic to the plane square $[0, 1]^2$;

(III) $S_j^k \subset \partial Y$ for all $j \in \mathbb{N}$, $k = 1, \ldots, k_j$.

The construction of Y with properties (I)–(III) can be carried out, for example, as follows: As the surface of the zeroth step, take a sphere containing O and such that A and D are inside the sphere. At the jth step, a small neighborhood of the point O of our surface is smoothly deformed so that the modified surface is still smooth, homeomorphic to a sphere, and contains all strips S_j^k, $k = 1, \ldots, k_j$. Besides, we make sure that, at the each step, the so-obtained surface be disjoint from the half-intervals $[A, O[$ and $[D, O[$, and, as above, contain all strips S_i^k, $i \leqslant j$, already included therein. Since the neighborhood we are deforming contracts to the point O as $j \to \infty$, the so-constructed sequence of surfaces converges (for example, in the Hausdorff metric) to a limit surface which is the boundary of a C^0-manifold Y with properties (I)–(III).

Property (I) guarantees that $\rho_Y(A, O) = \rho_Y(A, D) = 1$ and $\rho_Y(O, x) \leqslant 1 + \rho_Y(A, x)$ for all $x \in Y$. Property (II) implies the estimate $\rho_Y(x, y) < \infty$ for all $x, y \in Y \setminus \{O\}$, which, granted the previous estimate, yields $\rho_Y(x, y) < \infty$ for all $x, y \in Y$. However, property (III) and the assertion $(*_4)$ imply that $\rho_Y(A, D) \geqslant \dfrac{12}{5} > 2 = \rho_Y(A, O) + \rho_Y(A, D)$. Theorem 2.2 is proved. $\square$

If ρ_Y is a metric (the dimension n ($\geqslant 2$) is arbitrary) then the question of the existence of geodesics is solved in the following assertion, which implies that ρ_Y is the *intrinsic metric* (see, for example, §6 in [11]).

Theorem 2.3. *Assume that ρ_Y is a finite function and is a metric on Y. Then any two points $x, y \in Y$ can be joined in Y by a shortest curve $\gamma : [0, L] \to Y$ in the metric ρ_Y; i.e., $\gamma(0) = x$, $\gamma(L) = y$, and*

$$\rho_Y(\gamma(s), \gamma(t)) = t - s, \quad \forall s, t \in [0, L], \quad s < t. \tag{2.3}$$

Proof. Fix a pair of distinct points $x, y \in Y$ and put $L = \rho_Y(x, y)$. Now, take a sequence of paths $\gamma_j : [0, L] \to \mathrm{Int}\, Y$ such that $\gamma_j(0) = x_j$, $\gamma_j(L) = y_j$, $x_j \to x$, $y_j \to y$, and $l(\gamma_j) \to L$ as $j \to \infty$. Without loss of generality, we may also assume that the parametrizations of the curves γ_j are their natural parametrizations up to a factor (tending to 1) and the mappings γ_j converge uniformly to a mapping $\gamma : [0, L] \to Y$ with $\gamma(0) = x$, $\gamma(L) = y$. By these assumptions,

$$\lim_{j \to \infty} l(\gamma_j|_{[s,t]}) = t - s \quad \forall s, t \in [0, L], \quad s < t. \tag{2.4}$$

Take an arbitrary pair of numbers $s, t \in [0, L]$, $s < t$. By construction, we have the convergence $\gamma_j(s) \in \mathrm{Int}\, Y \to \gamma(s)$, $\gamma_j(t) \in \mathrm{Int}\, Y \to \gamma(t)$ as $j \to \infty$. From here and the definition of the

metric $\rho_Y(\cdot, \cdot)$ it follows that $\rho_Y(\gamma(s), \gamma(t)) \leqslant \lim_{j \to \infty} l(\gamma_j|_{[s,t]})$. By (2.4),

$$\rho_Y(\gamma(s), \gamma(t)) \leqslant t - s \quad \forall s, t \in [0, L], \ s < t. \tag{2.5}$$

Prove that (2.5) is indeed an equality. Assume that $\rho_Y(\gamma(s'), \gamma(t')) < t' - s'$ for some $s', t' \in [0, L]$, $s' < t'$. Then, applying the triangle inequality and then (2.5), we infer

$$\rho_Y(x, y) \leqslant \rho_Y(x, \gamma(s')) + \rho_Y(\gamma(s'), \gamma(t')) + \rho_Y(\gamma(t'), y) < s' + (t' - s') + (L - t') = L,$$

which contradicts the initial equality $\rho_Y(x, y) = L$. The so-obtained contradiction completes the proof of identity (2.3). $\qquad\square$

Remark 2.3. Identity (2.3) means that the curve of Theorem 2.3 is a geodesic in the metric ρ_Y, i.e., the length of its fragment between points $\gamma(s)$, $\gamma(t)$ calculated in ρ_Y is equal to $\rho_Y(\gamma(s), \gamma(t)) = t - s$. Nevertheless, if we compute the length of the above-mentioned fragment of the curve in the initial Riemannian metric then this length need not coincide with $t - s$; only the easily verifiable estimate $l(\gamma|_{[s,t]}) \leqslant t - s$ holds (see (2.4)). In the general case, the equality $l(\gamma|_{[s,t]}) = t - s$ can only be guaranteed if $n = 2$ (if $n \geqslant 3$ then the corresponding counterexample is constructed by analogy with the counterexample in the proof of Theorem 2.2, see above). In particular, though, by Theorem 2.3, the metric ρ_Y is always intrinsic in the sense of the definitions in [11, §6], the space (Y, ρ_Y) may fail to be *a space with intrinsic metric* in the sense of [ibid].

3.   Rigidity theorems for the boundaries of submanifolds in Riemannian manifolds

As in Sec. 2., let (X, g) be an n-dimensional smooth connected Riemannian manifold without boundary and let ρ_X be its intrinsic metric (i.e., let $\rho_X(x, y)$ be the infimum of the lengths $l(\gamma_{x,y,X})$ of smooth paths $\gamma_{x,y,X} : [0, 1] \to X$ joining points x and y in a manifold X).

Assume that Y is an n-dimensional compact connected C^0-submanifold $Y \subset X$ with nonempty boundary ∂Y satisfying condition (i) in Sec. 2., moreover, ρ_Y is a metric on Y. Then Y is called strictly convex in the metric ρ_Y if, for any $\alpha, \beta \in Y$, any shortest path $\gamma = \gamma_{\alpha,\beta,Y} : [0, 1] \to Y$ between α and β (in the metric ρ_Y) satisfies $\gamma(]0, 1[) \subset \operatorname{Int} Y$.

Theorem 3.1. *Let $n = 2$. Assume that condition* (i) *holds for a 2-dimensional compact connected C^0-submanifold Y_1 with nonempty boundary ∂Y_1 of a 2-dimensional smooth connected Riemannian manifold X without boundary which is strictly convex in the metric ρ_{Y_1}. Suppose that $Y_2 \subset X$ is also a 2-dimensional compact connected C^0-submanifold of X with $\partial Y_2 \neq \varnothing$ satisfying* (i); *moreover, ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j}, for $j = 1, 2$. Then, Y_2 is strictly convex with respect to ρ_{Y_2}.*

Proof. Suppose that, for points $x, y \in Y_2$, there exists a shortest path $\gamma_{x,y,Y_2} : [0, 1] \to Y_2$ in the metric ρ_{Y_2} joining x and y and such that $\{\gamma_{x,y,Y_2}(]0, 1[)\} \cap \partial Y_2 \neq \varnothing$, i.e., $x' = \gamma_{x,y,Y_2}(t') \in \{\gamma_{x,y,Y_2}(]0, 1[) \cap \partial Y_2\}$ for a point $t' \in]0, 1[$. By Theorem 2.3 and the fact that Y_2 is a 2-dimensional compact connected C^0-submanifold in X, for a sufficiently small neighborhood of x' in Y_2, we can find points $x_0, y_0 \in \partial Y_2$ and a shortest path $\gamma_{x_0,y_0,Y_2} : [0, 1] \to Y_2$ between x_0 and y_0 in the same metric satisfying the condition $x' \in \{\gamma_{x_0,y_0,Y_2}(]0, 1[) \cap \partial Y_2\}$. Further, we will suppose that $x = x_0$ and $y = y_0$.

Now, assume that $f : \partial Y_1 \to \partial Y_2$ is an isometry of ∂Y_1 and ∂Y_2 in the metrics ρ_{Y_1} and ρ_{Y_2} of the boundaries ∂Y_1 and ∂Y_2 of the submanifolds Y_1 and Y_2 of X. From Theorem 2.3, we have

$$\rho_{Y_2}(x, x') + \rho_{Y_2}(x', y) = l_1 + l_2 = l = \rho_{Y_2}(x, y).$$

Since f is an isometry,

$$\rho_{Y_1}(f^{-1}(x), f^{-1}(x')) + \rho_{Y_1}(f^{-1}(x'), f^{-1}(y)) = \rho_{Y_2}(x, x') + \rho_{Y_2}(x', y).$$

Next, consider shortest paths $\gamma_{f^{-1}(x), f^{-1}(x'), Y_1} : [0, 1/2] \to Y_1$ and $\gamma_{f^{-1}(x'), f^{-1}(y), Y_1} : [1/2, 1] \to Y_1$ in ρ_{Y_1} between (respectively) $f^{-1}(x)$ and $f^{-1}(x')$ and $f^{-1}(x')$ and $f^{-1}(y)$, and then construct a path $\gamma : [0, 1] \to Y_1$ by setting $\gamma(t) = \gamma_{f^{-1}(x), f^{-1}(x'), Y_1}(t)$ if $0 \leqslant t < 1/2$ and $= \gamma_{f^{-1}(x'), f^{-1}(y), Y_1}(t)$ for $1/2 \leqslant t \leqslant 1$. Let $l_{Y_1}(\delta)$ be the length of a path $\delta : [0, 1] \to Y_1$ in the metric ρ_{Y_1}. Since ρ_{Y_1} is a metric on Y_1, it is not difficult to show that

$$l_{Y_1}(\gamma) \leqslant l_{Y_1}(\gamma_{f^{-1}(x), f^{-1}(x'), Y_1}) + l_{Y_1}(\gamma_{f^{-1}(x'), f^{-1}(y), Y_1}) = l_1 + l_2.$$

Hence γ is a shortest path in ρ_{Y_1} joining $f^{-1}(x)$ and $f^{-1}(y)$ in Y_1. This contradicts the strict convexity of Y_1. The theorem is proved. $\square$

Corollary 3.1. *Suppose that the conditions of Theorem 3.1 hold and the manifold X has the following property: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y from every 2-dimensional compact connected C^0-submanifold $Y \subset X$ with $\partial Y \neq \varnothing$ satisfying condition* (i) *and strictly convex with respect to the metric ρ_Y. Then, ∂Y_1 and ∂Y_2 are isometric in the metric ρ_X on the ambient manifold X.*

Remark 3.1. The condition imposed on the manifold X in Corollary 3.1 can be reformulated as follows: in this manifold, every 2-dimensional compact connected C^0-submanifold Y with boundary satisfying condition (i) and strictly convex with respect to its intrinsic metric ρ_Y is a convex set in the ambient space X with respect to the metric ρ_X (for the notion of a convex set in a metric space the reader is referred, for example, to [11]).

We have the following analog of Theorem 3.1 and Corollary 3.1 (combined together) for $n \geqslant 3$:

Theorem 3.2. *Let $n \geqslant 3$. Suppose that (X, g) is an n-dimensional smooth connected Riemannian manifold without boundary and Y_1 and Y_2 are n-dimensional compact connected C^0-submanifolds with nonempty boundaries ∂Y_1 and ∂Y_2 in X satisfying conditions* (i),

(ii) *ρ_{Y_j} is a metric on Y_j ($j = 1, 2$),*
and

(iii) *for any two points $a, b \in Y_j$, there exist points $c, d \in \partial Y_j$ which can be joined in Y_j by a shortest path $\gamma : [0, 1] \to Y_j$ in the metric ρ_{Y_j} so that $a, b \in \gamma([0, 1])$.*

Furthermore, assume that Y_1 is strictly convex in the metric ρ_{Y_1}, X has the additional property that $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y in every n-dimensional compact connected C^0-submanifold $Y \subset X$ with $\partial Y \neq \varnothing$ satisfying conditions (i)–(iii) *and strictly convex with respect to ρ_Y and the boundaries ∂Y_1 and ∂Y_2 of the submanifolds Y_1 and Y_2 are isometric with respect to the metrics ρ_{Y_j}, where $j = 1, 2$. Then, ∂Y_1 and ∂Y_2 are isometric with respect to ρ_X.*

Remark 3.2. For a submanifold Y in X, (i) and (ii) can be considered as conditions of generalized regularity near its boundary.

Remark 3.3. Theorem 3.1, Corollary 3.1, and Theorem 3.2 are closely related to a theorem of A. D. Aleksandrov about the rigidity of the boundary ∂U of a strictly convex domain U in Euclidean n-space $\mathbb{R}^n$ by the relative metric $\rho_{\partial U, U}$ on the boundary. The following is an important particular case of this theorem:

Theorem 3.3 (A. D. Aleksandrov ([6])). *Let U_1 be a strictly convex domain in $\mathbb{R}^n$ (i.e., for any $\alpha, \beta \in \mathrm{cl}\, U_1$ every shortest path $\gamma = \gamma_{\alpha, \beta, \mathrm{cl}\, U_1} : [0, 1] \to \mathrm{cl}\, U_1$ between α and β (in the metric $\rho_{\mathrm{cl}\, U_1}$) satisfies $\gamma(]0, 1[) \subset U_1$). Assume that $U_2 \subset \mathbb{R}^n$ is any domain whose closure is a Lipschitz manifold (such that $\partial(\mathrm{cl}\, U_2) = \partial U_2 \neq \varnothing$); moreover, ∂U_1 and ∂U_2 are isometric in their relative metrics $\rho_{\partial U_1, U_1}$ and $\rho_{\partial U_2, U_2}$. Then ∂U_1 and ∂U_2 are isometric in the Euclidean metric.*

We say that an n-dimensional compact (closed) connected C^0-submanifold Y with boundary $\partial Y \neq \varnothing$ of an n-dimensional smooth connected (respectively, n-dimensional smooth complete connected) Riemannian manifold X without boundary has property ($\circ$) if $\gamma_{x,y,Y}(]0,1[) \subset \operatorname{Int} Y$ for any two points $x, y \in \partial Y$ and for every shortest path $\gamma_{x,y,Y} : [0,1] \to Y$ in the metric ρ_Y joining these points.

Theorem 3.4. *Let $n = 2$. Suppose that* (i) *holds for a 2-dimensional compact connected C^0-submanifold Y_1 with boundary $\partial Y_1 \neq \varnothing$ in a 2-dimensional smooth connected Riemannian manifold X without boundary; moreover, Y_1 has property* ($\circ$). *Assume that $Y_2 \subset X$ is a 2-dimensional compact connected C^0-submanifold with $\partial Y_2 \neq \varnothing$ in X and ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} ($j = 1, 2$). Then ∂Y_2 also has property* ($\circ$).

This theorem has the following generalization.

Theorem 3.5. *Let $n = 2$. Suppose that* (i) *holds for a 2-dimensional closed connected C^0-submanifold Y_1 with boundary ∂Y_1 ($\neq \varnothing$) in a 2-dimensional smooth complete connected Riemannian manifold X without boundary satisfying* ($\circ$). *Assume that $Y_2 \subset X$ is a 2-dimensional closed connected C^0-submanifold with $\partial Y_2 \neq \varnothing$ in X; moreover, ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} ($j = 1, 2$). Then Y_2 has the property* ($\circ$) *as well.*

Corollary 3.2 (of Theorem 3.4)**.** *Assume that the hypothesis of Theorem* 3.4 *hold and that the manifold X has the following property: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every 2-dimensional compact connected C^0-submanifold $Y \subset X$ with $\partial Y \neq \varnothing$ satisfying* (i) *and* ($\circ$). *Then ∂Y_1 and ∂Y_2 are isometric in the metric ρ_X of the ambient manifold X.*

Corollary 3.3 (of Theorem 3.5)**.** *Assume that the hypothesis of Theorem* 3.5 *hold and that the manifold X has the following property: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every 2-dimensional closed connected C^0-submanifold $Y \subset X$ with $\partial Y \neq \varnothing$ satisfying* (i) *and* ($\circ$). *Then ∂Y_1 and ∂Y_2 are isometric with respect to ρ_X.*

Theorem 3.6. *Let $n \geqslant 3$. Suppose that (X, g) is an n-dimensional smooth connected Riemannian manifold whithout boundary and Y_1 and Y_2 are n-dimensional compact connected C^0-submanifolds with nonempty boundaries ∂Y_1 and ∂Y_2 in X satisfying conditions* (i) *and* (ii) *(in Theorem 3.2). Assume that Y_1 has property* ($\circ$) *and X satisfies the following condition: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every n-dimensional compact connected C^0-submanifold $Y \subset X$ with $\partial Y \neq \varnothing$ satisfying* (i), (ii), *and* ($\circ$). *Suppose also that ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j}, where $j = 1, 2$. Then ∂Y_1 and ∂Y_2 are isometric in ρ_X.*

Theorem 3.7. *Let $n \geqslant 3$. Suppose that (X, g) is an n-dimensional smooth complete connected Riemannian manifold without boundary and Y_1 and Y_2 are n-dimensional closed connected C^0-submanifolds with nonempty boundaries ∂Y_1 and ∂Y_2 in X satisfying* (i) *and* (ii). *Assume that ∂Y_1 has property* ($\circ$) *and X satisfies the following condition: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every n-dimensional closed connected C^0-submanifold Y with $\partial Y \neq \varnothing$ in X satisfying* (i), (ii), *and* ($\circ$). *Suppose also that ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} ($j = 1, 2$). Then ∂Y_1 and ∂Y_2 are isometric in ρ_X.*

Proofs of Theorems 3.2 *and* 3.4–3.7 *are similar to the proof of Theorem 3.1 (Theorems 3.2 and 3.4–3.7 can be proved using the corresponding analogs of Theorems 2.1 and 2.3).*

In conclusion, note that main results of our article were earlier announced in [1] and [2].

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Об условия жесткости границ подмногообразий риманового многообразия

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В процессе развития идей академика А. Д. Александрова первым автором был предложен следующий подход к изучению проблем жесткости для краёв C^0-подмногообразий в некотором гладком римановом многообразии. Пусть Y_1 представляет собой двумерное компактное связное C^0-подмногообразие с непустым краем в некотором гладком двумерном римановом многообразии (X, g) без края. Рассмотрим внутреннюю метрику (инфимум длин путей, соединяющих данную пару точек) внутренности $\operatorname{Int} Y_1$ многообразия Y_1 и продолжим ее по непрерывности (операцией $\underline{\lim}$) на краевые точки ∂Y_1. В настоящей статье рассматривается вопрос о жесткости, т.е. когда указанная метрика определяет ∂Y_1 с точностью до изометрии в объемлющем пространстве (X, g). Рассматривается также случай $\dim Y_j = \dim X = n$, $n > 2$.

Ключевые слова: риманово многообразие, внутренняя метрика, индуцированная метрика на крае, строгая выпуклость многообразия, геодезические, условия жесткости.