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Equilibrium and Stability of a Free Liquid Film in a Longitudinal Gravitational Field

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Deformation of the free vertical liquid film by combined effect of gravity and thermocapillary forces is investigated. The system of equations connecting the flow rate, the film thickness and its temperature is found in the framework of thin-layer approximation. Using the shooting method, one-dimensional stationary problem is solved numerically for values of contact angle close to a right angle. The existence of a film having the constant thickness is found. It is shown that this solution is stable when the gravity is low. Using numerical continuation method, value of the gravity for which this solution becomes unstable is found.

Keywords: liquid film, free surface, thin-layer approximation, thermocapillary effect, small disturbances, stability of solution.
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Introduction

This paper motivated to interpret results of experiments that were carried out by professors O. Kabov, L. Tadrist and there colleagues from Marseille [1]. This experiments result in free vertical liquid films that can be used in water desalination technology [2].

The fundamental tool to solve problems for liquid films is thin-layer approximation. The most part of papers on this subject considers films trickling down along the solid wall.

Thermocapillary effect plays a significant part in liquid motions [3–5].

In [6,7] deformation of a free liquid film by thermocapillary forces is considered when the gravity is absent. In [6] temperature distribution for film is prescribed, meanwhile in [7] temperature is unknown function.

In [8] a deformation and a rupture of a thin liquid film which is hanging between two solid flat walls under the action of concentrated thermal load action are considered under micro-gravity conditions.

The paper [9] is devoted to the problem for infinite non-isothermal liquid film with thermoisolated free surfaces under gravitation. In the two-dimensional case when the thickness is constant, the exact solution of Navie - Stokes equations is found. A solution having the constant thickness is investigated for stability when wave number is small. Spectral problem for perturbations gives the solution in the form of damped oscillations.

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1. Mathematical model

Let the viscous incompressible thermo-conducting liquid occupy the infinite layer $\Omega_t = \{x_1 \in (-\infty, \infty);\ x_2 \in (0, l);\ x_3 \in (-h(x_1, x_2, t), h(x_1, x_2, t))\}$, when acceleration of gravity $\mathbf{g} = (0 - g\ 0)$ is opposed to Ox_2 axis, $x_3 = \pm h(x_1, x_2, t)$ are unknown free boundaries. When $x_2 = 0$ and $x_2 = l$, the liquid is bounded by solid walls.

We assume that $\max h = \varepsilon l$, $|\nabla h| = O(\varepsilon)$, $l\Delta h = O(\varepsilon)$ when $\varepsilon \to 0$, where ∇ and Δ are the two-dimension gradient and the Laplacian. The longitudinal scale of the problem l is much larger than the transversal scale εl, i.e $\varepsilon << 1$.

Let us suppose that liquid density ρ, kinematic coefficient of viscosity ν and thermodiffusion coefficient χ are constant while the surface tension coefficient σ is a linear function of temperature T:

$$\sigma = \sigma_0 - \kappa(T - T_0),$$

where σ_0, κ and T_0 are positive constants. Let us denote by δT the characteristic temperature difference and let suppose that $\kappa\delta T/\sigma_0 = O(\varepsilon^2)$ when $\varepsilon \to 0$. We will assume the flow to be symmetric with respect to the plane $x_3 = 0$.

It is known ([10]), that for isothermal flow, stable film's length is no more than capillary radius. So now let the temperature is not constant.

Reasoning by analogy with [6] we obtain two equations connecting the flow rate vector $\mathbf{q}$ through the cross-section of the layer, the film thickness h and the mean value $T^*(x_1, x_2, t)$ of its temperature using thin-layer approximation:

$$h_{tt} + \frac{\sigma_0}{\rho}\nabla \cdot (h\nabla\Delta h) = \frac{\kappa}{\rho}\Delta T^* - \mathbf{g}\nabla h, \tag{1}$$

$$h_t + \nabla\cdot\mathbf{q} = 0. \tag{2}$$

It is follows from impermeability condition for solid walls that

$$\mathbf{q} = 0 \ \text{ when } \ x_2 = 0,\ x_2 = l. \tag{3}$$

We obtain other pair of boundary conditions from prescribing three-phase contact angle:

$$\frac{\partial h}{\partial x_2} = \alpha_0 \ \text{ when } \ x_2 = 0, \quad \frac{\partial h}{\partial x_2} = \alpha_1 \ \text{ when } \ x_2 = l. \tag{4}$$

We note that condition $|\alpha_0|, |\alpha_1| << 1$ must hold true in the framework of thin-layer approximation.

We assume that the free surfaces are thermo-isolated:

$$\frac{\partial T}{\partial \mathbf{n}} = 0 \ \text{ when } \ x_3 = h(x_1, x_2, t), \tag{5}$$

where $\mathbf{n}$ is unit normal vector to the interfacial boundary.

The heat conduction equation can be written in the form:

$$T_t + (\mathbf{v} \cdot \nabla_3)T = \chi\Delta_3 T, \tag{6}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is velocity vector, ∇_3 and Δ_3 are the three-dimension gradient and the Laplacian.

It follows from mean value theorem that there exist $\xi \in [0, h]$ and $\eta \in [x_3, \xi]$ such, that

$$T(x_1, x_2, x_3, t) - T^*(x_1, x_2, t) = \frac{\partial T}{\partial x_3}(x_1, x_2, \eta, t) \cdot (x_3 - \xi).$$

We note that $|x_3 - \xi| \leqslant h \leqslant \varepsilon l$. So

$$T = T^* + O(\varepsilon).$$

Now we integrate left hand side of (6) over the interval [0,h]:

$$\int_0^h (T_t + (\mathbf{v} \cdot \nabla_3)T)dz = hT_t^* + \mathbf{q}\nabla T^* + O(\varepsilon) = hT_t^* + \nabla \cdot (\mathbf{q}T^*) - T\nabla \cdot \mathbf{q} + O(\varepsilon) = \\ = (hT^*)_t + \nabla \cdot (\mathbf{q}T^*) + O(\varepsilon),$$

The last equality follows from (2).

Integration of the right hand side of (6) gives:

$$\int_0^h \chi \Delta_3 T dz = \chi h \Delta_3 T^* + O(\varepsilon).$$

It is follows from (5) that $\nabla h \cdot \nabla T^* = 0$. So, $h \cdot \Delta T^* = \nabla \cdot (h \nabla T^*)$ and

$$\int_0^h \chi \Delta_3 T dz = \chi \nabla \cdot (h \nabla T^*) + O(\varepsilon).$$

The members of the order of $O(\varepsilon)$ can be neglected. We put $T(x_1, x_2, t) := T^*(x_1, x_2, t)$ and obtain the equation

$$(hT)_t + \nabla \cdot (\mathbf{q}T) = \chi \nabla \cdot (h \nabla T). \tag{7}$$

The system of equations (1)–(2), (7) is the closed system connecting the flow rate, the film thickness and its temperature.

2. Numerical solution of the one-dimensional stationary problem

We consider the one-dimensional stationary problem. The system (1)–(2), (7) is written as

$$\frac{\sigma_0}{\rho}(h'''h)' = \frac{\kappa}{\rho}T'' + gh', \quad (hT')' = 0, \; q = 0.$$

We integrate the first two equations and eliminate T. Using (3) we obtain

$$\frac{\sigma_0}{\rho}(h'''h) = \frac{\kappa b}{\rho h} + gh, \tag{8}$$

where prime denotes differentiation in $x_2 := x$, b=const.

We rewrite (8) in dimensionless form:

$$h'''h = \frac{K}{\eta h} + \frac{h}{\eta}, \tag{9}$$

where $\dfrac{1}{\eta} = \dfrac{gl^3\rho}{\sigma_0 \delta}, \; \dfrac{K}{\eta} = \dfrac{\kappa l^3 b}{\sigma_0 \delta^3}, \; \delta = \varepsilon \mathrm{MaCr}, \; \mathrm{Ma} = \kappa \delta T l / \rho \nu^2, \; \mathrm{Cr} = \kappa \delta T / \sigma_0.$

We make the change $s = \dfrac{x}{\eta^{1/3}}$ of the independent variable for convenience in solving problem numerically. Equation (9) will be written as

$$h\frac{d^3h}{ds^3} = \frac{K}{h} + h. \tag{10}$$

In addition to the boundary conditions (4) we specify condition

$$h(0) = h_0, \tag{11}$$

that is equivalent to definition of the volume of the flow.

Note, that if $h_0 = \sqrt{-K}$ and $\alpha_0 = \alpha_1 = 0$ (the contact angle equals to $\pi/2$), then the boundary value problem (10), (11), (4) has the constant solution $h = \sqrt{-K}$.

Now we will disturb the contact angle α_1. The problem (10), (11), (4) is solved numerically using the shooting method. The numerical results are presented in Fig.1.

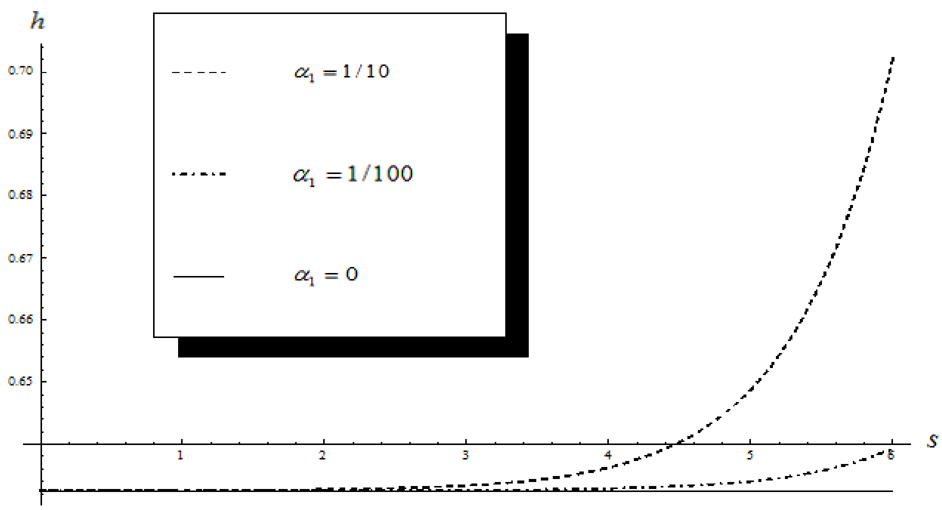

Fig. 1. The dependence of the thickness from the coordinate for various α_1 when $\alpha_0 = 0$, $K = -2/5$

As can be seen from the calculation results, when the deviate of the contact angle from $\pi/2$ is small, the solution is still exists, but when it grows, the solution begin to collapse.

3. Stability of the solution

As it is found, the system (1)–(2), (7) with boundary conditions (3)–(5) has the solution with a constant thickness. Let

$$h_0 = 1, \tag{12}$$

Then

$$q = 0,\ T = bx + c, \tag{13}$$

where $b = -\dfrac{\beta}{\gamma}$, $\beta = \mathrm{GaCr}/(\varepsilon\mathrm{Ma})$, $\gamma = \lim\limits_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{Cr}$, $\mathrm{Ga} = gl^3/\nu^2$, c is a constant which means a characteristic value of the temperature.

We linearize the problem (1)–(2), (7), (3)–(5), closely to the solution (12)–(13) and obtain the system of equations for disturbances:

$$\delta h_{tt} + h_{xxxx} = \gamma T_{xx} + \beta h_x, \quad T_t + bq = \alpha(bh_x + T_{xx}), \quad h_t + q_x = 0 \tag{14}$$

with boundary conditions

$$h_x(0) = h_x(1) = 0, \quad T_x(0) = T_x(1) = 0, \quad q(0) = q(1) = 0, \tag{15}$$

where $\alpha = 1/(\varepsilon\mathrm{MaPr})$, $\mathrm{Pr} = \nu/\chi$.

This system (14)-(15) is reduced to the equation for h:

$$\left(\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\alpha\partial t}\right)\left(\frac{\partial^4}{\partial x^4} + \delta\frac{\partial^2}{\partial t^2}\right) - 2\beta\frac{\partial^3}{\partial x^3}\right) h = 0 \tag{16}$$

with boundary conditions

$$h_{xxx} - \beta h = 0, \quad h_x = 0, \quad \alpha h_{xxxxx} = \beta(2\alpha h_{xx} + h_t) \;\; \text{when} \;\; x = 0, \; x = 1 \tag{17}$$

We seek the solution of the problem (16)–(17) in the form

$$h(x,t) = e^{\lambda t} f(x) \tag{18}$$

and obtain the spectral problem

$$\left(\left(\frac{\partial^2}{\partial x^2} - \frac{\lambda}{\alpha}\right)\left(\frac{\partial^4}{\partial x^4} + \delta\lambda^2\right) - 2\beta\frac{\partial^3}{\partial x^3}\right) f = 0 \tag{19}$$

$$\begin{aligned} &f_x(0) = f_x(1) = 0, \quad f_{xxx}(0) - \beta f(0) = f_{xxx}(1) - \beta f(1) = \\ &= f_{xxxxx}(0) - \left(2\beta f_{xx}(0) + \frac{\lambda}{\alpha} f(0)\right) = f_{xxxxx}(1) - \left(2\beta f_{xx}(1) + \frac{\lambda}{\alpha} f(1)\right) = 0. \end{aligned} \tag{20}$$

Assuming the parameter β to be small, we expand the solution into series in the β:

$$f = f_0 + \beta f_1 + ..., \; \lambda = \lambda_0 + \beta\lambda_1 + ...$$

From the boundary value problem (19), (20) in the zero and the first order, we obtain

$$(\lambda_0)_n = i\frac{n^2\pi^2}{\sqrt{\delta}}, \quad n \in Z$$

$$\lambda_1 = 0.$$

In the second order for $\alpha = 0.002$, $\delta = 0.01$, it is obtained that

$$\lambda_2 = -0.0000202508 - 0.000000267649i$$

for the first mode (n=1). It means that the solution with a constant thickness is stable in the second approximation.

Then we make the parameter continuation for β using orthogonalization method described in [11, 12]. The values of λ obtained are presented in Tab. 1.

That is for $g < 0,282406\, m/s^2$ the solution is stable and for $g > 0,282406\, m/s^2$ it becomes unstable.

Table 1. The values of λ obtained using orthogonalization method for various β

$g,\ m/s^2$	β	λ
0	0	$i \cdot 42,9075872459633$
0,01	1,33	$-0,000021 + i \cdot 42,9083737144401$
0,2	26,66	$-0,000009146 + i \cdot 42,9079533490524$
0,282406	37,6541	$i \cdot 42,9072745305347$
0,4	53,33	$0,0000072862 + i \cdot 42,9073447541626$
1,0	133,33	$0,00004554646 + i \cdot 42,9060715500626$
4,0	533,33	$0,00163450436266563 + i \cdot 95,8899780393908$
7,0	933,33	$0,00380961170250427 + i \cdot 72,4009517411667$
9,8	1306,66	$0,00637246347482759 + i \cdot 61,0867150009782$

Conclusion

Deformation of the free vertical liquid film by combined effect of gravity and thermocapillary forces is considered in the framework of thin-layer approximation. The system of equations connecting the flow rate, the film thickness and its temperature is obtained.

The solution $h = const$, $T = bx$, $q = 0$ of this system is found. Using the shooting method, one-dimensional stationary problem is solved numerically for values of contact angle close to $\pi/2$.

Parameter β is proportional to the gravitation. When β is small, it is obtained, that the solution is stable in the second approximation. Using numerical continuation method, value of the gravity for which this solution became unstable is found.

Then it is planned to study the stability of the free liquid film taking into account the evaporation effect.

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Равновесие и устойчивость свободной жидкой пленки в продольном поле тяжести

Оксана А. Бурмистрова

Исследуется деформация свободной вертикальной жидкой пленки при совместном действии силы тяжести и термокапиллярных сил. В приближении тонкого слоя получена система уравнений, связывающая расход жидкости, толщину плёнки и её температуру. Методом стрельбы одномерная стационарная задача численно решена при значениях краевого угла, близких к прямому. Обнаружено существование плёнки с постоянной толщиной. Для этого решения показана устойчивость при пониженной гравитации. С помощью метода продолжения по параметру найдено значение ускорения тяжести, при котором решение становится неустойчивым.

Ключевые слова: жидкая плёнка, свободная поверхность, приближение тонкого слоя, термокапиллярный эффект, малые возмущения, устойчивость решения.