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On Interpolation in the Class of Analytic Functions in the Unit Disk with the Nevanlinna Characteristic from Lp-spaces

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In this paper we solve the interpolation problem for the class of analytic functions in the unit disk with the Nevanlinna characteristic from Lp-spaces under the condition that interpolation nodes are contained in a finite union of Stolz angles.

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Introduction

Let $\mathbb{C}$ be the complex plane, D be the unit disk on $\mathbb{C}$, $H(D)$ be the set of all functions, holomorphic in D. For all $0 < p < +\infty$, $\alpha > -1$ we define the class S^p_α as (see [14]):

$$S^p_\alpha := \left\{ f \in H(D) : \int_0^1 (1-r)^\alpha T^p(r,f)dr < +\infty \right\},$$

where $T(r,f) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \ln^+ |f(re^{i\varphi})|d\varphi$ is the Nevanlinna characteristic of the function f, $\ln^+ |a| = \max(0, \ln |a|)$, $a \in \mathbb{C}$ (see [8]).

Note that S^p_α-classes are a natural generalization of the Nevanlinna-Djrbashian classes. In this paper we investigate the questions of interpolation in S^p_α-spaces. In solving the problem of free interpolation, that is, with minimal restrictions imposed on the interpolated function, it is important to find a natural class to which the restriction of on the interpolation set to belong. We denote it l^p_α.

In [9] it was set that if $f \in S^p_\alpha$, then

$$\ln^+ M(r,f) = o\left(\frac{1}{(1-r)^{\frac{\alpha+1}{p}+1}} \right), r \to 1-0, \tag{1}$$

where $M(r,f) = \max\limits_{|z|=r} |f(z)|$.

It is clear that if $f \in S^p_\alpha$ and $\{\alpha_k\}_{k=1}^{+\infty}$ is a sequence of points from the unit disk, then the operator $R(f) = (f(\alpha_1), ..., f(\alpha_k), ...)$ maps the class S^p_α into the class of sequences

$$l^p_\alpha = \left\{ \gamma = \{\gamma_k\}_{k=1}^{+\infty} : \ln^+ |\gamma_k| = o\left(\frac{1}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}} \right), k \to +\infty \right\}.$$

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In this article we answer the question under what conditions on the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ the operator $R(f)$ maps the class S_α^p *onto* the class l_α^p.

Definition 1. *A sequence $\{\alpha_k\}_{k=1}^{+\infty}$ is called interpolating for S_α^p if $R(S_\alpha^p) = l_\alpha^p$.*

Let us note that the solution of interpolation problems in the various classes of analytic functions has been widely discussed by Russian and foreign scientists: A. G Naftalevic [7], H. Shapiro and A. Shields [11], S. A. Vinogradov, V. P. Havin [16], M. Djrbashian [4], N. A. Shirokov and A. M. Kotochigov [6], K. Seip [12], A. Hartmann [5], V. A. Bednazh and F. A. Shamoyan [1] and etc. The fundamental result in this area belongs to the L. Carleson [2]. This work continues the research started in [10] in solving the interpolation problem in the classes of analytic functions in the unit disk with power growth of the Nevanlinna characteristic.

The paper is organized as follows: in the first section we present the formulation of main result of the article and give some auxiliary results, in the second section we present the proof of main result.

1. Formulation of main result and proof of auxiliary results

To formulate and proof the results of the work we introduce some more notation and definitions. For any $\beta > -1$ we denote $\pi_\beta(z, \alpha_k)$ as M. M. Djrbashian's infinite product with zeros at points of the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ (see [3]):

$$\pi_\beta(z, \alpha_k) = \prod_{k=1}^{+\infty} \left(1 - \frac{z}{\alpha_k}\right) \exp(-U_\beta(z, \alpha_k)),$$

where

$$U_\beta(z, \alpha_k) = \frac{2(\beta+1)}{\pi} \int_0^1 \int_{-\pi}^{\pi} \frac{(1-\rho^2)^\beta \ln|1 - \frac{\rho e^{i\theta}}{\alpha_k}|}{(1 - z\rho e^{-i\theta})^{\beta+2}} d\theta \rho d\rho. \tag{2}$$

We denote $\pi_{\beta,n}(z, \alpha_k)$ as infinite product $\pi_\beta(z, \alpha_k)$ without n-th factor. As stated in [3], the infinite product $\pi_\beta(z, \alpha_k)$ is absolutely and uniformly convergent in the unit disk D if and only if the series converges:

$$\sum_{k=1}^{+\infty} (1 - |\alpha_k|)^{\beta+2} < +\infty.$$

Definition 2. *The angle of the $\pi\delta$, $0 < \delta < 1$, contained in D, with vertex at the point $e^{i\theta}$ and with bisector $re^{i\theta}$, $0 \leqslant r < 1$ is said to be the Stolz angle $\Gamma_\delta(\theta)$.*

For any $0 < p < \infty$ we denote by $B_{1,p}^s$ the O. Besov space on $\mathbb{T}$ of $0 < s < 2$ order.

The main result of this article is the proof of the following theorem:

Theorem 1.1. *Let $\alpha > -1$, $0 < p < +\infty$, $\{\alpha_k\}_{k=1}^{+\infty}$ be the arbitrary sequence of complex numbers from D, which is contained in a finite union of Stolz angles, i.e.*

$$\{\alpha_k\} \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s),$$

with certain $0 < \delta < \dfrac{p}{2(\alpha+p+1)}$.
The following statements are equivalent:
i) $\{\alpha_k\}_{k=1}^{+\infty}$ is an interpolating sequence in S_α^p;

ii) the series converges:

$$\sum_{k=1}^{+\infty} \frac{n_k^p}{2^{k(\alpha+p+1)}} < +\infty, \tag{3}$$

$n_k = card\left\{z_k : |z_k| < 1 - \frac{1}{2^k}\right\}$, $k = 1, 2, ...$, *and there exists a sequence* $\{\varepsilon_k\}_{k=1}^{+\infty}$, $\varepsilon(k) \to 0$, $k \to +\infty$ *such that*

$$|\pi'_\beta(\alpha_k)| \geqslant \exp \frac{-\varepsilon(k)}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}}, \tag{4}$$

for all $\beta > \dfrac{\alpha+1}{p}$.

The proof of the theorem is based on the following statements.

Theorem 1.2 (see [14, 15])**.** *Let* $\alpha > -1$, $\beta > \dfrac{\alpha+1}{p}$. *The following assertions are equivalent:*
1) $f \in S_\alpha^p$;
2) function f *allows the following representation in* D

$$f(z) = c_\lambda z^\lambda \pi_\beta(z, z_k) \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\psi(e^{i\theta})d\theta}{(1-e^{-i\theta}z)^{\beta+1}}\right), \ z \in D,$$

where $\{z_k\}_{k=1}^{+\infty}$ *is an arbitrary sequence from* D, *satisfying the condition*

$$\sum_{k=1}^{+\infty} \frac{n_k^p}{2^{k(p+\alpha+1)}} < +\infty,$$

$\psi \in B_{1,p}^s$, $s = \beta - \dfrac{\alpha+1}{p}$, $\lambda \in \mathbb{Z}$, $c_\lambda \in \mathbb{C}$.

Here and in the sequel, unless otherwise noted, we denote by c, $c_1, ..., c_n(\alpha, \beta, ...)$ some arbitrary positive constants depending on α, $\beta, ...$, whose specific values are immaterial.

For the further exposition of the results we introduce metrics in spaces S_α^p and l_α^p as follows:

$$\rho(f,g) = \int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln\left(1 + |f(re^{i\theta}) - g(re^{i\theta})|\right) d\theta\right)^p dr \ \text{ for } 0 < p \leqslant 1,$$

$$\rho(f,g) = \left(\int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln\left(1 + |f(re^{i\theta}) - g(re^{i\theta})|\right) d\theta\right)^p dr\right)^{\frac{1}{p}} \ \text{ for } p > 1.$$

for all $f, g \in S_\alpha^p$.

$$\rho_{l_\alpha^p}(a,b) = \sup_{k\geqslant 1}\left\{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1} \ln(1+|a_k - b_k|)\right\}.$$

for all $a = \{a_k\}$, $b = \{b_k\} \in l_\alpha^p$. It is easy to check that these spaces are complete metric spaces with respect to these metrics.

The following statement is valid:

Lemma 1.3. *If the operator* $R(f) = (f(\alpha_1), ..., f(\alpha_n), ...)$ *maps space* S_α^p *onto space* l_α^p, *then there exists the sequence of the functions* $\{g_n(z)\}_{n=1}^{+\infty} \in S_\alpha^p$ *such that*

$$\sup_{n\geqslant 1} \rho_{S_\alpha^p}(g_n, 0) \leqslant C, \ C > 0$$

and

$$g_n(\alpha_k) = \gamma_k^{(n)}, \ \textit{where } \gamma_k^{(n)} = \begin{cases} 0, \textit{ for all } k \neq n, \\ \exp \frac{\delta(k)}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}}+1}, \ \textit{for } k = n, \end{cases}$$

where $k, n = 1, 2, ...,$ $\delta(k) = o(1)$, $k \to +\infty$.

The proof of Lemma 1.3 repeats the reasoning conducted in the proof of the corresponding assertions from [10] with $\frac{\alpha+1}{p} + 1$ index.

2. Proof of main result

Let prove the implication i) $\to$ ii).

We assume that $\{\alpha_k\}_{k=1}^{+\infty} \in D$ is an interpolation sequence in the class S_α^p, ($\alpha > -1$, $0 < p < \infty$), i.e. for any $\{\gamma_k\} \in l_\alpha^p$ there exists a function $f \in S_\alpha^p$ such that $f(\alpha_k) = \gamma_k$, $k = 1, 2, \dots$.

Let consider the sequence $\{\gamma_k\}_{k=1}^{+\infty}$: $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = ... = 0$. Evidently, $\{\gamma_k\}_{k=1}^{+\infty} \in l_\alpha^p$. Since $\{\alpha_k\}_{k=2}^{+\infty}$ is zero-sequence for the function $f \in S_\alpha^p$, then we have an estimate (3) as follows from Theorem 1.2.

In order to show (4) we fix $n \in \mathbb{N}$ and take the sequence $\gamma_k^{(n)} = 0$, $k \neq n$, $\gamma_k^{(n)} = \exp \dfrac{\delta(n)}{(1-|\alpha_n|)^{\frac{\alpha+1}{p}}+1}$, $k = n$, where $\delta(n) \to 0$, $n \to +\infty$. By Lemma 1.3 there exists a function $g_n \in S_\alpha^p$ such that $\rho_{S_\alpha^p}(g_n, 0) \leqslant C$ and $g_n(\alpha_k) = \gamma_k^{(n)}$ for all $k = 1, 2, ...$, where the constant $C > 0$ is independent on n. In particular, $g_n(\alpha_n) = \gamma_n^{(n)}$. According to Theorem 1.2, any function $g_n \in S_\alpha^p$ can be represented as

$$g_n(z) = c_{\lambda_n} z^{\lambda_n} \pi_{\beta,n}(z, \alpha_k) \exp\{h_n(z)\}, \ z \in D,$$

where $h_n(z) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \frac{\psi_n(e^{i\theta})}{(1 - ze^{-i\theta})^{\beta+1}} d\theta$, $\beta > \dfrac{\alpha+1}{p}$.

So,

$$|g_n(\alpha_n)| = |\gamma_n^{(n)}| = \exp \frac{\delta(n)}{(1-|\alpha_n|)^{\frac{\alpha+1}{p}}+1} = |c_{\lambda_n}| \, |\alpha_n|^{\lambda_n} |\pi_{\beta,n}(\alpha_n, \alpha_k)| \, |\exp\{h_n(\alpha_n)\}|,$$

where $\delta(n) \to 0$, $n \to +\infty$.

Since $\exp\{h_n(z)\} \in S_\alpha^p$, then taking into account the estimate (1) we have:

$$|g_n(\alpha_n)| = \exp \frac{\delta(n)}{(1-|\alpha_n|)^{\frac{\alpha+1}{p}}+1} \leqslant c_1 |\pi_{\beta,n}(\alpha_n, \alpha_k)| \exp \frac{\nu(n)}{(1-|\alpha_n|)^{\frac{\alpha+1}{p}}+1},$$

where the sequence $\nu(n) \to 0$, $n \to +\infty$, is chosen so that $\nu(n) > \delta(n)$.

From the last inequality we obtain:

$$|\pi_{\beta,n}(\alpha_n)| \geqslant \exp \frac{-\varepsilon(n)}{(1-|\alpha_n|)^{\frac{\alpha+1}{p}}+1}, \tag{5}$$

where $\varepsilon(n) = (\nu(n) - \delta(n))$ is positive infinitesimal sequence.

The same manner as in [10], it can be shown that (5)$\Rightarrow$(4). Thus, the implication i) $\to$ ii) is established.

Now we prove that ii) $\to$ i). Suppose that $\{\alpha_k\}_{k=1}^{+\infty}$ be the arbitrary sequence of complex numbers from D, which is contained in a finite union of Stolz angles, and the estimates (3), (4)

are valid. Let us show that there exists a function $\Psi \in S^p_\alpha$ such that $\Psi(\alpha_k) = \gamma_k$, $k = 1, 2, ...$ for each $\{\gamma_k\}_{k=1}^{+\infty} \in l^p_\alpha$, where $\{\gamma_k\}_{k=1}^{+\infty} \in l^p_\alpha$, i.e.

$$\gamma_k = \exp \frac{\delta(k)}{(1 - |\alpha_k|)^{\frac{\alpha+1}{p}+1}}, \tag{6}$$

$\delta(k) \to 0$, $k \to +\infty$.

We denote by $\{q_k\}$ the sequence with general term $q_k = \delta_k + \varepsilon_k$, $k = 1, 2, ...$, where $\{\varepsilon_k\}$ is an infinitesimal sequence from the estimate (4), $\{\delta_k\}$ is an infinitesimal sequence from the estimate (6).

We construct a function $\Psi(z)$ as follows:

$$\Psi(z) = \sum_{k=1}^{+\infty} \gamma_k \frac{\pi_\beta(z, \alpha_j)}{(z - \alpha_k)} \frac{1}{\pi'_\beta(\alpha_k, \alpha_j)} \left(\frac{1 - |\alpha_k|}{1 - \overline{\alpha_k} z}\right)^m \frac{f(z)}{f(\alpha_k)}, \tag{7}$$

where $\beta > \dfrac{\alpha+1}{p}$, $m > \dfrac{\alpha+1}{p} + 1$,

$$f(z) = \exp \sum_{s=1}^{n} \sum_{m=1}^{+\infty} q_k^m \frac{(1 - {\rho_m}^2)^{\hat\beta}}{(1 - z\rho_m e^{-i\theta_s})^{\hat\beta + \frac{\alpha+1}{p}+1}}, \ z \in D,$$

where $0 < \hat\beta < \dfrac{\alpha+1}{p} + 1$, $q_k \to 0$, $k \to +\infty$. Here the sequence $\{q_k\}$ is depending on the sequence of nodes $\{\alpha_k\}$, therefore the function f depends on k, that is $f = f_k$.

For brevity $\beta' = \hat\beta + \dfrac{\alpha+1}{p} + 1$. It is obvious, that $\Psi(\alpha_n) = \gamma_n$, $n = 1, 2, ...$

Without lost of generality we assume that interpolating nodes are contained in the Stolz angle $\Gamma_\delta(\theta)$. Then we have

$$f(z) = f_k(z) = \exp \sum_{m=1}^{+\infty} q_k^m \frac{(1 - {\rho_m}^2)^{\hat\beta}}{(1 - z\rho_m e^{-i\theta})^{\beta'}}, \ z \in D.$$

Let us show that $\Psi \in S^p_\alpha$. First we estimate $f(\alpha_k)$ in the angle of $\Gamma_\delta(\theta)$.

$$\begin{aligned} |f(\alpha_k)| &= \exp \sum_{m=1}^{+\infty} q_k^m \Re \frac{(1 - {\rho_m}^2)^{\hat\beta}}{(1 - \alpha_k \rho_m e^{-i\theta})^{\beta'}} = \\ &= \exp \sum_{m=1}^{+\infty} q_k^m (1 - {\rho_m}^2)^{\hat\beta} \frac{\Re(1 - \overline{\alpha_k}\rho_m e^{-i\theta})^{\beta'}}{|1 - \alpha_k \rho_m e^{-i\theta}|^{2\beta'}}. \end{aligned}$$

But

$$\begin{aligned} \Re(1 - \overline{\alpha_k}\rho_m e^{-i\theta})^{\beta'} &= \Re\Big(1 - r_k \rho_m e^{-i(\varphi_k - \theta)}\Big)^{\beta'} = \\ = \Re\Big(1 - \rho_m r_k + \rho_m r_k (1 - e^{-i(\varphi_k - \theta)})\Big)^{\beta'} &= \Re\Big(1 - \rho_m r_k + \rho_m r_k (1 - e^{-i(\varphi_k - \theta)})\Big)^{\beta'} = \\ &= (\rho_m r_k \rho)^{\beta'} \cdot \Re\left(\frac{1 - \rho_m r_k}{\rho_m r_k \rho} + e^{-i\varphi}\right)^{\beta'}, \end{aligned}$$

where $\alpha_k = r_k e^{i\varphi_k}$, $(1 - e^{-i(\varphi_k - \theta)}) = \rho e^{-i\varphi}$, $|\varphi| < \dfrac{\pi\delta}{2} < \dfrac{\pi}{2\beta'}$. So we have $\Re(1 - \overline{\alpha_k}\rho_m e^{-i\theta})^{\beta'} \geqslant c_1(\rho_m r_k \rho)^{\beta'}$, $c_1 > 0$, by Lemma, established in [13].

From the other hand,

$$|1-e^{-i(\varphi_k-\theta)}|^{\beta'} = 2^{\beta'}\sin^{\beta'}\left(\frac{\theta-\varphi_k}{2}\right) = 2^{\beta'}\sin^{\beta'}\frac{\varphi}{2},$$

therefore

$$\Re\frac{1}{(1-\alpha_k\rho_m e^{-i\theta})^{\beta'}} \geqslant \frac{c_1(\rho_m r_k)^{\beta'}2^{\beta'}\sin^{\beta'}\left(\frac{\theta-\varphi_k}{2}\right)}{\left((1-\rho_m r_k)^2+4\sin^2\left(\frac{\theta-\varphi_k}{2}\right)\rho_m r_k\right)^{\beta'}} \geqslant \frac{2^{\beta'}\sin^{\beta'}\frac{\varphi}{2}}{(1-\rho_m r_k)^{2\beta'}\cdot\left(1+\frac{4\sin^2\frac{\varphi}{2}}{(1-\rho_m r_k)^2}\right)^{\beta'}}.$$

Since $\{\alpha_k\}\subset\Gamma_\delta(\theta)$, we have

$$\frac{\left|\sin\left(\frac{\theta-\varphi_k}{2}\right)\right|}{(1-r_k)} \leqslant C.$$

Hence,

$$\Re\frac{1}{(1-\alpha_k\rho_m e^{-i\theta})^{\beta'}} \geqslant \frac{c(\beta')}{(1-\rho_m r_k)^{\beta'}}.$$

Thus we have the following estimate for $f(\alpha_k)$ in the angle of $\Gamma_\delta(\theta)$:

$$|f(\alpha_k)| \geqslant \exp c(\beta')\sum_{m=1}^{+\infty} q_k^m\frac{(1-{\rho_m}^2)^{\hat\beta}}{(1-r_k\rho_m)^{\beta'}}.$$

We will continue this estimate. For this purpose we split the interior sum into two parts:

$$S=\sum_{m=1}^{+\infty} q_k^m\frac{(1-{\rho_m}^2)^{\hat\beta}}{(1-r_k\rho_m)^{\beta'}} = \sum_{(1-\rho_m)\leqslant(1-r_k)}(\ldots)+\sum_{(1-\rho_m)>(1-r_k)}(\ldots) = S_1(k)+S_2(k).$$

We estimate each of them separately. Let estimate the sum S_2.

$$\begin{aligned} S_2(k) &= \sum_{(1-\rho_m)>(1-r_k)} q_k^m\frac{(1-{\rho_m}^2)^{\hat\beta}}{(1-r_k\rho_m)^{\beta'}} \geqslant \\ &\geqslant \sum_{\rho_m<r_k} q_k^m\frac{1}{(1-\rho_m^2)^{\beta'-\hat\beta}} \geqslant \\ &\geqslant \frac{1}{2}\sum_{\rho_m<r_k} q_k^m\frac{1}{(1-\rho_m)^{\frac{\alpha+1}{p}+1}}. \end{aligned}$$

Without lost of generality we assume that $1-\rho_m=\hat q^m$, $0<\hat q<1$, and we find that the sum above converges to $\delta_2(k)=\dfrac{q_k}{\hat q^{(\beta'-\hat\beta)}-q_k}=o(1),\ k\to+\infty$.

Now we get lower estimate for the sum S_1.

$$\begin{aligned} S_1(k) &= \sum_{(1-\rho_m)\leqslant(1-r_k)} q_k^m\frac{(1-{\rho_m}^2)^{\hat\beta}}{(1-r_k\rho_m)^{\beta'}} \geqslant \\ &\geqslant \sum_{\rho_m\geqslant r_k} q_k^m\frac{(1-{\rho_m}^2)^{\hat\beta}}{(1-r_k\rho_m)^{\frac{\alpha+1}{p}+1}(1-r_k\rho_m)^{\hat\beta}} \geqslant \\ &\geqslant \frac{1}{(1-r_k^2)^{\frac{\alpha+1}{p}+1}}\sum_{\rho_m\geqslant r_k} q_k^m\frac{(1-{\rho_m}^2)^{\hat\beta}}{(1-r_k\rho_m)^{\hat\beta}}. \end{aligned}$$

Since

$$\sum_{\rho_m \geqslant r_k} q_k^m \frac{(1-{\rho_m}^2)^{\hat{\beta}}}{(1-r_k\rho_m)^{\hat{\beta}}} \leqslant \sum_{m=1}^{+\infty} (q_k)^m ,$$

then $S_1(k) \geqslant \dfrac{\delta_1(k)}{(1-r_k^2)^{\frac{\alpha+1}{p}+1}}$, where $\delta_1(k) \approx \dfrac{q_k}{1-q_k}$, $\delta_1(k) \to 0$, $k \to +\infty$.

From the estimation for S_1, S_2 we finally set:

$$S \geqslant \frac{\delta_1(k)}{(1-r_k^2)^{\frac{\alpha+1}{p}+1}} + \delta_2(k),$$

whence we conclude that

$$|f(\alpha_k)| \geqslant \exp \frac{\delta_0(k)}{(1-r_k^2)^{\frac{\alpha+1}{p}+1}}, \tag{8}$$

where $\delta_0(k) = \sup\limits_j \delta_j(k)$, $j = 1, 2$. It is obvious, that $\delta_0(k) \sim \dfrac{q_k}{1-q_k}$.

From condition (3) we conclude (see [15])

$$\sum_{k=1}^{+\infty} (1-|\alpha_k|)^m < +\infty \tag{9}$$

for all $m > \dfrac{\alpha+1}{p} + 1$. Taking into account the convergence of the series (9), we obtain that the infinite product $\pi_\beta(z, \alpha_j)$ and the series (7) are absolutely and uniformly convergent in D.

Now we need to prove that function $\Psi(z)$ is analytic in D and $\Psi \in S_\alpha^p$.

Now we get an upper estimate for the function $|\Psi(z)|$. Since $\{\gamma_k\}_{k=1}^{+\infty} \in l_\alpha^p$ and condition (4) is valid, we have:

$$|\Psi(z)| \leqslant \sum_{k=1}^{+\infty} |\gamma_k| \frac{|\pi_\beta(z,\alpha_j)|}{|z-\alpha_k|} \frac{1}{|\pi'_\beta(\alpha_k,\alpha_j)|} \left(\frac{1-|\alpha_k|}{|1-\overline{\alpha_k}z|}\right)^m \frac{|f(z)|}{|f(\alpha_k)|} \leqslant$$

$$\leqslant \sum_{k=1}^{+\infty} \exp \frac{\delta(k)}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}} \frac{|\pi_\beta(z,\alpha_j)|}{|z-\alpha_k|} \exp \frac{\varepsilon(k)}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}} \left(\frac{1-|\alpha_k|}{|1-\overline{\alpha_k}z|}\right)^m \frac{|f(z)|}{|f(\alpha_k)|}.$$

We estimate the factor $\dfrac{|\pi_\beta(z,\alpha_j)|}{|z-\alpha_k|}$. Using the well-known estimate for the Djrbashian product (see [13]):

$$\ln^+ |\pi_{\beta,k}(z,\alpha_j)| \leqslant c_\beta \sum_{k=1}^{+\infty} \left(\frac{1-|\alpha_k|}{|1-\overline{\alpha}_k z|}\right)^{\beta+2},$$

we get:

$$\frac{|\pi_\beta(z,\alpha_j)|}{|z-\alpha_k|} = \frac{1}{|z-\alpha_k|} |\pi_{\beta,k}(z,\alpha_j)| \frac{|\alpha_k - z|}{|\alpha_k|} |\exp(-U_\beta(z,\alpha_k))| \leqslant \tilde{c}_\beta \frac{|\pi_{\beta,k}(z,\alpha_j)|}{|1-\overline{\alpha}_k z|},$$

$$\frac{|\pi_{\beta,k}(z,\alpha_j)|}{|1-\overline{\alpha}_k z|} \leqslant \frac{\tilde{c}_\beta}{|1-\overline{\alpha}_k z|} \exp\left(c_\beta \sum_{n=1}^{+\infty} \left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n z|}\right)^{\beta+2}\right)$$

for all $\beta > \dfrac{\alpha+1}{p}$.

Therefore

$$|\Psi(z)|\leqslant \exp\left(c_\beta\sum_{n=1}^{+\infty}\left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n z|}\right)^{\beta+2}\right)\times|f(z)|\times\widetilde{c}_\beta\cdot\sum_{k=1}^{+\infty}\exp\frac{\delta(k)+\varepsilon(k)}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}}\cdot\frac{1}{|f(\alpha_k)|}\frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k}z|^{m+1}}.$$

Now we consider the last factor in the product:

$$\sum_{k=1}^{+\infty}\exp\frac{q_k}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}}\cdot\frac{1}{|f(\alpha_k)|}\frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k}z|^{m+1}},$$

where $q_k=\delta(k)+\varepsilon(k)$.

We split the sum into n parts:

$$\sum_{s=1}^{n}\sum_{\alpha_k\in\Gamma_\delta(\theta_s)}\exp\frac{q_k}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}}\cdot\frac{1}{|f(\alpha_k)|}\frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k}z|^{m+1}}.$$

Since $\{\alpha_k\}\subset\bigcup\limits_{s=1}^{n}\Gamma_\delta(\theta_s)$ for certain $0<\delta<\dfrac{p}{2(\alpha+p+1)}$, we can apply for each part the equation (8). Thus we have:

$$\sum_{s=1}^{n}\sum_{\alpha_k\in\Gamma_\delta(\theta_s)}\exp\frac{q_k-\delta_0(k)}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}}\cdot\frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k}z|^{m+1}}.$$

Since $q_k-\delta_0(k)=q_k-\dfrac{q_k}{1-q_k}=-q_k^2<0$, we have the following estimate:

$$\exp\frac{q_k-\delta_0(k)}{(1-|\alpha_k|)^{\frac{\alpha+1}{p}+1}}\leqslant 1,$$

for all $k=1,2,\dots$.

Thus we have:

$$|\Psi(z)|\leqslant\exp\left(c_\beta\sum_{n=1}^{+\infty}\left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n z|}\right)^{\beta+2}\right)\times|f_k(z)|\times\widetilde{c}_\beta\cdot\sum_{k=1}^{+\infty}\frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k}z|^{m+1}}.$$

Taking into account the convergence of the series (9), we have:

$$\sum_{k=1}^{+\infty}\frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k}z|^{m+1}}\leqslant\frac{c}{(1-|z|)^{m+1}}\sum_{k=1}^{+\infty}(1-|\alpha_k|)^m\leqslant\frac{c_1}{(1-|z|)^{m+1}}$$

for all $m>\dfrac{\alpha+1}{p}+1$.

The estimate of function $|\Psi(z)|$ takes form:

$$|\Psi(z)|\leqslant\exp\left(c_\beta\sum_{n=1}^{+\infty}\left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n z|}\right)^{\beta+2}\right)\times|f_k(z)|\times\frac{c_2}{(1-|z|)^{m+1}}.$$

From the works of F. A. Shamoyan (see [14], also [15, p. 132]) and of the author (see [9]) it follows that a majorizing function in the last inequality belongs to S_α^p space, and hence $\Psi\in S_\alpha^p$ for all $\alpha>-1$, $0<p<+\infty$.

This shows that ii) $\to$ i). □

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Об интерполяции в классах аналитических в круге функций с характеристикой Р. Неванлинны из Lp-весовых пространств

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В статье получено решение интерполяционной задачи в классе аналитических функций в единичном круге, характеристика Р. Неванлинны которых принадлежит Lp-весовым пространствам, при условии, что узлы интерполяции принадлежат конечному числу углов Штольца.

Ключевые слова: интерполяция, аналитические функции, характеристика Р. Неванлинны, углы Штольца.