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Carleman's Formula for a Matrix Polydisk

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In this work an integral formula for a matrix polydisk is obtained. For a function from the Hardy class it allows to recover its value at any interior point from its values on a part of the Shilov boundary.

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1. Statement of the problem and preliminaries

Consider a class of holomorphic functions in a domain D that behave reasonably well near the boundary ∂D . Carleman's formulas solve the problem of recovery of a function from such a class from its values on a set of uniqueness $M \subset \partial D$ for this class, which does not contain the Shilov boundary of D .

One-dimensional and multidimensional formulas were studied in the monograph [1]. In the paper [2] a new method is proposed for finding Carleman's formulas in homogeneous domains using a domain's automorphisms. In the present paper we use this method for a matrix polydisk.

Let $Z = (Z^1, Z^2, \dots, Z^n) \in \mathbb{C}^n[m \times m]$ be a vector of quadratic matrices of order m over the field of complex numbers \mathbb{C} . The unit matrix disk is defined as the set

$$\tau = \{Z \in \mathbb{C}[m \times m] : ZZ^* < I\},$$

where $Z^* = \overline{Z}'$ is the conjugate transpose of the matrix Z , the notation $ZZ^* < I$ (I is the unit $[m \times m]$ -matrix) means that the Hermitian matrix $I - ZZ^*$ is positive definite. The skeleton of a matrix disk is the set

$$S(\tau) = \{Z \in \mathbb{C}[m \times m] : ZZ^* = I\}.$$

The direct product of matrix disks

$$T = T_n = \tau^n = \{Z = (Z^1, \dots, Z^n) : Z^j \in \tau, j = 1, \dots, n\}$$

is called *the matrix unit polydisk* $T = T_n$ in the space $\mathbb{C}^n[m \times m]$.

The set $S(T) = S(\tau) \times \dots \times S(\tau)$ is called *the skeleton* of T (see [3]).

To formulate the main result, we need the following Carleman formula for the unit matrix disk obtained in [2]. Let the set $M \subset S(\tau)$ have positive measure $\mu_1 M > 0$. Denote

$$M_{0,u} = \{\xi : \xi \in M, \xi = \lambda u, |\lambda| = 1\}, u \in SU(m), \quad (1)$$

$$M'_0 = \{u : u \in SU(m), \mu_1 M_{0,u} > 0\},$$

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where $SU(m)$ is the group of special unitary matrices, i.e. $\det(u) = 1$, m_1 is the normed Lebesgue measure on the unit circle ∂U . Further on, let

$$\varphi_0 = \exp \psi_0, \quad \text{where } \psi_0(\xi) = \frac{1}{2\pi i} \int_{M_{0,u}} \frac{\eta + \lambda}{\eta - \lambda} \cdot \frac{d\eta}{\eta}.$$

The Hardy class $H^1(D)$ is defined as follows: a holomorphic in D function $f(z)$ belongs to the class $H^1(D)$ if

$$\sup_{0 \leq r < 1} \int_{S(D)} |f(r\xi)| d\mu_1 < \infty,$$

where μ_1 is the Lebesgue measure on the skeleton $S(D)$.

Lemma 1 ([2]). *Let $f \in H^1(\tau)$. Then the formula*

$$f(0) = \frac{m}{\int_{M'_0} d\mu_1} \lim_{l \rightarrow \infty} \int_M f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu_\xi \quad (2)$$

holds.

2. The main part

Let $\Phi_A(Z)$ be an automorphism of a unit matrix polydisk T_n rearranging the points $A \in T_n$ and 0. Such an automorphism has the form (see [4, p. 84])

$$\Phi_A(Z) = (\Phi_A^1(z^1), \dots, \Phi_A^n(z^n)),$$

where

$$\Phi_A^j(Z^j) = Q^j(Z^j - A^j)(I - (A^j)^* Z^j)^{-1} (R^j)^{-1},$$

Q^j and R^j are $[m \times m]$ -matrices satisfying the following conditions

$$\begin{aligned} \bar{Q}^j (I - \bar{A}^j A^{j'}) Q^{j'} &= I, \quad \bar{R}^j (I - A^{j'} \bar{A}^j) R^{j'} = I, \\ Q^j A^j + A^j R^j &= 0. \end{aligned}$$

In particular, for $A = 0$

$$\Phi_0(Z) = (\Phi_0^1(z^1), \dots, \Phi_0^n(z^n))$$

and

$$\Phi_0^j(Z^j) = Q^j Z^j (R^j)^{-1}.$$

Let $E = (E^1, \dots, E^n) \subset S(T)$ and $\mu E > 0$; $\xi = (\xi^2, \dots, \xi^n) \in S(T_{n-1})$.

Define the sets

$$E_{0,\xi} = \left\{ Z : Z \in E, \Phi_0^1(Z^1) = \lambda, \Phi_0^j(Z^j) = \lambda \Phi_0^j(\xi^j), j = \overline{2, n}, \lambda \in S(\tau) \right\},$$

$$\tilde{E}_0 = \{ Z \in E : \mu_1 E_{0,\xi} > 0 \}.$$

Lemma 2. *Let $f \in H^1(T_n)$. Then the following formula*

$$f(0) = \frac{m}{\int_{\tilde{E}_0} d\mu_{m(n-1)}} \lim_{l \rightarrow \infty} \int_E f(Z) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu \quad (3)$$

holds. Here $\varphi_0 = \exp \psi_0$ and

$$\psi_0(\xi) = \frac{1}{2\pi i} \int_{E_{0,\xi^1}^1} \frac{\eta + \lambda}{\eta - \lambda} \frac{d\eta}{\eta},$$

where E_{0,ξ^1}^1 is defined the same as in (1).

Proof. Let $\xi = (\xi^2, \dots, \xi^n) \in S(T_{n-1})$. Then by Lemma 1 we have

$$f(0) = \frac{m}{\int_{\tilde{E}_{0,\xi}} d\mu} \lim_{l \rightarrow \infty} \int_{E_{0,\xi}} f(\xi) [\varphi_0(\xi)/\varphi_0(0)]^l d\mu_1.$$

Integrating both parts of this equation over \tilde{E}_0 , we obtain formula (3) since

$$\begin{aligned} \left| \int_{E_{0,\xi}} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu_1 \right| &\leq \left| \int_{S(\tau)} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu_1 \right| + \left| \int_{S(\tau) \setminus E_{0,\xi}} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu_1 \right| \leq \\ &\leq |f(0)| + \int_{S(\tau)} |f(\xi)| d\mu_1. \end{aligned}$$

Here we use the facts that $\varphi_0(Z)$ is the ‘‘quenching’’ function, and the Cauchy-Szegő formula is applicable for the function f in τ (see [4, p.93]). Therefore the transition to the limit under the integral sign is possible by the Lebesgue theorem. \square

Formula (3) recovers the value of the function f at the point 0 from its values on E . Now, with the help of this formula, we shall prove a formula recovering the value of f at an arbitrary point of the domain T_n .

Denote

$$\xi = (\xi^2, \dots, \xi^n) \in S(T_{n-1}),$$

$$E_{A,\xi} = \left\{ Z : Z \in E, \Phi_A^1(Z^1) = \lambda, \Phi_A^j(\xi^j) = \lambda \Phi_A^j(\xi^j), j = \overline{2, n}, \lambda \in S(\tau) \right\}.$$

This set is measurable with respect to the measure μ_1 for almost all A and ξ . Denote by \tilde{E}_A the set $\{\xi : \xi \in S(T_{n-1}), \mu_1 E_{A,\xi} > 0\}$. The Fubini theorem implies that the $(mn - m)$ -dimensional Lebesgue measure of this set is positive. Denote

$$\varphi_A = \exp \psi_A, \quad \psi_A(\xi) = \frac{1}{2\pi i} \int_{E_{A^1, W^1}} \frac{\eta + \lambda d\eta}{\eta - \lambda} \frac{d\eta}{\eta},$$

where $E_{A^1, W^1} = \{\xi^1 \in E^1, \xi = (\Phi_A^1)^{-1}(\lambda(\Phi_A^1)^{-1}(w)), |\lambda| = 1\}$, $W \in \Phi_A^1(SU(m))$.

Theorem 1. *Let $f \in H^1(T_n)$, $E \subset S(T_n)$, $\mu(E) > 0$. Then for an arbitrary point $A \in T_n$ we have the following formula*

$$f(A) = \frac{m}{\mu_{mn-m}(\Phi_A^{-1}(\tilde{E}_A))} \lim_{l \rightarrow \infty} \int_E f(Z) \left[\frac{\varphi_A(Z)}{\varphi_A(A)} \right]^l \prod_{j=1}^n H(A^j, \bar{Z}^j) d\mu(Z). \quad (4)$$

Here

$$H(A^j, \bar{Z}^j) = \frac{1}{\det(I^{(m)} - A^j Z^j)^m}$$

is the Cauchy-Szegő kernel for the (matrix) generalized disk.

Proof. For $A = 0$ the statement of Theorem 1 was proved in Lemma 2. Let $f_A(\xi) = f(\Phi_A^{-1}(\xi))$. Then $f_A(0) = f(A)$. It is known that $f_A \in H^1(T_n)$ (see [2]). By applying Lemma 2 to f_A and then performing the reverse transformation, we obtain

$$f(A) = \frac{m}{\int_{\tilde{E}_A} d_{mn-m}(\Phi_A^{-1}(\tilde{E}_A))} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[\frac{\varphi_A(\xi)}{\varphi_A(A)} \right] d\mu(\Phi_A(\xi)). \quad (5)$$

Since

$$d\mu(\Phi_A(\xi)) = |\det J(\xi, A)| d\mu(\xi),$$

where $J(A, \xi)$ is the Jacobi matrix of the transformation $\Phi_A(Z)$, and (see [4])

$$\det |J(\xi, A)| = \frac{\prod_{j=1}^n H(A^j, \bar{\xi}^j) H(\xi^j, \bar{A}^j)}{\prod_{j=1}^n H(A^j, \bar{A}^j)},$$

the application of formula (5) to the function $f(\xi) \prod_{j=1}^n H^{-1}(\xi^j, \bar{A}^j) \in H^1(T_n)$ gives (4). \square

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Формула Карлемана для матричного полидиска

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В работе получена интегральная формула для матричного полидиска. Для функций из класса Харди дано утверждение, позволяющее восстановить значения функции во внутренних точках полидиска по ее значениям на части границы Шилова.

Ключевые слова: формула Карлемана, матричный полидиск.