Unification and Inference Rules in the Multi-modal Logic of Knowledge and Linear Time LTK

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We study unification of formulas in multi-modal LTK logic and give a syntactic description of all formulas which are non-unificable in this logic. Passive inference rules are considered, it is shown that in LTK logic there is a finite basis for passive rules.

Keywords: unification, modal temporal logic, passive inference rules.


Introduction

The research of unification for various logic systems is one of the most rapidly developing areas of modern mathematical logic. Arisen in the field of Computer Science, primarily in the form of a question about the possibility to transform two different terms in syntactically equivalent by replacing the variables of certain other terms ([1, 2]), from the time the task has changed course on the study of semantic equivalence ([3, 4]).

For most of the non-classical logics (modal, pseudo-boolean, temporal, etc.), there are special dual equational theories of special algebraic systems, so their problems are reduced to the corresponding logical-unificational counterparts ([5–7]). Basic unificational problem can be viewed as a complex issue: whether the formula is to be transformed into a theorem after replacing the variables (keeping the same values of the coefficients parameters). This issue was investigated and partly resolved (including V. V. Rybakov [8–10]), for intuitionistic and modal logics S4 and Grz.

Unification in intuitionistic logic and in propositional modal logic over the K4 investigated by S. Ghilardi [11–15] (with applications of projective algebra ideas and technology based on projective formulas). In these works the problem of constructing the finite complete sets of unifiers was solved for the considered logic, efficient algorithms were found. Such an approach proved to be a a useful and effective in dealing with the admissibility and the basis of admissible rules (Jerabek [16–18], Iemhoff, Metcalfe [19, 20]). Indeed, the existence of computable finite sets of unifiers follows directly solution of the admissibility problem.

Temporal logic is also very dynamic area of mathematical logic and computer science (including Gabbay и Hodkinson [21–23]). In particular, LTL (linear temporal logic) has a significant application in the field of Computer Science (Manna, Pnueli [24, 25], Vardi [26, 27]). Solving the problem of admissibility of rules in the LTL was proposed by Rybakov [28], basis of admissible
rules in LTL by Babenyshev and Rybakov in [29] (without the operator Until [30]). Solution of the unificational problem for LTL has also been found by Rybakov [31, 32] and proposed for basic modal and intuitionistic logic in [33, 34]. Particularly, in [31] it proved that not all unified in LTL formula are projective, and in [32] proved the projectivity of any unified formula in $LTL_u$ (it is a fragment of LTL, only with the operator Until). In the paper of Dzik and Wojtylak [35] they obtained the same result for S4.

Research conducted in the present paper, primarily based on the approach proposed in [36]. The key focus here is on the description of non-unifiable formulas in a wide class of modal logics. Especially, it proposed the criteria of non-unifiable (with the proofs) for modal extensions of S4 (Theorem 1.4 below) and $[K4 + □ ⊥ ≡ ⊥]$ (Theorem 1.5). The aim of this article is to investigate the question of unification in linear temporal logic (LTK).

1. Fundamental definitions and notations

Before describing the main results, we introduce the most important definitions and notations. Proofs for the most of propositions, consequences and the theorems in this section are detailed in [36].

First, we define a unified formula in this logic. Let $λ$ is a logic with the formula $φ(p, q)$ which describes the equivalent formula. We say that $α$ is equivalent to $β$ in $λ$, and we write $α ≡ β$ if $\vdash_λ φ(α, β)$. For convenience, $φ(α, β)$ we denote $α \equiv β$.

**Definition 1.1.** Formula $α(p_1, \ldots, p_n)$ is unifiable in an algebraic logic $λ$ iff there is a tuple of formulas $δ_1, \ldots, δ_n$ such that $\vdash_λ α(δ_1, \ldots, δ_n)$.

**Definition 1.2.** Formulas $α(p_1, \ldots, p_n)$ and $β(p_1, \ldots, p_n)$ are said to be unifiable in algebraic logic $λ$ iff there is a tuple of formulas $δ_1, \ldots, δ_n$ such that $\vdash_λ α(δ_1, \ldots, δ_n) \equiv β(δ_1, \ldots, δ_n)$. In this case, the tuple $δ_1, \ldots, δ_n$ is called an unifier for these two formulas.

**Corollary 1.3 (2.7 from [36]).** For all logics $SIL$, $S_4^{ext}$, $K4 + □ ⊥ ≡ ⊥$ unifiers for formulas can be effectively found among sequences of formulas $Γ$ end $⊥$.

For example, by setting $⊤$ everywhere in the performance of the variable $p$ and $⊥$ otherwise.

**Theorem 1.4 (2.10 from [36]).** For any modal logic $λ$ extending $S4$ and any modal formula $α$, $α$ is not unifiable in $λ$ iff the formula $□ α \rightarrow \bigvee_{p \in Var(α)} ♦ p \land ♦ \neg p$ is provable in $λ$.

**Theorem 1.5 (2.11 from [36]).** For any modal logic $λ$ extending $K4$, where $□ ⊥ ≡ ⊥ \in λ$ and any modal formula $α$, $α$ is not unifiable at $λ$ iff formula $□ α \land α \rightarrow \bigvee_{p \in Var(α)} ♦ p \land ♦ \neg p$ is provable at $λ$.

**Definition 1.6.** Rule $r := A/B$ is a consequence of the rules $r_1 := A_1/B_1, \ldots, r_n := A_n/B_n$ in logic $L \Leftrightarrow \forall A \in Var(L) = \{A|A \vdash (α = ⊤), \forall α \in L\}$: if

\[ \forall i A \vdash (α_i = ⊤) \Rightarrow (β_i = ⊤), \]

then

\[ A \vdash (α = ⊤) \Rightarrow (β = ⊤). \]

Let us recall the definition of algebra formulas, Lindenbaum algebra. Let $For$ is the set of all formulas in the language of logic. We will use the following notation: $A \equiv B \Leftrightarrow (A \rightarrow B) \land (B \rightarrow A)$. We write $A \equiv_L B$, if $A \equiv B \in L$. Suppose that the logic $L$ has theorem of replacing equivalent.
Namely, if ⊕ is any binary logic connective (for example →, ∧, ∨), and $A_1$, $A_2$, $B_1$ and $B_2$ are the formulas, then

$$(A_1 \equiv L B_1, A_2 \equiv L B_2) \Rightarrow A_1 \oplus A_2 \equiv L B_1 \oplus B_2,$$

and if ⊗ is any unary logical connective, then

$$(A_1 \equiv L B_1) \Rightarrow \otimes A_1 \equiv L \otimes B_1.$$
d) $R_\leq$ is linear, reflexive, transitive binary temporal relation on $W_F$, specifying linear order of clusters (simple chain):

$$\forall i, z \in W_F (wR_\leq z \iff \exists i, j \in N ((v \in C^i) \& (z \in C^j) \& (i \leq j)))$$

Also hold the following properties of matching these relations:

1) $wR_\leq z \iff (wR_\leq z) \& (zR_\leq w)$;
2) $wR_\leq z \Rightarrow wR_\leq z$.

We denote class of all such frames $LTK$.

**Definition 2.4.** For two $R_\leq$-clusters $C^m$ and $C^j$ notation $C^mR_\leq C^j$ indicates that $\forall w \in C^m, \forall z \in C^j$ is performed $(wR_\leq z)$. Thus, $C^m$ is $R_\leq$-precursor of cluster $C^j$, and $C^j$ is $R_\leq$-follower of cluster $C^m$.

Frames of this class model a situation in which each agent has all the information in the current temporary state $C^t$. Any temporary state $C^t$ (i.e $R_\leq$-cluster) consists of a set of information points available at $t$. The relation $R_\leq$ is a connection into a linear stream of information points, wherein for two points $w$ and $z$ term $wR_\leq z$ means that either $w$ and $z$ are available at the same moment of time, or $z$ will be available at subsequent times in relation to $w$. Relation $R_\leq$ connects all information points potentially available at the same moment of time, thus it represents knowledge that is potentially available at any given time. Each relation $R_i$, $i = 1, ..., n$, reflects the information available to a particular agent $i$.

**Definition 2.5.** Model $M_F$ on a $LTK$-frame $F$ is a tuple $M_F = \langle F, V \rangle$, where $V$ is a valuation of a set of propositional letters $p \in P$ on the frame, i.e $\forall p \in P \ [V(p) \subseteq W_F]$. Given a model $M_F = \langle F, V \rangle$, where $F$ is a $LTK$-frame $W_F$. Then $\forall w \in W_F$:

a) $\langle F, w \rangle \models_V p \iff w \in V(p)$;

b) $\langle F, w \rangle \models_V \Box_\leq A \iff \forall z \in W_F (wR_\leq z \Rightarrow \langle F, z \rangle \models_V A)$;

c) $\langle F, w \rangle \models_V \Diamond_\leq A \iff \forall z \in W_F (wR_\leq z \Rightarrow \langle F, z \rangle \models_V A)$;

d) $\forall i \in I, (F, w) \models_V \Box \neg A \iff \forall z \in W_F (wR_\leq z \Rightarrow (F, z) \models_V A)$.

The relation $\models_V$ here means truth relation on the element $w$ of model $M$. Namely, $\langle F, w \rangle \models_V A$ means that $A$ true on the element $w$ in model $\langle F, V \rangle$. If the formula $A$ true on any element of the frame $F$ with any valuation $V$, we called $A$ true on the frame $F$ and write $F \models V A$.

**Definition 2.6.** The logic $LTK$ is the set of all $LTK$-valid formulae on all frames: $LTK := \{ A \in Fm\alpha(LTK) | \forall F \in LTK (F \models_A) \}$. If $A$ belongs to $LTK$, then we say that $A$ is a theorem of $LTK$.

### 3. A criterion of non-unifiability

We immediately begin with the proof of the main statement of this section.

**Theorem 3.1.** Any modal formula $A$ is non-unifiable in $LTK$ iff formula

$$\Box_\leq A \rightarrow \bigvee_{p \in V ar(A)} \Diamond_\leq p \land \Diamond_\leq \neg p$$

is a theorem in $LTK$. 

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- 152 -
Proof. 1. Prove the theorem by contradiction. Assume that

$$\Box \xi A \rightarrow \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p \in LTK,$$

but at the same time, the formula $A$ is unifiable in $LTK$.

Then by definition of unifier, there is a substitution (unifier) $g$ s.t. $g(A) \in LTK$. By the fact that $LTK$ is closed under substitution, we obtain $g \left( \Box \xi A \rightarrow \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p \right) \in LTK$.

Let us consider $LTK$-frame $F_1$ with all single element clusters (i.e. $\forall t : C^t = a$). Consider the valuation $V$ for all variables $q$ of formulas $g(p)$, where $p \in Var(A)$, on the $F_1 : V(q) = \emptyset$. Then it is easy to check by the induction on the length of any formula $B$ constructed on variables $q$ that:

$$\forall b \in F_1, \forall c \in F_1 : b \not\Vdash V B \Leftrightarrow c \not\Vdash V B.$$ Consequently,

$$\forall b \in F_1 : b \not\Vdash V \bigvee_{p \in Var(A)} \diamond \xi g(p) \land \diamond \xi \neg g(p).$$

At the same time,

$$\forall b \in F_1 : b \not\Vdash \Box \xi g(A).$$

Thereby,

$$\forall b \in F_1 : b \not\Vdash g \left( \Box \xi A \rightarrow \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p \right),$$

which contradicts the hypothesis:

$$g \left( \Box \xi A \rightarrow \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p \right) \in LTK.$$

2. On the contrary, say that the formula $A$ is non-unifiable in $LTK$, but at the same time $\Box \xi A \rightarrow \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p \notin LTK$. Then, by finitary approximability of $LTK$, there is a certain root frame $F$ that disproves this formula:

$$\exists a \in F : \langle F, a \rangle \not\Vdash \Box \xi A \rightarrow \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p.$$

That is $\langle F, a \rangle \not\Vdash \Box \xi A$ and $\langle F, a \rangle \not\Vdash \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p$. Assume this element $a$ as the root of the frame $F_1$ ($F_1 = a^\xi$). By $\langle F, a \rangle \not\Vdash \bigvee_{p \in Var(A)} \diamond \xi p \land \diamond \xi \neg p$, $\forall p \in Var(A)$: either

1. $\forall b \in F_1 a R b : b \not\Vdash V p$,

or

2. $\forall b \in F_1 a R b : b \not\Vdash V p$.

Choose a substitution $g$ for all of the variables $p$ from the formula $A$ as follows: $\forall p \in Var(A) : g(p) = \top$ if (1) and $g(p) = \bot$ in the case of (2). Then $g$ is a unifier of the formula $A$. Indeed, if we take any frame $F_2$, any cluster $a^\xi \in F_2$ and any valuation $V_2$:

$$a^\xi \not\Vdash V_2 A \Leftrightarrow a \not\Vdash V A.$$

Therefore, the formula $A$ is unifiable in $LTK$. 

\[\square\]
4. Passive inference rules

**Definition 4.1.** Let \( r := A_1, \ldots, A_n / \beta \) be an inference rule in the logic \( LTK \). The rule \( r \) called passive for \( LTK \) if for any substitution \( g \) of formulas instead of variables in \( r \) never \( g(A_1) \in LTK \& \ldots \& g(A_n) \in LTK \). In other words \( r \) is a passive rule if formulas from its premise have no common unifiers.

**Proposition 4.2.** For multi-modal logic \( LTK \) the rules \( r_n := \frac{\bigvee_{1 \leq i \leq n} \diamond \leq p_i \land \diamond \leq \neg p_i}{\bot} \) form a basis for all passive rules for \( LTK \).

**Proof.** It is true that \( \square \leq \bigwedge_{1 \leq i \leq n} \diamond \leq p_i \land \diamond \leq \neg p_i \rightarrow \bigvee_{p \in Var(A_1 \land \cdots \land A_n)} \diamond \leq p \land \diamond \leq \neg p \) \( \in LTK \), and hence by Theorem 3.1 formula \( A_n = \bigvee_{1 \leq i \leq n} \diamond \leq p_i \land \diamond \leq \neg p_i \) does not unifiable in modal logic \( LTK \), i.e. any rule \( r_n \) is passive. Let us assume that a rule \( t_1 := A_1, \ldots, A_n / \beta \) is passive for \( LTK \). Then the rule \( t_2 := A_1 \land \cdots \land A_n / \beta \) is also passive for \( LTK \) and formula \( A_1 \land \cdots \land A_n \) is not unifiable in \( LTK \). Applying Theorem 4.1 we conclude

\[
\square \leq (A_1 \land \cdots \land A_n) \rightarrow \left[ \bigvee_{p \in Var(A_1 \land \cdots \land A_n)} \diamond \leq p \land \diamond \leq \neg p \right] \in LTK.
\]

Using the premise of rule \( t_2 \) we conclude

\[
\bigvee_{p \in Var(A_1 \land \cdots \land A_n)} \diamond \leq p \land \diamond \leq \neg p
\]

and then applying the rule \( r_n \), where \( n \) is the number of variables in the conjunction of \( A_1 \land \cdots \land A_n \), we can derive the formula \( \bot \). From \( \bot \rightarrow \beta \in LTK \), in its turn holds \( \beta \). Thus, all \( r_n \) really represent all passive rules in \( LTK \). \( \square \)

Now we consider the possibility of reduction infinite (due to an unlimited number of variables) basis of passive rules in \( LTK \) that was obtained in the Proposition 4.1 to a finite and more simple form.

Let us remind that the rule \( r \) is a consequence of the rules \( r_i \in X, i \in I \) in a logic \( L \), if for any algebra \( A \in \text{Var}(L) \) and \( \forall i \in I : A \models r_i \Rightarrow A \models r \). Accordingly, a rule \( r \) is not a consequence of the rules \( r_i \in X, i \in I \) otherwise. A rule \( r \) true in the algebra \( A \) if for any substitution of elements from algebra instead of the variables of a rule \( r \) if all formulas from the premise of a rule \( r \) is true, then a conclusion formula of \( r \) is also true.

**Theorem 4.3.** In multi-modal logic \( LTK \) the rule \( r := \frac{\diamond \leq p \land \diamond \leq \neg p}{\bot} \) is a basis for all passive inference rules.

**Proof.** According to Proposition 4.2, it suffices to show that the rules \( r_n (\forall n) \) are a consequence of \( r (r \vdash r_n, \forall n) \).

Suppose that it is not true:

\[
r_n := \frac{\bigvee_{1 \leq i \leq n} \diamond \leq p_i \land \diamond \leq \neg p_i}{\bot}
\]

is not a consequence of the rule \( r \). Hence there is a finitely generated algebra \( A \), in which the rule \( r \) is valid \( (A \models r) \), but \( r_n \) is not \( (A \not\models r_n) \), thus \( \forall i \in \{1, \ldots, n\} \) there is \( a_i \in A : \bigvee_{1 \leq i \leq n} \diamond \leq a_i \land \diamond \leq \neg a_i = \top \). Get a subalgebra \( A_1 \) of algebra \( A \) generated by such elements \( a_i, 1 \leq i \leq n, \)
(\(A_1 = A_1(a_1, \ldots, a_n) \subseteq A\)). \(A_1\) is a \(S4.3\)-algebra on \(\Box \leq\). By Lemma 4.3.18 from [10] Kripke-frame \(A_1^+\), associated to \(A_1\) has a single element reflexive maximal cluster \(C\). By the definition \(A_1^+\), \(\forall a \in A_1\), \(a \subseteq A_1^+\). By hypothesis of proof, \(\bigvee_{1 \leq i \leq n} \Diamond \leq a_i \land \Diamond \leq \neg a_i \in A_1\), because \(A_1\) is a subalgebra \(A\), on the construction. Then \(\bigvee_{1 \leq i \leq n} \Diamond \leq a_i \land \Diamond \leq \neg a_i = T = A_1^+\), but it is impossible on a single element reflexive maximal cluster \((C \not\models \bigvee_{1 \leq i \leq n} \Diamond \leq a_i \land \Diamond \leq \neg a_i)\), and hence \(\bigvee_{1 \leq i \leq n} \Diamond \leq a_i \land \Diamond \leq \neg a_i \notin A_1^+\) that contradict with the proof conditions. □

References


Унификация и правила вывода в многомодальной логике знания и линейного времени LTK

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В статье исследуется унификация формул в многомодальной логике LTK и предложено синтаксическое описание всех формул, которые не являются унифицируемыми в данной логике. Рассмотрен вопрос пассивных правил вывода, показано, что в логике LTK есть конечный базис для пассивных правил.

Ключевые слова: унификация, модальная темпоральная логика, пассивные правила вывода.