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Lyapunov Exponents in 1D Anderson Localization with Long-range Correlations

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The Lyapunov exponents for Anderson localization are studied in a one dimensional disordered system. A random Gaussian potential with the power law decay $\sim 1/|x|^q$ *of the correlation function is considered. The exponential growth of the moments of the eigenfunctions and their derivative is obtained. Positive Lyapunov exponents, which determine the asymptotic growth rate are found.*

Keywords: long-range correlations, Furutsu-Novikov formula, fractional derivatives.

In this paper we consider Anderson localization [1] in a one dimensional disordered system with a long-range correlations. The recent realization of disordered systems by using ultra cold atoms [2, 3] in optical lattices and microwave realization of the Hofstadter butterfly [4] show that the random potential in the experiments are highly correlated. The increased interest in the problem of Anderson localization in random potentials with long-range correlations is also relevant to studies of the metal-insulator transition [5, 6].

Anderson localization in a one dimensional disordered system is described in the framework of the eigenvalue problem

$$\epsilon\phi(x) = -\frac{d^2}{dx^2}\phi(x) - V(x)\phi(x)\,, \tag{1}$$

with a Gaussian random potential $V(x)$. The long-range correlations of the disorder is modelled by the two point correlation function $\mathcal{C}(x)$ with the power law decay at the large scale

$$\langle V(x')V(x)\rangle = \mathcal{C}_q(x-x') = \frac{C_q}{|x-x'|^q}\,, \tag{2}$$

where $q > 0$. Spectral properties of the random operator of Eq. (1) (and its discrete counterpart) were studied [7–9], and a rigorous result on localization of Eq. (1) with the power-law correlation functions was stated in [8]. It has also been shown by various techniques under study of the metal-insulator transition. that all eigenfunctions are localized for correlated potentials with the correlation decay rate $0 < q < 1$ [6, 10–12]. Due to the physical interpretation, see discussion in Ref. [5], one of the main results is the absence of the absolutely continuous spectra for the random Schrödinger operator (1) with the correlation properties due to Eq. (2). This means that the eigenfunctions $\phi(x)$ are localized, and investigation of Lyapunov exponents is a serious task related to localization of the eigenfunctions.

The Lyapunov exponents are important in spectral theory, since they govern the asymptotic behavior of the wave functions. They are defined on the asymptotic behavior of the averaged

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envelope $\gamma_s(\epsilon) = \lim_{x\to\infty} \frac{\langle \ln \phi^2(x)\rangle}{2x}$. It was shown by rigorous analysis that the positive Lyapunov exponents are absent for the absolutely continuous spectrum, while the positiveness of the Lyapunov exponents ensures that the spectrum is pure point [9, 13].

In this paper, we calculate $\langle\phi^2(x)\rangle$ of solutions of Eq. (1) for a certain energy ϵ, with given boundary conditions at some point, for example $\phi(x=0)$ and $\phi'(x=0)$, where prime means the derivative with respect to x. Since the distribution of random potentials is translationally invariant, it is independent of the choice of the initial point as $x=0$. It will be shown that this quantity grows exponentially with the rate $\gamma(\epsilon) = \lim_{x\to\infty} \frac{\ln\langle\phi^2(x)\rangle}{x} > 0$. Note that it is different from γ_s, which supposes a knowledge of all the even moments [14–17].

We develop a general procedure which is suitable for calculation of all moments of the wave function and its first derivative. To this end the Schrödinger equation (1) is considered as the Langevin equation and the x coordinate as a formal time. For the δ correlated process it can be easily mapped on the Fokker-Planck (diffusion) equation for the probability distribution function $\mathcal{P}(\phi, \phi')$ [13, 18]. Unlike this, the two point correlation function (2), which corresponds to the stationary process, leads to additional integration over the formal "time" with a memory kernel. The method of consideration enables one to observe the exponential growth of $\langle\phi^2(x)\rangle$ with the Lyapunov exponent $\gamma(\epsilon) > 0$.

Since the Schrödinger equation (1) is a linear stochastic equation, equations for the $2n$ moments of the type

$$M_{k,l}(x) = \langle[\phi(x)]^k[\phi'(x)]^l\rangle\,, \quad k+l = 2n, \quad k,l = 0,1,2,\dots\,, \tag{3}$$

can be obtained in the closed form. To this end we rewrite Eq. (1) in the form of the Langevin equation. The x coordinate is considered as a formal time on the half axis $x \equiv \tau$, $\tau \in [0,\infty)$ and the new dynamical variables $u(\tau) = \phi(x)$, $v(\tau) = \dot{u} = \phi'(x)$ are defined. In the new variables the Langevin equation reads

$$\dot{u} = v\,, \quad \dot{v} = -[\epsilon + V(\tau)]u\,, \tag{4}$$

where $V(\tau)$ is now the long-range correlated noise

$$\mathcal{C}_\alpha(\tau) = \frac{C_\alpha}{\tau^{1+\alpha}}\,. \tag{5}$$

It is convenient to set $q = 1+\alpha$ and $C_q \equiv C_\alpha$. In the new variables the expectation values of Eq. (3) are now $M_{k,l}(\tau) = \langle u^k v^l\rangle$. Solutions of Eq. (4) are obtained as functionals

$$v(t) = -\int_0^t [\epsilon + V(\tau)]u(\tau)d\tau\,, \quad u(t) = \int_0^t v(\tau)d\tau\,. \tag{6}$$

Following [16] we obtain a temporal equation for the moments from the Langevin equation (4) and its solutions (6). Differentiating $M_{k.l}(\tau)$ with respect to τ, we obtain

$$\dot{M}_{k,l} = kM_{k-1,l+1} - l\epsilon M_{k+1,l-1} - l\langle V(t)u^{k+1}v^{l-1}\rangle\,. \tag{7}$$

Eq. (7) can be obtained in a closed form. The application of the Furutsu-Novikov formula [19] to the last term in Eq. (7) yields

$$\langle V(t)\mathcal{F}[V(t)]\rangle = \int_0^t d\tau' \langle V(t)V(\tau')\rangle \Big\langle \frac{\delta\mathcal{F}[V(\tau)]}{\delta V(\tau')}\Big\rangle\,. \tag{8}$$

Eq. (8) can be found explicitly for the δ correlated noise. The correlation function can be expanded in the δ function and its derivatives $\mathcal{C}(t-\tau) = \sum_p c_p \delta^{(p)}(t-\tau)$. To this end the correlation function is truncated $\mathcal{C}(x-x') = 0$ for $|x-x'| > X_0$, where X_0 is an arbitrary large. Substituting this expansion in Eq. (8), we obtain*

$$\int_0^t d\tau \mathcal{C}(t-\tau)\Big\langle \frac{\delta \mathcal{F}[V(\tau)]}{\delta V(\tau)} \Big\rangle = -(l-1)\int_0^t \mathcal{C}_\alpha(t-\tau) M_{k+2,l-2}(\tau) d\tau \,. \tag{9}$$

We also used here that the variational derivative equals zero when $\tau > t$. Here the solution of Eq. (6) is used to obtain the functional derivative of the functional $\mathcal{F}[V(\tau)] = u^{k+1}v^{l-1}$. Substituting the solution of Eq. (9) in Eq. (7), we obtain that the temporal behavior of the moments is described by the fractional–differential equation

$$\dot{M}_{k,l} = kM_{k-1,l+1} - l\epsilon M_{k+1,l-1} + l(l-1)D_t^\alpha M_{k+2,l-2} \,, \tag{10}$$

where the convolution integral in Eq. (9) is the fractional derivative $D_t^\alpha f(t)$

$$D_t^\alpha f(t) = C_\alpha \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1+\alpha}} \,. \tag{11}$$

Here the correlation function $\mathcal{C}_\alpha(t)$ defines the memory kernel, or the causal function. Eqs. (10) and (11) are relevant to the fractional Fokker-Planck equations which describe a variety of physical processes related to fractional diffusion [20–22]. An important technique for the treatment of the fractional equation is the Laplace transform. It is worth stressing that both analytical properties of this fractional integration and the Laplace transform depend on α.

For $-1 < \alpha < 0$ Eq. (10) is readily solved by means of the Laplace transform. Defining $\hat{\mathcal{L}}[M_{k,l}(t)] = \tilde{M}_{k,l}(s)$, one obtains from Eq. (11) $\hat{\mathcal{L}}[D_t^\alpha M_{k,l}(t)] = C_\alpha \Gamma(-\alpha) s^\alpha \tilde{M}_{k,l}(s)$, where $\Gamma(\alpha)$ is the gamma function. For simplicity, disregarding the sign of the correlation function (5), we set $C_\alpha = 2\sigma^2/\Gamma(-\alpha)$, where the variance $\sigma^2 = \langle V^2(0)\rangle$ determines the amplitude of the noise. Then, we introduce $2n+1$-dimensional vectors $\mathbf{M}_n(t) = \Big(M_{2n,0}, M_{2n-1,1}, \dots, M_{1,2n-1}, M_{0,2n}\Big)$ in the "time" space and $\tilde{\mathbf{M}}_n(s) = \hat{\mathcal{L}}[\mathbf{M}_n(t)]$ in the Laplace space, correspondingly. Then the solution of Eq. (10) is the Laplace inversion of the following vector

$$\tilde{\mathbf{M}}_n(s) = \frac{1}{s - A_n(s)}\mathbf{M}_n(0) \,, \tag{12}$$

where $(2n+1)\times(2n+1)$ matrix $A_n(s)$ consists of coefficients from the matrix equation (10). In the limit $s \to 0$ the disorder term of order of $s^\alpha \to \infty$ is dominant, and the maximal eigenvalues of A_n can be evaluated at the energy $\epsilon \approx 0$. Following Ref. [17], it can be proven that for $\epsilon = 0$ the maximal eigenvalues of A_n behaves for large n as $\Omega(s) \approx s^{\alpha/3}\sigma^{2/3}(2n)^{4/3}$. Expanding the initial condition $M_n(0)$ over the eigenfunctions of A_n, we obtain that the maximal growth of the nth moment is

$$M_n(t) = \hat{\mathcal{L}}^{-1}\Big[\frac{s^{-\alpha/3}}{s^{1-\alpha/3} - \sigma^{2/3}(2n)^{4/3}}\Big]M_\Omega(0) \,. \tag{13}$$

*The following sequence of equalities is used
$\int d\tau' \mathcal{C}(t-\tau')\Big\langle \frac{\delta\mathcal{F}[V(\tau)]}{\delta V(\tau')}\Big\rangle = \sum_p c_p \Big(-\frac{d}{dt}\Big)^p \Big\langle \frac{\delta\mathcal{F}[V(t)]}{\delta V(t)}\Big\rangle = \sum_p c_p\Big(-\frac{d}{dt}\Big)^p M_{k+2,l-2}(t) = \int d\tau \mathcal{C}(t-\tau) M_{k+2,l-2}(\tau)$.

The inverse Laplace transform is the definition of the Mittag-Leffler function [23]:

$$E_{1-\alpha/3}\Big(\frac{3}{4}\sigma^{2/3}(2n)^{4/3}t^{1-\alpha/3}\Big).$$

Asymptotic behavior of the Mittag-Leffler function for $t \to \infty$ is determined by the exponential function $\exp\Big[(2n\sqrt{\sigma})^{4/(3-\alpha)}t\Big]$. Therefore the exponential growth of the nth moment is due to the Lyapunov exponent

$$\gamma(0) \sim (2n\sqrt{\sigma})^{4/(3-\alpha)} \tag{14}$$

for $-1 < \alpha < 0$.

For $\alpha > 0$ the fractional integral diverges. Therefore, to obtain the Lyapunov exponents avoiding the difficulty one discards the causality principle and extend the consideration of the random process on the entire x axis $x \in (-\infty, +\infty)$. For this formal consideration, the Furutsu-Novikov formula in Eq. (9) reads

$$\begin{aligned} &-(l-1)C_\alpha \int_{-\infty}^{x} \frac{M_{k+2,l-2}(y)}{(x-y)^{1+\alpha}}dy \quad \text{for } x > 0\,, \\ &-(l-1)C_\alpha \int_{x}^{\infty} \frac{M_{k+2,l-2}(y)}{(y-x)^{1+\alpha}}dy \quad \text{for } x < 0\,. \end{aligned} \tag{15}$$

Setting again $C_\alpha = 1/\Gamma(-\alpha)$, we obtain that Eq. (15) is the definition of the Riesz/Weyl fractional derivative $\mathcal{W}_x^\alpha$ see e.g., [21, 22, 24]. Therefore, Eq. (10) now reads

$$\frac{d}{dx}M_{k,l} = kM_{k-1,l+1} + l\epsilon M_{k+1,l-1} + l(l-1)\mathcal{W}_x^\alpha M_{k+2,l-2}\,. \tag{16}$$

A specific property that we use is the fractional differentiation of an exponential $\mathcal{W}_x^\alpha \exp(\gamma x) = \gamma^\alpha \exp(\gamma x)$. Substituting this in Eq. (7), one seeks the solution for the maximal moment growth

$$M_{k,l}(x) = \exp(\pm\gamma x)M_{k,l}(x=0)\,, \tag{17}$$

where plus stays for $x > 0$ and minus for $x < 0$, respectively. One readily checks that the both cases yield the same algebraic equation

$$\gamma\mathbf{M}_n = A_n(\gamma)\mathbf{M}_n\,, \tag{18}$$

where the moment vector $\mathbf{M}_n$ is defined above and the matrix $A_n(\gamma)$ is defined from Eq. (16). Therefore, $\Omega(\gamma) = \gamma^\alpha\sigma^2/\epsilon$, where conditions $\gamma \ll \epsilon/\sigma^2$ and $\gamma^\alpha \ll \epsilon/\sigma^2$ are used. Finally, one obtains

$$\gamma(\epsilon) \sim \Big(\frac{\epsilon}{\sigma^2}\Big)^{1/(\alpha-1)}\,. \tag{19}$$

This solution for γ also yields conditions of validity for different values of energy ϵ. Indeed, for $0 < \alpha < 1$ Eq. (18) describe an exponential growth for asymptotically large energies $\epsilon \gg \sigma^2$, since, in this case, $\gamma(\epsilon) \ll \epsilon/\sigma^2$. On the contrary, when $\alpha > 1$ the solution of Eq. (19) is valid for $\epsilon \ll \sigma^2$. This follows from the condition $\gamma^\alpha \ll \epsilon/\sigma^2$. Note that for large negative values of the energy $\Omega \sim 2\sqrt{|\epsilon|}/|\sigma|$, and this is just the Lyapunov exponent $\gamma(\epsilon) \sim 2\sqrt{|\epsilon|}/|\sigma|$.

In conclusion, we studied the Lyapunov exponents for Anderson localization in a one-dimensional disordered system with a long-range correlations. The averaged behavior of the second moment of the eigenfunction is calculated, and its asymptotic exponential growth for $|x| \to \infty$ is determined by the Lyapunov exponents for different values of the energy ϵ. The main result of the study is the existence of the positive Lyapunov exponents $\gamma(\epsilon) > 0$ for the rate $q = 1+\alpha > 0$ of the power law decay of the correlation function. It is relevant to the exponential localization of the eigenfunctions of the random Schrödinger operator of Eq. (1).

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References

[1] P.W. Anderson, *Phys. Rev.*, **109**(1958), 1492.

[2] L. Sanchez-Palencia et al., *Phys. Rev. Let.*, **98**(2007), 210401 ; J. Billy et al., *Nature*, **453**(2008), 891.

[3] G. Roati et al., *Nature*, **453**(2008), 895.

[4] U. Kuhl et al., *Appl. Phys. Let.*, **77**(2000), 633; U. Kuhl, H.-J. Stöckmann, *Phys. Rev. lett.*, **80**(1998), 3232.

[5] A.M. Garcia-Garcia, E. Cuevas, Absence of localization in one-dimensional disordered systems, cond-mat.0808.3757.

[6] F.A.B.F. de Moura, M.L. Lyra, *Phys. Rev. Let.* **81**(1998), 3735.

[7] S. Kotani, Proc. Kyoto Stoch. Com., (1982); B. Simon, *Comm. Math. Phys.*, **89**(1983), 227.

[8] S. Kotani, B. Simon, *Comm. Math. Phys.*, **112**(1987), 103.

[9] L.A. Pastur, A.L. Figotin, Spectra of Random and Almost-Periodic Operators, Springer, Berlin, 1992.

[10] F.A.B.F. de Moura, M.L. Lyra, *Physica A*, **266**(1999), 465; S. Russ, et al., *Physica A*, **266**(1999), 492.

[11] U. Kuhl, F.M. Izrailev, A.A. Krokhin, *Phys. Rev. Lett.*, **100**(2008), 126402.

[12] H. Shima, T. Nomura, T. Nakayama, *Phys. Rev. B*, **70**(2004), 075116.

[13] I.M. Lifshits, S.A. Gredeskul, L.A. Pastur, Introduction to the theory of disordered systems, Wiley-Interscience, New York, 1988.

[14] K. Mallick, P. Marcq, *Phys. Rev. E*, **66**(2002), 041113.

[15] H. Schomerus, M. Titov, *Phys. Rev. E*, **66**(2002), 066207.

[16] R. Zilmer, A. Pikovsky, *Phys. Rev. E*, **67**(2003), 061117.

[17] S. Fishman, A. Iomin, K. Mallick, *Phys. Rev. E*, **78**(2008), 066605.

[18] B.I. Halperin, *Phys. Rev.*, **139A**(1965), 104.

[19] V.I. Kliatskin, Stochastic equations and waves in randomly inhomogeneous media, Nauka, Moskva, 1980 (in Russian).

[20] J.-P. Bouchaud, A. Georges, *Phys. Rep.*, **195**(1990), 127.

[21] R. Metzler, J. Klafter, *Phys. Rep.*, **339**(2000), 1.

[22] G.M. Zaslavsky, *Phys. Rep.*, **371**(2002), 461.

[23] H. Bateman, A. Erdélyi, Higher Transcendental functions, Mc Graw-Hill, New York, V.3, 1955.

[24] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

Ляпуновские экспоненты в локализации Андерсона с длинными корреляциями

Александр Иомин

В работе изучается показатель Ляпунова, характеризующий локализацию Андерсона в одномерном случае. Рассматривается случайный потенциал в виде гауссовского случайного процесса с корреляционной функцией, затухающей по степенному закону. Получен экспоненциальный рост четных моментов собственных волновых функций. Показано, что асимптотический рост четных моментов собственных волновых функций определяется положительной ляпуновской экспонентой. Характерные значения показателя Ляпунова найдены для разных режимов случайного потенциала.

Ключевые слова: дальние корреляции, формула Фурутцу-Новикова, фрактальные производные.