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On the Dynamics of a Class of Kolmogorov Systems

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In this paper we charaterize the integrability and the non-existence of limit cycles of Kolmogorov systems of the form

$$\begin{cases} x' = x\left(P(x,y) + \sqrt{R(x,y)}\right), \\ y' = y\left(Q(x,y) + \sqrt{R(x,y)}\right), \end{cases}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, homogeneous polynomials of degree n, n, m, respectively.

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Introduction

The autonomous differential system on the plane given by

$$\begin{cases} x' = \dfrac{dx}{dt} = xF(x,y), \\ y' = \dfrac{dy}{dt} = yG(x,y), \end{cases} \tag{1}$$

known as Kolmogorov system, the derivatives are performed with respect to the time variable and F , G are two functions in the variables x and y. If F and G are linear (Lotka- Volterra-Gause model), then it is well known that there is at most one critical point in the interior of the realistic quadrant ($x \geqslant 0, y \geqslant 0$), and there are no limit cycles [10,14,16]. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics [12,17,18] chemical reactions, plasma physics [13], hydrodynamics [5], etc.

In the qualitative theory of planar dynamical systems [7–9,15], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem. There is a huge literature about the limit cycles, most of it deals essentially with their detection, number and stability [1–4,11]. We recall that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1).

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant continuously differentiable function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} xF(x,y) + \frac{\partial H(x,y)}{\partial y} yG(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait [6].

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In this paper we are intersted in studying the integrability and the periodic orbits of the 2-dimensional Kolmogorov systems of the form

$$\begin{cases} x' = x\left(P(x,y) + \sqrt{R(x,y)}\right), \\ y' = y\left(Q(x,y) + \sqrt{R(x,y)}\right), \end{cases} \tag{2}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, homogeneous polynomials of degree n, n, m respectively.

We define the trigonometric functions $f_1(\theta) = P(\cos\theta, \sin\theta)\cos^2\theta + Q(\cos\theta, \sin\theta)\sin^2\theta$, $f_2(\theta) = \sqrt{R(\cos\theta, \sin\theta)}$ and $f_3(\theta) = (Q(\cos\theta, \sin\theta) - P(\cos\theta, \sin\theta))\cos\theta\sin\theta$.

1. Main result

Our main result on the integrability and the periodic orbits of the Kolmogorov system (2) is the following.

Theorem 1. *For Komogorov system* (2) *the following statements hold.*

(a) *If* $f_3(\theta) \neq 0$, $R(\cos\theta, \sin\theta) \geqslant 0$ *and* $m \neq 2n$, *then system* (2) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{2n-m}{4}} \exp\left(\frac{m-2n}{2}\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ &+ \left(\frac{m-2n}{2}\right)\int_0^{\arctan\frac{y}{x}} \exp\left(\frac{m-2n}{2}\int_0^{w} A(\omega)\,d\omega\right) B(w)\,dw, \end{aligned}$$

where $A(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$ *and the curves which are formed by the trajectories of the differential system* (2), *in Cartesian coordinates are written as*

$$\begin{aligned} x^2+y^2 = \Bigg(& h\exp\left(\frac{2n-m}{2}\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ & + \frac{2n-m}{2}\exp\left(\frac{2n-m}{2}\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) \times \\ & \times \int_0^{\arctan\frac{y}{x}} \exp\left(\frac{m-2n}{2}\int_0^{w} A(\omega)\,d\omega\right) B(w)\,dw \Bigg)^{\frac{2}{2n-m}}, \end{aligned}$$

where $h \in \mathbb{R}$. *Moreover, the system* (2) *has no a limit cycle.*

(b) *If* $f_3(\theta) \neq 0$, $R(\cos\theta, \sin\theta) \geqslant 0$ *and* $m = 2n$, *then system* (2) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} (A(\omega)+B(\omega))\,d\omega\right),$$

and the curves which are formed by the trajectories of the differential system (2), *in Cartesian coordinates are written as*

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}} (A(\omega)+B(\omega))\,d\omega\right) = 0,$$

where $h \in \mathbb{R}$. *Moreover, the system* (2) *has no a limit cycle.*

(c) If $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$ *and* $R(\cos\theta, \sin\theta) \geqslant 0$, *then system* (2) *has the first integral* $H = \dfrac{y}{x}$ *and the curves which are formed by the trajectories of the differential system* (2), *in Cartesian coordinates are written as* $y - hx = 0$, *where* $h \in \mathbb{R}$. *Moreover, the system* (2) *has no a limit cycle.*

Proof. In order to prove our results we write the polynomial differential system (2) in polar coordinates (r, θ), defined by $x = r\cos\theta$, and $y = r\sin\theta$, then system (2) becomes

$$\begin{cases} r' = f_1(\theta)\, r^{n+1} + f_2(\theta)\, r^{\frac{1}{2}m+1}, \\ \theta' = f_3(\theta)\, r^{n}, \end{cases} \tag{3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction, $r' = \dfrac{dr}{dt}$ and $\theta' = \dfrac{d\theta}{dt}$.

If $f_3(\theta) \neq 0$, $R(\cos\theta, \sin\theta) \geqslant 0$ and $n \neq 2m$.

Taking as new independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta} = A(\theta)\, r + B(\theta)\, r^{\frac{m-2n+2}{2}}, \tag{4}$$

which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{\frac{2n-m}{2}}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = \left(\frac{2n-m}{2}\right)\left(A(\theta)\,\rho + B(\theta)\right). \tag{5}$$

The general solution of linear equation (5) is

$$\begin{aligned} \rho(\theta) &= \exp\left(\frac{2n-m}{2}\int_0^{\theta} A(\omega)\, d\omega\right) \times \\ &\quad \times \left(\alpha + \frac{2n-m}{2}\int_0^{\theta} \exp\left(\frac{m-2n}{2}\int_0^{w} A(\omega)\, d\omega\right) B(w)\, dw\right), \end{aligned}$$

where $\alpha \in \mathbb{R}$, then system (2) has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{2n-m}{4}} \exp\left(\frac{m-2n}{2}\int_0^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) + \\ &\quad + \left(\frac{m-2n}{2}\right)\int_0^{\arctan\frac{y}{x}} \exp\left(\frac{m-2n}{2}\int_0^{w} A(\omega)\, d\omega\right) B(w)\, dw. \end{aligned}$$

Let γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\gamma = H(\gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates written as

$$\begin{aligned} x^2 + y^2 = \Bigg(& h \exp\left(\frac{2n-m}{2}\int_0^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) + \\ & + \frac{2n-m}{2} \exp\left(\frac{2n-m}{2}\int_0^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) \times \\ & \times \int_0^{\arctan\frac{y}{x}} \exp\left(\frac{m-2n}{2}\int_0^{w} A(\omega)\, d\omega\right) B(w)\, dw \Bigg)^{\frac{2}{2n-m}}, \end{aligned}$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit γ is contained in the curve

$$r(\theta) = \left(h_\gamma \exp\left(\frac{2n-m}{2} \int_0^\theta A(\omega)\, d\omega \right) + \right.$$
$$\left. + \frac{2n-m}{2} \exp\left(\frac{2n-m}{2} \int_0^\theta A(\omega)\, d\omega \right) \int_0^\theta \exp\left(\frac{m-2n}{2} \int_0^w A(\omega)\, d\omega \right) B(w)\, dw \right)^{\frac{2}{2n-m}}.$$

But this curve can not contain the periodic orbit γ, and consequently no limit cycle contained in one of the open quadrants because this curve has at most one point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$. Hence statement (a) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$, $R(\cos\theta, \sin\theta) \geqslant 0$ and $m = 2n$.

Taking as new independent variable the coordinate θ, this differential system (3) is written in the form

$$\frac{dr}{d\theta} = (A(\theta) + B(\theta))\, r. \tag{6}$$

The general solution of equation (6) is

$$r(\theta) = \alpha \exp\left(\int_0^\theta (A(\omega) + B(\omega))\, d\omega \right),$$

where $\alpha \in \mathbb{R}$, then system (2) has the first integral

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega))\, d\omega \right).$$

Let γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\gamma = H(\gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$\left(x^2 + y^2\right)^{\frac{1}{2}} - h \exp\left(\int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega))\, d\omega \right) = 0,$$

where $h \in \mathbb{R}$. Therefore the periodic orbit γ is contained in the curve

$$r(\theta) = h_\gamma \exp\left(\int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega))\, d\omega \right).$$

But this curve cannot contain the periodic orbit γ, and consequently no limit cycle contained in one of open quadrants because this curve at most have one point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$. This completes the proof of statement (b) of Theorem 1.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$ and $R(\cos\theta, \sin\theta) \geqslant 0$, then from system (3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system (2), then curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, and consequently no limit cycle.

Hence statement (c) of Theorem 1 is proved. □

Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar differential systems of ODEs in order to obtain explicit expression for a first integral which characterizes its trajectories. This is one of the classical tools in the classification of all trajectories of dynamical systems.

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О динамике одного класса систем Колмогорова

Рашид Букуша

В этой статье мы характеризуем интегрируемость и отсутствие предельных циклов систем Колмогорова вида

$$\begin{cases} x' = x\left(P(x,y) + \sqrt{R(x,y)}\right), \\ y' = y\left(Q(x,y) + \sqrt{R(x,y)}\right), \end{cases}$$

где $P(x,y)$, $Q(x,y)$, $R(x,y)$ *— однородные многочлены степени* n, n, m *соответственно.*

Ключевые слова: система Колмогорова, первый интеграл, периодические орбиты, предельный цикл.