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Solvability of One Nonlinear Boundary-value Problem for a System of Differential Equations of the Theory of Shallow Timoshenko-type Shells

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Solvability of a system of nonlinear second order partial differential equations with given initial conditions is considered in the paper. Reduction of the initial system of equations to one nonlinear operator equation is used to study the problem. The solvability is established with the use of the principle of contracting mappings.

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1. Problem formulation

Let us introduce in the plane simply connected bounded domain Ω and consider a system of nonlinear partial differential equations in the form

$$
\begin{gathered}
T^{i\lambda}_{\alpha^\lambda} + R^i = 0, \quad i = 1, 2, \\
T^{\lambda 3}_{\alpha^\lambda} + k_\lambda T^{\lambda\lambda} + (T^{\lambda\mu} w_{3\alpha^\mu})_{\alpha^\lambda} + R^3 = 0, \\
M^{i\lambda}_{\alpha^\lambda} - T^{i3} + L^i = 0, \quad i = 1, 2
\end{gathered} \tag{1}
$$

under the following conditions at the boundary Γ:

$$w_1 = \psi_1 = 0, \tag{2}$$

$$T^{12} d\alpha^2/ds - T^{22} d\alpha^1/ds = P^2(s), \tag{3}$$

$$
\begin{aligned}
T^{13} d\alpha^2/ds - T^{23} d\alpha^1/ds + T^{11} w_{3\alpha^1} d\alpha^2/ds - T^{22} w_{3\alpha^2} d\alpha^1/ds + \\
+ T^{12}(w_{3\alpha^2} d\alpha^2/ds - w_{3\alpha^1} d\alpha^1/ds) = P^3(s),
\end{aligned} \tag{4}
$$

$$M^{12} d\alpha^2/ds - M^{22} d\alpha^1/ds = N^2(s). \tag{5}$$

From this point on the index α^λ means differentiation with respect to α^λ.

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In (1)–(5) the following notations are used:

$$T^{ij} \equiv T^{ij}(a) = D_0^{ijkn}\gamma_{kn}^0, \quad M^{ij} \equiv M^{ij}(a) = D_2^{ijkn}\gamma_{kn}^1, \quad a = (w_1, w_2, w_3, \psi_1, \psi_2),$$
$$D_m^{ijkn} = \int_{\frac{-h}{2}}^{\frac{h}{2}} B^{ijkn}(\alpha^3)^m d\alpha^3, \quad B^{1111} = B^{2222} = E/(1-\mu^2), \quad B^{1122} = \mu E/(1-\mu^2), \tag{6}$$
$$B^{1212} = E/(2(1+\mu)), \quad B^{1313} = B^{2323} = Ek^2/(2(1+\mu));$$

the remainder $B^{ijkn} = 0$; $\alpha^j = \alpha^j(s) (j = 1, 2)$ is the equation of the curve Γ, s is the length of the arc Γ;

$$\gamma_{jj}^0 = w_{j\alpha^j} - k_j w_3 + w_{3\alpha^j}^2/2 \ (j = 1, 2), \quad \gamma_{12}^0 = w_{1\alpha^2} + w_{2\alpha^1} + w_{3\alpha^1}w_{3\alpha^2},$$
$$\gamma_{jj}^1 = \psi_{j\alpha^j} \ (j = 1, 2), \quad \gamma_{12}^1 = \psi_{1\alpha^2} + \psi_{2\alpha^1}, \tag{7}$$
$$\gamma_{j3}^0 = w_{3\alpha^j} + \psi_j \ (j = 1, 2), \quad \gamma_{33}^0 = \gamma_{k3}^1 \equiv 0, \quad k = \overline{1,3}.$$

The system (1) together with the boundary conditions (2)–(5) describes the state of equilibrium of shallow isotropic elastic homogeneous shell with simply supported edges within the framework of Timoshenko shear model [1]. Here T^{ij} are stresses, M^{ij} are moments; $\gamma_{ij}^k(i, j = \overline{1,3}, \ k = 0, 1)$ are components of deformation of the shell middle surface S_0 that is homeomorphic to Ω; $w_i(i = 1, 2)$ and w_3 are tangential and normal displacements of the points of S_0; $\psi_i(i = 1, 2)$ are rotation angles of normal cross-section of S_0; a is the vector of generalized displacements; $R^j(j = \overline{1,3})$, $L^k(k = 1, 2), N^2, P^2$, P^3 are components of the external forces acting on the shell; $\mu = \text{const}$ is the Poisson coefficient, $E = \text{const}$ is Young‘s modulus, $k_1, k_2 = \text{const}$ are principal curvatures; $k^2 = \text{const}$ is the shear coefficient; $h = \text{const}$ is the shell width; α^1, α^2 are the Cartesian coordinates of the points in the domain Ω.

We assume summation over repeating Latin indices from 1 to 3 and over Greek indices from 1 to 2 in (1), (6) and in what follows.

System (1) is written in terms of generalized displacements $w_1, w_2, w_3, \psi_1, \psi_2$:

$$w_{1\alpha^1\alpha^1} + \mu_1 w_{1\alpha^2\alpha^2} + \mu_2 w_{2\alpha^1\alpha^2} = f_1,$$
$$\mu_1 w_{2\alpha^1\alpha^1} + w_{2\alpha^2\alpha^2} + \mu_2 w_{1\alpha^1\alpha^2} = f_2,$$
$$k^2\mu_1(w_{3\alpha^1\alpha^1} + w_{3\alpha^2\alpha^2} + \psi_{1\alpha^1} + \psi_{2\alpha^2}) + k_3 w_{1\alpha^1} + k_4 w_{2\alpha^2} - k_5 w_3 +$$
$$+ k_3 w_{3\alpha^1}^2/2 + k_4 w_{3\alpha^2}^2/2 + \beta_2\left[(T^{\lambda\mu}w_{3\alpha^\lambda})_{\alpha^\mu} + R^3\right] = 0, \tag{8}$$
$$\psi_{1\alpha^1\alpha^1} + \mu_1\psi_{1\alpha^2\alpha^2} + \mu_2\psi_{2\alpha^1\alpha^2} = g_1,$$
$$\mu_1\psi_{2\alpha^1\alpha^1} + \psi_{2\alpha^2\alpha^2} + \mu_2\psi_{1\alpha^1\alpha^2} = g_2,$$

where

$$f_1 \equiv f_1(w_3) = k_3 w_{3\alpha^1} - w_{3\alpha^1}w_{3\alpha^1\alpha^1} - \mu_2 w_{3\alpha^2}w_{3\alpha^1\alpha^2} - \mu_1 w_{3\alpha^1}w_{3\alpha^2\alpha^2} - \beta_2 R^1,$$
$$f_2 \equiv f_2(w_3) = k_4 w_{3\alpha^2} - w_{3\alpha^2}w_{3\alpha^2\alpha^2} - \mu_2 w_{3\alpha^1}w_{3\alpha^1\alpha^2} - \mu_1 w_{3\alpha^2}w_{3\alpha^1\alpha^1} - \beta_2 R^2, \tag{9}$$
$$g_j \equiv g_j(w_3) = k_0(w_{3\alpha^j} + \psi_j) - \beta_1 L^j, j = 1, 2,$$

$$\mu_1 = (1-\mu)/2, \ \mu_2 = (1+\mu)/2, \ k_3 = k_1 + \mu k_2, \ k_4 = k_2 + \mu k_1, k_5 = k_1^2 + k_2^2 + 2\mu k_1 k_2,$$
$$k_0 = 6k^2(1-\mu)/h^2, \ \beta_1 = 12(1-\mu^2)/(h^3 E), \beta_2 = (1-\mu^2)/(Eh).$$

System (8) is a system of second order partial differential equations. It is linear with respect to tangential displacements w_1, w_2, rotation angles ψ_1, ψ_2 and it is a nonlinear system with respect to deflection w_3.

Problem A. Find a solution to system (8) under boundary conditions (2)–(5).

Solvability of the system of nonlinear differential equations that describes shell equilibrium in the framework of the Kirchhoff-Love model has been well studied [2–5]. The questions of the existence of solutions of nonlinear problems in the framework of more general shell models, not based on the Kirchhoff-Love hypotheses, are in the well-known Volovich list of unresolved problems of the mathematical theory of shells [2]. These questions have not been clarified yet. There are a number of works devoted to the solvability of nonlinear problems in the framework of the Timoshenko displacement model [6–10]. The method used in these studies is based on the integral representations of the desired solution of system (8) that contain arbitrary holomorphic functions. These representations are constructed with the use of general solutions to the inhomogeneous Cauchy-Riemann equation. Holomorphic functions are defined so that the desired solution of system of differential equations (8) satisfies given boundary conditions. At the present time, existence theorems of solutions of nonlinear problems for Timoshenko-type shell with rigidly clamped edges [6, 7] and with free edges [8, 9] are obtained. Method developed in [6–9] was applied to system (8) with boundary conditions $w_1 = w_3 = \psi_1 = 0$ that describe the state of equilibrium of Timoshenko-type shell with simply supported edges [10]. The study presented in this paper develops results obtained in [10]. The more complicated case of boundary conditions $w_1 = \psi_1 = 0$ is considered. These conditions describe elastic bearing against transverse deflection.

Consider boundary-value problem A in a generalized formulation. Let the following conditions hold true:

a) Ω is a simply connected domain with the boundary $\Gamma \in C^1_\beta$ (see, for example, [11, p. 23]);

b) external forces $R^j(j = \overline{1,3})$, $L^k(k = 1,2) \in L_p(\Omega)$, $N^2, P^2, P^3 \in C_\beta(\Gamma)$; in what follows $p > 2$, $0 < \beta < 1$.

Definition 1. *The vector of generalized displacements $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W^{(2)}_p(\Omega)$, $p > 2$, is a generalized solution to the problem A if the vector satisfies almost everywhere the equations of system (8) and it satisfies boundary conditions (2)–(5) in pointwise fashion.*

Here $W^{(2)}_p(\Omega)$ is a Sobolev space. Let us note that due to embedding theorems for Sobolev spaces $W^{(2)}_p(\Omega)$ with $p > 2$, the generalized solution a belongs to $C^1_\alpha(\overline{\Omega})$. In what follos $\alpha = (p-2)/p$.

2. Solution to problem A with respect to tangential displacements and angles of rotation

Let us consider the first two equations in (8) and initially assume that w_3 is fixed. In terms of the complex function $\omega = w_{1\alpha^1} + w_{2\alpha^2} + i\mu_1(w_{2\alpha^1} - w_{1\alpha^2})$ these equations can be represented in the form

$$\omega_{\overline{z}} = f, \tag{10}$$

where $f = (f_1 + if_2)/2$, $\omega_{\overline{z}} = (\omega_{\alpha^1} + i\omega_{\alpha^2})/2$, $z = \alpha^1 + i\alpha^2$.

Equation (10) is an inhomogeneous Cauchy–Riemann equation. It has general solution [11]:

$$\omega(z) = \Phi_1(z) + Tf(z), \quad Tf = -\frac{1}{\pi}\iint_\Omega \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \ \zeta = \xi + i\eta, \tag{11}$$

where $\Phi_1(z)$ is an arbitrary holomorphic function that belongs to the space $C_\alpha(\overline{\Omega})$.

It is well-known [11, pp. 41, 53] that Tf is a completely continuous operator which acts in $L_p(\Omega)$, $p > 2$, $C^k_\alpha(\overline{\Omega})$. It maps these spaces into $C_\alpha(\overline{\Omega})$ and $C^{k+1}_\alpha(\overline{\Omega})$, respectively. Besides, there exist the generalized derivatives [11, pp. 33–34, 53–67]

$$\frac{\partial Tf}{\partial \overline{z}} = f, \quad \frac{\partial Tf}{\partial z} \equiv Sf = -\frac{1}{\pi}\iint_\Omega \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta, \tag{12}$$

where the integral exists in the principal value sense of Cauchy (almost everywhere when $f \in L_p(\Omega), p > 1$) and Sf is a linear bounded operator in $L_p(\Omega)$, $C^k_\alpha(\overline{\Omega})$.

With the function $\omega_0(z) = w_2 + iw_1$, relation (11) can be also rewritten in the form of an inhomogeneous Cauchy–Riemann equation

$$\omega_{0\overline{z}} = i(d_1\omega + d_2\overline{\omega}) \equiv id[\omega], \quad d_j = (\mu_1 + (-1)^j)/(4\mu_1), \ j = 1, 2, \tag{13}$$

The general solution of this equation is

$$\omega_0(z) = \Phi_2(z) + iTd[\Phi_1 + Tf](z), \tag{14}$$

where Φ_2 is an arbitrary holomorphic function of the class $C^1_\alpha(\overline{\Omega})$.

Thus, for fixed w_3 the general solution of the two first equations (8) is of the form (14) and contains two arbitrary holomorphic functions $\Phi_j(z), j = 1, 2$. We define these functions so that tangential displacements w_1 and w_2 will satisfy boundary conditions (2), (3). First, we find $\Phi_2(z)$ from the condition $w_1 = 0$ on Γ. We have a Rimann-Hilbert problem with the boundary condition $\mathrm{Re}[i\Phi_2(t)] = \mathrm{Re}Td[\omega](t)$, $t \in \Gamma$ for the holomorphic function $\Phi_2(z)$. Second, we assume that domain Ω is the unit disk: $|z| \leqslant 1$. Then the solution of the Riemann-Hilbert problem has the form [12]

$$\Phi_2(z) = -\frac{1}{2\pi}\int_\Gamma \mathrm{Re}Td\left[\Phi_1 + Tf\right](t)\frac{t+z}{t-z}\frac{dt}{t} + c_0, \quad z \in \Omega, \tag{15}$$

where c_0 is a arbitrary real constant.

We define the holomorphic function $\Phi_1(z)$ with the use of boundary condition (3). Let us represent this condition in terms of displacements:

$$\mu_1(w_{1\alpha^2} + w_{2\alpha^1})(t)d\alpha^2/ds - (\mu w_{1\alpha^1} + w_{2\alpha^2})(t)d\alpha^1/ds = \varphi(w_3)(t), \tag{16}$$

$$t = t(s) = \alpha^1(s) + i\alpha^2(s) \in \Gamma,$$

where

$$\begin{gathered}\varphi(t) \equiv \varphi(w_3)(t) = \beta_2 P^2(s) + \left[(\mu w^2_{3\alpha^1} + w^2_{3\alpha^2})/2 - \mu k_1 w_3 - k_2 w_3\right] d\alpha^1/ds- \\ -\mu_1 w_{3\alpha^1}w_{3\alpha^2}d\alpha^2/ds.\end{gathered} \tag{17}$$

We substitute relations for the tangential displacements w_1, w_2 from (14) into (16). Taking into account (10), (11) and (14), we obtain

$$w_{j\alpha^j} = \mathrm{Re}\{\Phi_1(z) + Tf(z)\}/2 - (-1)^j\mathrm{Im}\{\Phi_2'(z) + iSd[\Phi_1 + Tf](z)\}, \ j = 1, 2, \tag{18}$$

$$w_{1\alpha^2} + w_{2\alpha^1} = 2\mathrm{Re}\{\Phi_2'(z) + iSd[\Phi_1 + Tf](z)\}.$$

Hence, boundary conditions (16) take the form

$$\mathrm{Re}\{t\Phi_2'(t)\} + \mathrm{Re}\{itSd[\Phi_1]^+(t)\} - \mu_3 d\alpha^1/ds \ \mathrm{Re}\Phi_1(t) = \varphi(t)/(1-\mu) + h_1 f(t), \ t \in \Gamma, \tag{19}$$

where

$$h_1 f(t) = \mathrm{Im}\{tSd[Tf]^+(t)\} + \mu_3 d\alpha^1/ds \ \mathrm{Re}Tf(t), \ \mu_3 = (1+\mu)/(2(1-\mu)); \tag{20}$$

the symbol $\Psi^+(t)$ means the limit of the function $\Psi(z)$ as $z \to t \in \Gamma$ from the interior of the domain Ω.

Let us transform relation (19). Representing holomorphic in the domain Ω function $\Phi_1(z)$ by the Cauchy integral and using (4.7), (8.8a) from [11], we have

$$Sd[\Phi_1]^+(t) = d_1\overline{t}^2[\Phi_1(t) - \Phi_1(0)], \tag{21}$$

where constant d_1 is defined in (13).

Further, we differentiate relation (15) with respect to z and use (13), (11) for $d[\Phi_1]$, $Td[\Phi_1]$. Rearranging the order of integration in the repeated integrals and applying the Cauchy theorem and formula, we have

$$\Phi_2'(z) = (-i)\{d_1\overline{\Phi_1(0)} + d_2\Phi_1(z) + 2S_\Gamma(\mathrm{Re}Td[Tf])(z)\}, \quad z \in \Omega, \tag{22}$$

$$S_\Gamma f(z) \equiv \frac{\partial T_\Gamma f(z)}{\partial z} = \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)}{(\tau - z)^2}d\tau.$$

In the limit $z \to \Gamma$ taken in (22) from the interior of the domain Ω, we find

$$\Phi_2'(t) = (-i)\{d_1\overline{\Phi_1(0)} + d_2\Phi_1(t) + 2(S_\Gamma(\mathrm{Re}Td[Tf]))^+(t)\}, \quad t \in \Gamma, \tag{23}$$

where constants d_k are defined in (13).

Now we substitute (21) and (23) in (19). Then we obtain the Riemann-Hilbert problem for function $\Phi_1(z)$ in Ω with the boundary condition

$$\mathrm{Re}[it\Phi_1(t)] = h[f;\varphi](t), \quad t \in \Gamma, \tag{24}$$

where

$$h[f;\varphi](t) = (\mu - 1)[h_1 f(t) + 2\mathrm{Re}\{it(S_\Gamma(\mathrm{Re}Td[Tf]))^+(t)\}] - \varphi(t); \tag{25}$$

operators $h_1 f$, $S_\Gamma g$ are defined in (20), (22).

The index of problem (23) equals -1. Therefore, the solution of this problem is [12]

$$\Phi_1(z) = -\frac{1}{\pi}\int_\Gamma \frac{h[f;\varphi](t)}{t - z}\frac{dt}{t} \equiv \Phi_1[f;\varphi](z), \tag{26}$$

and the solvability condition

$$\int_\Gamma \frac{h[f;\varphi](t)}{t}dt = 0 \tag{27}$$

of problem (24) should be fulfilled.

We substitute expression (26) into (15) and (22) to obtain

$$\begin{gathered}\Phi_2(z) = -\frac{1}{2\pi}\int_\Gamma (\mathrm{Re}Td[\Phi_1[f;\varphi]](t) + \mathrm{Re}Td[Tf](t))\frac{t+z}{t-z}\frac{dt}{t} + c_0 \equiv \Phi_2[f;\varphi](z) + c_0,\\ \Phi_2'(t) = (-i)\{d_1\overline{\Phi_1[f;\varphi](0)} + d_2\Phi_1[f;\varphi](z) + 2S_\Gamma(\mathrm{Re}Td[Tf])(z)\} \equiv \Phi_2'[f;\varphi](z), \ z \in \Omega.\end{gathered} \tag{28}$$

Consider tangential displacements w_1 and w_2 that satisfy the first two equations (8) and conditions (2), (3). Upon substituting (26), (28) into (14) and assuming that condition (27) is true, we obtain

$$\begin{gathered}\omega_0(z) = H_0 w_3 + c_0,\\ H_0 w_3 \equiv H_0[f(w_3);\varphi(w_3)] = \Phi_2[f;\varphi](z) + iTd[\Phi_1[f;\varphi] + Tf](z).\end{gathered} \tag{29}$$

Let us obtain integral representations for the derivatives of w_1 and w_2 (up to second order inclusively). Using (11) and (18), we find

$$\begin{gathered}w_{j\alpha^j} = \mathrm{Re}\{\Phi_1[f;\varphi] + Tf\}/2 - (-1)^j\mathrm{Im}\{\Phi_2'[f;\varphi] + iSd[\Phi_1[f;\varphi] + Tf]\} \equiv\\ \equiv H_{jj}[f(w_3);\varphi(w_3)] \equiv H_{jj}w_3,\\ w_{j\alpha^k} = \mathrm{Re}\{\Phi_2'[f;\varphi] + iSd[\Phi_1[f;\varphi] + Tf]\} + (-1)^j Im\{\Phi_1[f;\varphi] + Tf\}/(2\mu_1) \equiv\\ \equiv H_{jk}[f(w_3);\varphi(w_3)] \equiv H_{jk}w_3, \quad j,k = 1,2;\end{gathered} \tag{30}$$

$f \equiv f(w_3), \varphi \equiv \varphi(w_3)$ are defined in (9), (17).

Upon differentiating relation (13) with respect to $z, \overline{z}$, we obtain

$$\omega_{0\overline{z}z} = i\left\{d_1(\Phi_1'[f;\varphi] + Sf) + d_2\overline{f(w_3)}\right\} \equiv P_1[f(w_3);\varphi(w_3)] \equiv P_1 w_3, \tag{31}$$

$$\omega_{0\overline{z}\overline{z}} = i\left\{d_1 f(w_3) + d_2(\overline{\Phi_1'[f;\varphi]} + \overline{Sf})\right\} \equiv P_2[f(w_3);\varphi(w_3)] \equiv P_2 w_3.$$

Using formula (8.20) from [11], we obtain

$$Sd[\Phi_1 + Tf](z) = T_\Gamma(d[\Phi_1 + Tf]/t^2)(z) + T(d_1[\Phi_1' + Sf] + d_2\overline{f})(z). \tag{32}$$

Now we differentiate (14) two times with respect to z. Taking into account (32), we have

$$\begin{gathered}\omega_{0zz} = \Phi_2''[f;\varphi] + iS_\Gamma\left\{d[\Phi_1[f;\varphi] + Tf]\overline{t^2}\right\} + \\ +iS\{d_1(\Phi_1'[f;\varphi] + Sf) + d_2\overline{f(w_3)}\} \equiv P_3[f(w_3);\varphi(w_3)] \equiv P_3 w_3.\end{gathered} \tag{33}$$

We use the following designations in (31), (33):

$$\Phi_1'[f;\varphi](z) = -\frac{1}{\pi}\int_\Gamma \frac{h[f;\varphi](t)}{(t-z)^2 t}dt; \quad \Phi_2''[f;\varphi](z) = (-i)\{d_2\Phi_1'[f;\varphi](z) + 2S_\Gamma'\mathrm{Re}Td[Tf](z)\}, \tag{34}$$

$$\begin{gathered}S_\Gamma'\mathrm{Re}Td[Tf](z) = \{d_1\overline{f(z)} + d_2 Sf(z) - S(d_1\overline{Sf} + d_2 f)(z) - S_\Gamma\{\overline{T(d_1 Sf + d_2\overline{f})}\overline{t}^2\}(z) - \\ -S_\Gamma\overline{d[Tf]}(z)/2 - S_\Gamma\{\overline{T_\Gamma(d[Tf]\overline{\tau}^2)}\}(z)\}/2, \quad T_\Gamma f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}dt.\end{gathered}$$

One can express derivatives

$$w_{j\alpha^k\alpha^k} = \mathrm{Im}\{i^{j-1}[2\omega_{0z\overline{z}} + (-1)^{k-1}(\omega_{0zz} + \omega_{0\overline{z}\overline{z}})]\}, \; w_{j\alpha^1\alpha^2} = \mathrm{Re}[i^{j-1}(\omega_{0zz} - \omega_{0\overline{z}\overline{z}})], \; j,k = 1,2$$

in terms of $\omega_{0z\overline{z}}$, $\omega_{0\overline{z}\overline{z}}$, ω_{0zz}.

Lemma 1. *Let conditions a), b) in Section 1 be fulfilled. Then 1) $P_j w_3 (j = \overline{1,3})$ are nonlinear bounded operators acting from $W_p^{(2)}(\Omega)$ to $L_p(\Omega)$, $2 < p < 2/(1-\beta)$; 2) $H_{jk} w_3 (j,k = 1,2)$ are nonlinear completely continuous operators acting from $W_p^{(2)}(\Omega)$ to $L_p(\Omega)$, $2 < p < 2/(1-\beta)$ and $H_0 w_3$ is nonlinear bounded operator acting from $W_p^{(2)}(\Omega)$ to $C_\alpha(\overline{\Omega})$, $C_\alpha^1(\overline{\Omega})$. For any $w_3^j (j = 1,2) \in W_p^{(2)}(\Omega)$ the following estimates hold*

$$\begin{gathered}\|P_j w_3^1 - P_j w_3^2\|_{L_p(\Omega)}, \|H_{jk} w_3^1 - H_{jk} w_3^2\|_{C_\alpha(\overline{\Omega})}, \|H_0 w_3^1 - H_0 w_3^2\|_{C_\alpha^1(\overline{\Omega})} \leqslant \\ \leqslant c(1 + \|w_3^1\|_{W_p^{(2)}} + \|w_3^2\|_{W_p^{(2)}})\|w_3^1 - w_3^2\|_{W_p^{(2)}}.\end{gathered} \tag{35}$$

Proof. Let us note that $f(w_3)$ defined in (9) is nonlinear bounded operator acting from $W_p^{(2)}(\Omega)$ to $L_p(\Omega)$, $p > 2$ and estimate (35) is true. Using formulas (6.10) from [11] and Sokhotski formulas [12], we obtain

$$\begin{gathered}S_\Gamma\mathrm{Re}Td[Tf](z) = \{\overline{d[Tf](z)} - T(d_1\overline{Sf} + d_2 f)(z) - T_\Gamma\{\overline{T(d_1 Sf + d_2\overline{f})}\overline{t}^2\}(z) - \\ -T_\Gamma\overline{d[Tf]}(z)/2 - T_\Gamma\{\overline{T_\Gamma(d[Tf]\overline{\tau}^2)}\}(z)\}/2.\end{gathered} \tag{36}$$

It is known that if condition *b)* is fulfilled then $T_\Gamma g$ and $S_\Gamma g$ are linear bounded operators acting from $C_\beta(\Gamma)$ to $C_\beta(\overline{\Omega})$ and to $L_p(\Omega)$ $(2 < p < 2/(1-\beta))$, respectively [11, pp. 26–27]. Besides, using formula (6.10) from [11], it can be shown that $S_\Gamma g$ is a linear bounded operator acting from

$W_p^{(1)}(\Omega)$ to $L_p(\Omega)$, $p > 2$. Taking into account this fact and (32), (34), (36), one can obtain that $Sd[Tf](z)$, $S_\Gamma ReTd[Tf](z) \in W_p^{(1)}(\Omega)$, $p > 2$. Taking into account $h_1 f$ in (20) and $\varphi(w_3)$ in (17), we obtain from (25) that $h[f;\varphi](t) = -\beta_2 P^2(s) + \widetilde{h}(t)$. Here $P^2(s) \in C_\beta(\Gamma)$ and $\widetilde{h}(t)$ is the boundary value of the function that belongs to the space $W_p^{(1)}(\Omega)$, $p > 2$. Therefore, from (34), (28), (26) we have 1) $\Phi_1'[f(w_3);\varphi(w_3)], \Phi_2''[f(w_3);\varphi(w_3)]$ are nonlinear bounded operators acting from $W_p^{(2)}(\Omega)$ to $L_p(\Omega), 2 < p < 2/(1-\beta)$; 2) $\Phi_1[f(w_3);\varphi(w_3)], \Phi_2'[f(w_3);\varphi(w_3)]$ are nonlinear completely continuous operators acting from $W_p^{(2)}(\Omega)$ to $L_p(\Omega), 2 < p < 2/(1-\beta)$ and they are nonlinear bounded operators acting from $W_p^{(2)}(\Omega)$ to $C_\alpha(\overline{\Omega})$ These operators satisfy estimates (35) (here $\min(\alpha,\beta) = \alpha$, when $(2 < p < 2/(1-\beta))$). Lemma 1 follows from (29)–(31) and (33).

Let us consider conditions of solvability (27). With (32) and (36) we find $Sd[Tf]^+(t)$, $S_\Gamma \mathrm{Re}Td[Tf]^+(t)$. Taking into account expressions for operators $h_1 f$, Tf, Sf in (20), (11), (12) and holomorphic function $T_\Gamma g(z)$, we obtain

$$\int_\Gamma \frac{h_1 f(t)}{t} dt = -\frac{i}{1-\mu} \iint_\Omega f_2(w_3)(z) d\alpha^1 d\alpha^2, \int_\Gamma \frac{\mathrm{Re}\{it(S_\Gamma \mathrm{Re}Td[Tf])^+(t)\}}{t} dt = 0. \tag{37}$$

Next using $f_2(w_3)$ in (9), we have

$$\begin{gathered}\iint_\Omega f_2(w_3)(z) d\alpha^1 d\alpha^2 = \int_\Gamma \{(\mu w_{3\alpha^1}^2/2 + w_{3\alpha^2}^2/2 - k_4 w_3) d\alpha^1/ds - \\ -\mu_1 w_{3\alpha^1} w_{3\alpha^2} d\alpha^2/ds\} ds - \beta_2 \iint_\Omega R^2 d\alpha^1 d\alpha^2.\end{gathered} \tag{38}$$

Upon substituting (25), (37), (38) into (27), the condition of solvability take the final form

$$\int_\Gamma P^2(s) ds + i \int_\Omega R^2 d\alpha^1 d\alpha^2 = 0, \tag{39}$$

where $P^2(s)$ and $R^2(\alpha^1;\alpha^2)$ are the components of external load.

We now turn to functions ψ_k $(k = 1,2)$ in the last two equations (8). These functions should satisfy boundary conditions (2), (5). Taking into account expressions for moments M^{jk} in (6), we write boundary condition (5) in the form

$$\mu_1(\psi_{1\alpha^2} + \psi_{2\alpha^1})(t) d\alpha^2/ds - (\mu\psi_{1\alpha^1} + \psi_{2\alpha^2})(t) d\alpha^1/ds = \widetilde{\varphi}(t), \quad \widetilde{\varphi}(t) = \beta_1 N^2(s); \tag{40}$$

β_1 is defined in (9), $N^2(s)$ is the component of the external forces.

Let us note that the structure of left-hand sides in the last two equations (8) coincides with the structure of left-hand sides in boundary conditions (2) and (40). Relations for tangential displacements differ only in the right-hand sides. Therefore at fixed right-hand sides for rotation angles we obtain

$$\psi = \psi_2 + i\psi_1 = H_0[g(\upsilon);\widetilde{\varphi}] + c_1, \tag{41}$$

where

$$\begin{gathered}\upsilon = \upsilon_2 + i\upsilon_1, \quad g(\upsilon) = (g_1(\upsilon) + i g_2(\upsilon))/2, \\ \upsilon_j = w_{3\alpha^j} + \psi_j, \quad g_j(\upsilon) = k_0 \upsilon_j - \beta_1 L^j, j = 1,2;\end{gathered} \tag{42}$$

c_1 an arbitrary real constant, operator $H_0[g(\upsilon);\widetilde{\varphi}]$ is defined in (29).

As this takes place, the condition of solvability is

$$\int_\Gamma \frac{h[g;\widetilde{\varphi}](t)}{t} dt = 0,$$

where $h[g;\widetilde{\varphi}](t)$ is given in (25). These conditions can be reduced to the form

$$\int_\Gamma \{N^2 + [k_1(\alpha^1)^2 - k_2(\alpha^2)^2]P^2/2 - k_1\alpha^1\alpha^2 T^1(a) - \alpha^2 P^3\}ds + \iint_\Omega \{L^2 + [k_1(\alpha^1)^2 - \\ - k_2(\alpha^2)^2 R^2/2 - k_1\alpha^1\alpha^2 R^1 - \alpha^2 R^3\}d\alpha^1 d\alpha^2 + \int_\Gamma P^2 w_3 ds + \iint_\Omega R^2 w_3 d\alpha^1 d\alpha^2 = 0, \tag{43}$$

where $T^1(a) = T^{11}(a)d\alpha^2/ds - T^{12}(a)d\alpha^1/ds$ ($T^{ij}(a)$ are defined in (6)); N^2, L^2, P^k $(k = 2, 3)$, R^j $(j = \overline{1, 3})$ are components of external load.

In a similar way to (30) we obtain the following relations

$$\psi_{j\alpha^k} = H_{jk}[g;\widetilde{\varphi}], \quad j, k = 1, 2, \tag{44}$$

where operators H_{jk} are defined in (30).

Lemma 2. *Let conditions a), b) in Section 1 be fulfilled. Then $H_{jk}[g(v);\widetilde{\varphi}]$ $(j, k = 1, 2)$ and $H_0[g(v);\widetilde{\varphi}]$ are linear completely continuous operators with respect to v that act from $W_p^{(1)}(\Omega)$ to $L_p(\Omega)$, $2 < p < 2/(1-\beta)$ and they are linear continuous operators that act from $W_p^{(1)}(\Omega)$ to $C_\alpha(\overline{\Omega})$ and to $C^1_\alpha(\overline{\Omega})$, respectively.*

Taking into account expressions for $g(v)$ in (42) and indicated above properties of operators $Tf, Sf, T_\Gamma f, S_\Gamma f$, we obtain from (29), (30) that Lemma 2 is true.

Problem A at fixed w_3, v_j $(j = 1, 2)$ is solvable with respect to tangential displacements and rotation angles under conditions (39) and (43). Solution of this problem is described in (29) and (41).

In conclusion of Section 2 we represent relationships (41) and (44) in the form convenient for further analysis. First of all we obtain relations for $\widetilde{\varphi}$ from (40) and for $g(v)$ from (42):

$$\widetilde{\varphi} = \widetilde{\varphi}^0 + \widetilde{\varphi}^1, \quad g(v) = g^0 + g^1(v), \quad \widetilde{\varphi}^0 = \beta_1 N^2(s), \quad \widetilde{\varphi}^1 = 0, \tag{45}$$

$$g^k = (g_{1k} + ig_{2k})/2, \quad k = 0, 1, \quad g_{j0} = -\beta_1 L^j, \quad g_{j1} = k_0 v_j, \quad j = 1, 2.$$

Let us note that $g^j(v)$ are homogenous operators of order j with respect to v.

Now if we substitute (45) into (41) and (44), we arrive at the desired representations for rotation angles and their derivatives

$$\psi \equiv \psi(v) = \psi^0 + \psi^1(v) + c_1, \quad \psi_{j\alpha^k} \equiv \psi_{j\alpha^k}(v) = \psi_{j0\alpha^k} + \psi_{j1\alpha^k}(v), \tag{46}$$

$$\psi^n(v) = \psi_{2n}(v) + i\psi_{1n}(v) = H_0[g^n(v);\widetilde{\varphi}^n], \quad \psi_{jn\alpha^k}(v) = H_{jk}[g^n(v);\widetilde{\varphi}^n], \ j, k = 1, 2, \ n = 0, 1.$$

It is easy to see that $\psi^n(v), \psi_{jn\alpha^k}(v)$ are homogenous operators of order n with respect to v.

3. Reduction of system (8) to a single equation and solvability analysis.

Before considering the third equation in (8) we express the deflection w_3 and its derivatives in terms of $v_j (j = 1, 2)$. Taking into account (42) and (46), we obtain

$$w_{3\alpha^j} \equiv w_{3\alpha^j}(v) = w_{30\alpha^j} + w_{31\alpha^j}(v) - (j-1)c_1, \tag{47}$$

$$w_{30\alpha^j} = -\psi_{j0}, \quad w_{31\alpha^j}(v) = v_j - \psi_{j1}(v), \quad j = 1, 2.$$

Using (47), we derive

$$w_3 \equiv w_3(v) = w_{30} + w_{31}(v) - c_1\alpha^2 + c_2, \tag{48}$$

$$w_{30} = -\int_{(0,0)}^{(\alpha^1,\alpha^2)} \psi_{10}d\alpha^1 + \psi_{20}d\alpha^2, \quad w_{31}(\upsilon) = \int_{(0,0)}^{(\alpha^1,\alpha^2)} [\upsilon_1 - \psi_{11}(\upsilon)]d\alpha^1 + [\upsilon_2 - \psi_{21}(\upsilon)]d\alpha^2.$$

Upon substituting expressions (47), (48) into (9), (17) and then the resulting expression into (29) and (30), we obtain the decomposition of tangential displacements and their derivatives into linear and nonlinear operators

$$\omega_0 \equiv \omega_0(\upsilon) = \omega_{01}(\upsilon) + \omega_{02}(\upsilon) + \omega_0^{\bullet}, \tag{49}$$

$$w_{j\alpha^k} \equiv w_{j\alpha^k}(\upsilon) = w_{j1\alpha^k}(\upsilon) + w_{j2\alpha^k}(\upsilon) + w^{\bullet}_{j\alpha^k}, \quad j = 1,2,$$

where

$$\omega_{0j}(\upsilon) = w_{2j}(\upsilon) + iw_{1j}(\upsilon) = H_0[f^j(\upsilon);\varphi^j(\upsilon)], \quad w_{jn\alpha^k}(\upsilon) = H_{jk}[f^n(\upsilon);\varphi^n(\upsilon)], \tag{50}$$

$$\omega_0^{\bullet} = w_2^{\bullet} + iw_1^{\bullet} = H_0[f^{\bullet};\varphi^{\bullet}] + c_0, \quad w^{\bullet}_{j\alpha^k} = H_{jk}[f^{\bullet};\varphi^{\bullet}], \quad j,k,n = 1,2,$$

$$f^j(\upsilon) = [f_{1j}(\upsilon) + if_{2j}(\upsilon)]/2, \quad f_{j1}(\upsilon) = k_{2+j}w_{31\alpha^j}(\upsilon), \quad f_{j2}(\upsilon) = k_{2+j}w_{30\alpha^j} - \beta_2R^j -$$
$$-w_{3\alpha^j}(\upsilon)w_{3\alpha^j\alpha^j}(\upsilon) - \mu_2 w_{3\alpha^{3-j}}(\upsilon)w_{3\alpha^1\alpha^2}(\upsilon) - \mu_1 w_{3\alpha^j}(\upsilon)w_{3\alpha^{3-j}\alpha^{3-j}}(\upsilon), \quad j = 1,2,$$
$$f^{\bullet} = -ic_1k_4/2, \quad \varphi^1(\upsilon) = -k_4w_{31}(\upsilon)d\alpha^1/ds, \quad \varphi^2(\upsilon) = \beta_2P^2(s) + \{-k_4w_{30} + \mu(w_{30\alpha^1}+$$
$$+w_{31\alpha^1})^2/2 + (w_{30\alpha^2} + w_{31\alpha^2})^2/2 - c_1(w_{30\alpha^2} + w_{31\alpha^2})\}d\alpha^1/ds - \mu_1\{w_{3\alpha^1}(w_{30\alpha^2}+$$
$$+w_{31\alpha^2}) - c_1(w_{30\alpha^1} + w_{31\alpha^2})\}d\alpha^2/ds, \quad \varphi^{\bullet} = (c_1^2/2 + c_1k_4\alpha^2 - k_4c_2)d\alpha^1/ds,$$

operators $H_0[f;g]$, $H_{jk}[f;g]$, $(j,k = 1,2)$ are defined in (29), (30).

After some cumbersome mathematical treatment one can derive the explicit expression

$$\omega_0^{\bullet} = -c_1k_4(\alpha^2)^2/2 + (c_2k_4 - c_1^2/2)\alpha^2 + c_1k_4/4 + c_0. \tag{51}$$

Now we turn to the third equation in (8). Replacing generalized displacements by relations (46)–(49), we reduce the third equation to the equivalent system with respect to $\upsilon = \upsilon_2 + i\upsilon_1$:

$$\partial\upsilon/\partial\bar{z} = [\psi_{2\alpha^1}(\upsilon) - \psi_{1\alpha^2}(\upsilon) + if_3(\upsilon)]/2 \equiv f_0(\upsilon), \tag{52}$$

$$f_3(\upsilon) \equiv f_3(w_3(\upsilon)) = -\{k_3w_{1\alpha^1}(\upsilon) + k_4w_{2\alpha^2}(\upsilon) - k_5w_3(\upsilon) + k_3w_{3\alpha^1}^2(\upsilon)/2 + k_4w_{3\alpha^2}^2(\upsilon)/2+$$
$$+\beta_2[T^{\lambda\mu}(\upsilon)w_{3\alpha^\lambda}(\upsilon)]_{\alpha^\mu} + \beta_2R^3\}/(k^2\mu_1), \ T^{\lambda\mu}(\upsilon) \equiv T^{\lambda\mu}(a(\upsilon)) \ (\lambda,\mu = 1,2).$$

Boundary condition (4) is transformed to

$$\upsilon_1 d\alpha^2/ds - \upsilon_2 d\alpha^1/ds = \varphi_0(\upsilon)(t), \ t \in \Gamma, \tag{53}$$

$$\varphi_0(\upsilon)(t) \equiv \varphi_0(w_3(\upsilon))(t) = \beta_3[P^3(s) - T^{11}(\upsilon)w_{3\alpha^1}(\upsilon)d\alpha^2/ds + T^{22}(\upsilon)w_{3\alpha^2}(\upsilon)d\alpha^1/ds-$$
$$-T^{12}(\upsilon)(w_{3\alpha^2}(\upsilon)d\alpha^2/ds - w_{3\alpha^1}(\upsilon)d\alpha^1/ds)], \ \beta_3 = 2(1+\mu)/(k^2Eh).$$

So, problem A is now to find solution to equation (52) under boundary condition (53).

Equivalent form of equation (52) is

$$\upsilon = \Phi(z) + Tf_0(\upsilon)(z), \tag{54}$$

where $\Phi(z)$ is an arbitrary holomorphic function of the class $C_\alpha(\overline{\Omega})$ and operator Tf is defined in (11).

We define the holomorphic function $\Phi(z)$ so that the function υ from (54) satisfy (53). We assume for the time being that $\varphi_0(\upsilon)$, $f_0(\upsilon)$ in the right-hand sides of (53), (54) are fixed. Substituting (54) into (53), we obtain the Riemann-Hilbert problem for $\Phi(z)$ in the unit disk.

The boundary condition for this problem is $\mathrm{Re}[(-i)t\Phi(t)] = l(v)(t),\ t \in \Gamma$. The solution of this problem is

$$\Phi(z) \equiv \Phi[l(v)](z) = \frac{1}{\pi}\int_\Gamma \frac{l(v)(t)}{t-z}\frac{dt}{t}, \tag{55}$$

where $l(v)(t)$ should satisfy the condition

$$\int_\Gamma \frac{l(v)(t)}{t}dt = 0, \quad l(v)(t) = \varphi_0(v)(t) + \mathrm{Re}[itTf_0(v)(t)].$$

This condition can be represented in the form

$$\int_\Gamma (k_1\alpha^1 T^1(a) + k_2\alpha^2 P^2 + P^3)ds + \iint_\Omega (k_1\alpha^1 R^1 + k_2\alpha^2 R^2 + R^3)d\alpha^1 d\alpha^2 = 0, \tag{56}$$

where $T^1(a)$ is defined in (43), $P^k (k = 1,2)$ and $R^j (j = \overline{1,3})$ are components of external load.

Substituting (55) into (54), we obtain the following equation for $v \in W_p^{(1)}$, $p > 2$

$$v - \Phi[l(v)] - Tf_0(v) = 0. \tag{57}$$

Now we represent equation (57) in a slightly different form. Taking into account relations (46), (48), (49), (51), we obtain for $f_3(v),\ f_0(v),\ l(v)$ the decompositions into linear and nonlinear terms:

$$f_3(v) = f_{31}(v) + f_{32}(v),\ f_0(v) = f_{01}(v) + f_{02}(v),\ l(v) = l_1(v) + l_2(v), \tag{58}$$

where

$$\begin{gathered} f_{31}(v) = -[k_3 w_{11\alpha^1}(v) + k_4 w_{21\alpha^2}(v) - k_5 w_{31}(v)]/(k^2\mu_1), \\ f_{32}(v) = -[k_3 w_{12\alpha^1}(v) + k_4 w_{22\alpha^2}(v) + k_{2+\lambda}(w_{30\alpha^\lambda} + w_{31\alpha^\lambda}(v))^2/2 - k_5 w_{30} - \\ -k_4 c_1(w_{30\alpha^2} + w_{31\alpha^2}(v)) + \beta_2(T^{\lambda\mu}(v)w_{3\alpha^\lambda}(v))_{\alpha^\mu} + \beta_2 R^3 + k_1^2(1-\mu^2)(c_1\alpha^2 - c_2)]/(k^2\mu_1), \\ f_{01}(v) = [\psi_{21\alpha^1}(v) - \psi_{11\alpha^2}(v) + if_{31}(v)]/2, \quad f_{02}(v) = [\psi_{20\alpha^1}(v) - \psi_{10\alpha^2}(v) + if_{32}(v)]/2, \\ l_1(v) = Re[itTf_{01}(v)], \quad l_2(v) = \varphi_0(v) + Re[itTf_{02}(v)],\ t \in \Gamma. \end{gathered} \tag{59}$$

Let us introduce the following operators

$$Kv = \Phi[l_1(v)] + Tf_{01}(v),\ Gv = \Phi[l_2(v)] + Tf_{02}(v). \tag{60}$$

Then equation (57) takes the form

$$v - Kv - Gv = 0. \tag{61}$$

Let us consider the solvability of equation (61) in the space $W_p^{(1)}(\Omega),\ p > 2$.

Lemma 3. *Let conditions a), b) in Section 1 be fulfilled. Then 1) Kv are linear completely continuous operators in $W_p^{(1)}(\Omega),\ p > 2$; 2) Gv are nonlinear bounded operators in $W_p^{(1)}(\Omega)$, $2 < p < 2/(1-\beta)$ and for any $v^j \in W_p^{(1)}(\Omega)$ $(j = 1,2)$ which belong to the ball $\|v\|_{W_p^{(1)}(\Omega)} < r$, the following estimate takes place*

$$\|Gv^1 - Gv^2\|_{W_p^{(1)}(\Omega)} \leqslant c[q_0 + (1 + \|w_3(0)\|_{W_p^{(2)}(\Omega)} + r)(\|w_3(0)\|_{W_p^{(2)}(\Omega)} + r)]\|v^1 - v^2\|_{W_p^{(1)}(\Omega)},$$

$$q_0 = \sum_{\lambda,\mu=1}^{2} \|T^{\lambda\mu}(0)\|_{C(\overline{\Omega})} + \sum_{\lambda=1}^{2} \|k_{2+\lambda}w_{3\alpha^\lambda}(0) + R^\lambda\|_{L_p(\Omega)},\ T^{\lambda\mu}(0) \equiv T^{\lambda\mu}(a(0)),$$

$a(0) = (w_1(0), w_2(0), w_3(0), \psi_1(0), \psi_2(0)),\ w_j(0)\ (j = \overline{1,3}),\ w_{3\alpha^\lambda}(0),\ \psi_\lambda(0)\ (\lambda = 1,2)$ *are defined in (49), (48), (47), (46) at $v = 0$.*

Lemma 3 follows from (60) and (59), in view of Lemmas 1, 2 and properties of operators $T_\Omega f$, $S_\Omega f$, $T_\Gamma f$ and $S_\Gamma f$.

Consider the homogenous equation

$$\upsilon - K\upsilon = 0. \tag{62}$$

Let $\upsilon \in W_p^{(1)}(\Omega)$, $2 < p < 2/(1-\beta)$ be nonzero solution of equation (62). In view of (46), (48), (50), this solution is associated with the generalized displacements $w_{j1}(\upsilon)$ $(j = \overline{1,3})$, $\psi_{j1}(\upsilon)$ $(j = 1, 2)$ which satisfy the system of linear homogenous equations

$$\begin{gathered} w_{1\alpha^1\alpha^1} + \mu_1 w_{1\alpha^2\alpha^2} + \mu_2 w_{2\alpha^1\alpha^2} - k_3 w_{3\alpha^1} = 0, \\ \mu_1 w_{2\alpha^1\alpha^1} + w_{2\alpha^2\alpha^2} + \mu_2 w_{1\alpha^1\alpha^2} - k_4 w_{3\alpha^2} = 0, \\ k^2\mu_1(w_{3\alpha^1\alpha^1} + w_{3\alpha^2\alpha^2} + \psi_{1\alpha^1} + \psi_{2\alpha^2}) + k_3 w_{1\alpha^1} + k_4 w_{2\alpha^2} - k_5 w_3 = 0, \\ \psi_{1\alpha^1\alpha^1} + \mu_1\psi_{1\alpha^2\alpha^2} + \mu_2\psi_{2\alpha^1\alpha^2} - k_0(w_{3\alpha^1} + \psi_1) = 0, \\ \mu_1\psi_{2\alpha^1\alpha^1} + \psi_{2\alpha^2\alpha^2} + \mu_2\psi_{1\alpha^1\alpha^2} - k_0(w_{3\alpha^2} + \psi_2) = 0 \end{gathered} \tag{63}$$

and homogenous static boundary conditions (2) and (16) with $\varphi(t) = 0$, boundary conditions (40) with $\widetilde{\varphi}(t) = 0$ and boundary conditions (53) with $\varphi_0(t) = 0$. We multiply equalities (63) by w_{11}, w_{21}, w_{31}, ψ_{11}, ψ_{21}, integrate the resulting relations over the domain Ω, and add up the result of integration. Then upon integrating by parts the resulting relation and taking into account boundary conditions, we obtain $\upsilon_j = 0$, $j = 1, 2$, i.e., $\upsilon = 0$ in $\overline{\Omega}$. Therefore, equation (62) has only zero solution in $W_p^{(1)}(\Omega)$, $2 < p < 2/(1-\beta)$. Thus, there exists the inverse operator $(I-K)^{-1}$ bounded in $W_p^{(1)}(\Omega)$, $2 < p < 2/(1-\beta)$. It reduces equation (61) to the equivalent form

$$\upsilon - G_*\upsilon = 0, \quad G_*\upsilon = (I-K)^{-1}G\upsilon. \tag{64}$$

It follows from the established above properties of the operator $G\upsilon$ that $G_*\upsilon$ is a nonlinear bounded operator in $W_p^{(1)}(\Omega)$, $2 < p < 2/(1-\beta)$. For any $\upsilon^j \in W_p^{(1)}(\Omega)$ $(j = 1, 2)$ which belong to the ball $||\upsilon||_{W_p^{(1)}} < r$, in view of Lemma 3, the following estimate holds

$$||G_*\upsilon^1 - G_*\upsilon^2||_{W_p^{(1)}(\Omega)} \leqslant q_*\|\upsilon^1 - \upsilon^2\|_{W_p^{(1)}(\Omega)},$$

where $q_* = c\|(I-K)^{-1}\|_{W_p^{(1)}(\Omega)}[q_0 + (1 + \|w_3(0)\|_{W_p^{(2)}(\Omega)} + r)(\|w_3(0)\|_{W_p^{(2)}(\Omega)} + r)]$.

Let us assume that the radius r of the ball and the external forces exerted on the shell are such that the following conditions hold

$$q_* < 1, \quad \|G_*(0)\|_{W_p^{(1)}(\Omega)} < (1-q_*)r, \tag{65}$$

where $G_*(0)$ is given by relations that follow from (53), (59), (60) at $\upsilon = 0$.

Let us note that to fulfill conditions (65) it is enough, for example, to require that the external load and the radius of the ball are sufficiently small.

Under these conditions we can apply the principle of contracting mappings to equation (64) [13]. According this principle equation (64) has the unique solution $\upsilon \in W_p^{(1)}(\Omega)$, $2 < p < 2/(1-\beta)$ in the ball $||\upsilon||_{W_p^{(1)}} < r$. This solution can be represented in the form $\upsilon = \Re G_*(0)$, where $\Re$ is the resolvent operator $G_*(\upsilon) - G_*(0)$.

Using $\upsilon = \Re G_*(0)$, (46), (48) and (49), we obtain the generalized displacements $w_j \in W_p^{(2)}(\Omega)$ $(j = \overline{1,3})$, $\psi_j \in W_p^{(2)}(\Omega)$ $(j = 1, 2)$, $2 < p < 2/(1-\beta)$. Finally we obtain the generalized solution $a = (w_1, w_2, w_3, \psi_1, \psi_2)$ of problem A. It can be represented in the form $a = a_0 + a^\bullet$, where $a^\bullet = (0, w_2^\bullet, -c_1\alpha^2 + c_2, 0, c_1)$ ($w_2^\bullet$ is defined in (51)); a_0 is the vector with components $w_{j1}(\upsilon) + w_{j2}(\upsilon)$ $(j = 1, 2)$, $w_{30} + w_{31}(\upsilon)$, $\psi_{j0} + \psi_{j1}(\upsilon)$ $(j = 1, 2)$, that are defined in (50), (48), (46).

Then we substitute the solution $\upsilon = \upsilon_2 + i\upsilon_1 = \Re G_*(0)$ of Eq. (64) into (58), (43). Taking into account relations

$$T^1(a) = T^1(a_0)(\upsilon) + T^1(a^\bullet) + c_1 l_0(\upsilon),$$

$$l_0(\upsilon) = \{(w_{30\alpha^1} + w_{31\alpha^1}(\upsilon))d\alpha^1/ds - \mu(w_{30\alpha^2} + w_{31\alpha^2}(\upsilon))d\alpha^2/ds\}/\beta_2$$

and calculating integrals that contain $T^1(a^\bullet)$, we transform the solvability conditions (58), (43) into the form

$$\begin{gathered}
\int_\Gamma (k_1\alpha^1 T^1(a_0) + k_2\alpha^2 P^2 + P^3)ds + \iint_\Omega (k_1\alpha^1 R^1 + k_2\alpha^2 R^2 + R^3)d\alpha^1 d\alpha^2 + \\
+ c_1 \int_\Gamma k_1\alpha^1 l_0(\upsilon)(s)ds + \pi c_2 k_1^2(\mu^2 - 1)/\beta_2 = 0, \\
\int_\Gamma \{N^2 + [k_1(\alpha^1)^2 - k_2(\alpha^2)^2]P^2/2 - k_1\alpha^1\alpha^2 T^1(a_0) - \alpha^2 P^3\}ds + \iint_\Omega \{L^2 + [k_1(\alpha^1)^2 - \\
- k_2(\alpha^2)^2 R^2/2 - k_1\alpha^1\alpha^2 R^1 - \alpha^2 R^3\}d\alpha^1 d\alpha^2 + \int_\Gamma P^2 w_3 ds + \iint_\Omega R^2 w_3 d\alpha^1 d\alpha^2 - \\
- k_1 c_1 \int_\Gamma \alpha^1\alpha^2 l_0(\upsilon)(s)ds - \pi c_1 k_1^2(1 - \mu^2)/(2\beta_2) = 0.
\end{gathered} \tag{66}$$

Let us note that relation (66) is the system of equations with respect to arbitrary constants c_1 and c_2. Thus, the solvability conditions (58), (43) depend on constants c_1, c_2. Note that at zero external load $c_1 = c_2 = 0$.

Therefore, we obtain the generalized solution of problem A, where components w_1, w_3, ψ_1, ψ_2 are defined uniquely and component w_2 depends on constant c_0.

Condition (39) is not only sufficient but also necessary for the solvability of problem A. Indeed, if $a = (w_1, w_2, w_3, \psi_1, \psi_2)$ is a generalized solution of problem A then, upon integrating by parts second equality in (1) over the domain Ω and taking into account condition (2), we come to condition (39).

Thus we have proved the following basic theorem.

Theorem 1. *Let conditions a), b) in Section 1 be fulfilled and inequality (65) holds. Then geometrically nonlinear boundary value problem for elastic shallow Timoshenko-type shell with simply supported edge is solvable if and only if condition (39) is satisfied. Then the problem has generalized solution* $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$. *Components* w_1, w_3, ψ_1, ψ_2 *are uniquely defined and component* w_2 *depends on constant* c_0.

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Исследование разрешимости одной нелинейной краевой задачи для системы дифференциальных уравнений теории пологих оболочек типа Тимошенко

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Работа посвящена исследованию разрешимости системы нелинейных дифференциальных уравнений с частными производными второго порядка при заданных граничных условиях. Метод исследования заключается в сведении исходной системы уравнений к одному нелинейному операторному уравнению, разрешимость которого устанавливается с помощью принципа сжатых отображений.

Ключевые слова: система нелинейных дифференциальных уравнений, уравнения равновесия, интегральные представления, теорема существования.