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Analytic Continuation of Power Series by Means of Interpolating the Coefficients by Meromorphic Functions

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We study the problem of analytic continuation of a power series across an open arc on the boundary of the circle of convergence. The answer is given in terms of a meromorphic function of a special form that interpolates the coefficients of the series. We find the conditions for the sum of the series to extend analytically to a neigbourhood of the arc, to a sector defined by the arc, or to the whole complex plane except some arc on the convergence disk.

Keywords: Power series, analytic continuation, interpolating meromorphic function, indicator function.

Introduction

The problem of analytic continuation and finding singular points of a function has a rich and long history. It has been studied by many prominent mathematicians such as Carlson, Polya, Hadamard, and others (see, for example, [1]). There are different approaches to studying such problems. In this paper we consider the question of continuation of a power series across the boundary of its circle of convergence. First, we recall some definitions and results.

Consider a power series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

in $z \in \mathbb{C}$, whose domain of convergence is the unit disk $D_1 := \{z \in \mathbb{C} : |z| < 1\}$.
The Cauchy-Hadamard theorem yields that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|f_n|} = 1. \tag{2}$$

We say that a function φ *interpolates* the coefficients of the series (1), if

$$\varphi(n) = f_n \ \text{ for all } \ n \in \mathbb{N}. \tag{3}$$

Recall (see, e.g. [2]) that the *indicator* function $h_\varphi(\theta)$ for an entire function φ is defined as the upper limit

$$h_\varphi(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln|\varphi(re^{i\theta})|}{r}, \ \ \theta \in \mathbb{R}.$$

Let Δ_σ be the sector $\{z = re^{i\theta} \in \mathbb{C} : |\theta| \leqslant \sigma\}$, $\sigma \in [0, \pi)$. By $\gamma_{\sigma,\rho}$ we denote the open arc $\partial D_\rho \setminus \Delta_\sigma$.

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There are at least three types of questions of analytic continuation of (1) across the arc γ_σ. The first one asks about the conditions for continuation to the whole complex plane except $\partial D_1 \setminus \Delta_\sigma$. The answer is given by Polya's theorem.

Theorem (Polya [3]). *The series* (1) *extends analytically to* $\mathbb{C}$, *possibly except the arc* $\partial D_1 \setminus \gamma_\sigma$, *if and only if there exists an entire function of exponential type* $\varphi(\zeta)$ *interpolating the coefficients* f_n *such that*

$$h_\varphi(\theta) \leqslant \sigma|\sin\theta| \quad \textit{for} \quad |\theta| \leqslant \pi.$$

Two other questions concern continuation to the sector $\mathbb{C} \setminus \Delta_\sigma$ defined by the arc $\gamma_\sigma = \partial D_1 \setminus \Delta_\sigma$, or to a neighbourhood of this arc. Both of them are answered by Arakelian's theorems.

Theorem (Arakelian [4, 5]). *The sum of the series* (1) *extends analytically to the sector* $\mathbb{C}\setminus\Delta_\sigma$ *if and only if there is an entire function* φ *of exponential type interpolating the coefficients of the series* f_n *whose indicator function* $h_\varphi(\theta)$ *satisfies the condition*

$$h_\varphi(\theta) \leqslant \sigma|\sin\theta| \quad \textit{for} \quad |\theta| < \frac{\pi}{2}. \tag{4}$$

The continuation property of $f(z)$ to a neighbourhood of the arc γ_σ was studied in [6] (see also [7]). In this case we refer to γ_σ as an *arc of regularity* for the series (1).

Theorem (Arakelian [7]). *The open arc* $\gamma_\sigma = \mathbb{C} \setminus \Delta_\sigma$ *is an arc of regularity of the series* (1) *if and only if there is an entire function* φ *of exponential type interpolating the coefficients of the series* f_n *whose indicator function* $h_\varphi(\theta)$ *satisfies the conditions:* $h_\varphi(0) = 0$ *and*

$$\overline{\lim_{\theta\to 0}} \frac{h_\varphi(\theta)}{|\theta|} \leqslant \sigma. \tag{5}$$

The inequality (4) implies (5), and (5) together with (2) and (3) gives $h_\varphi(0) = 0$.

Sometimes it can be easier to interpolate coefficients by meromorphic functions instead of entire ones. Here we consider interpolating functions of the form

$$\psi(\zeta) = \phi(\zeta)\frac{\prod_{j=1}^{p}\Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q}\Gamma(c_k\zeta + d_k)}, \tag{6}$$

where $\phi(\zeta)$ is entire, $a_j \geqslant 0$, $j = 1, ..., p$, and

$$\sum_{j=1}^{p} a_j = \sum_{k=1}^{q} c_k. \tag{7}$$

Denote also

$$l = \sum_{k=1}^{q} |c_k| - \sum_{j=1}^{p} a_j.$$

In this paper we find the conditions on a meromorphic interpolating function such that the conclusions of all theorems formulated above still hold.

Theorem 1. *The sum of the series* (1) *extends analytically to* $\mathbb{C} \setminus (\partial D_1 \cap \Delta_\sigma)$ *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (6) *interpolating the coefficient* f_n *such that the entire function*

$$\varphi(\zeta) := \phi(\zeta)\frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta}}{\prod_{k=1}^{q} |c_k|^{c_k\zeta}}$$

satisfies

$$h_\varphi(\theta) + \frac{\pi}{2}l|\sin\theta| \leqslant \sigma|\sin\theta| \quad \textit{for} \quad |\theta| \leqslant \pi.$$

Theorem 2. *The sum of the series* (1) *extends analytically to the open sector* $\mathbb{C} \setminus \Delta_\sigma$ *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (6) *interpolating the coefficients* f_n *such that the entire function*

$$\varphi(\zeta) := \phi(\zeta) \frac{\prod_{j=1}^{p} a_j{}^{a_j \zeta}}{\prod_{k+1}^{q} |c_k|^{c_k \zeta}}$$

satisfies the conditions

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \max\left\{h_\varphi\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} l, h_\varphi\left(\frac{\pi}{2}\right) + \frac{\pi}{2} l\right\} \leqslant \sigma.$$

Theorem 3. *The open arc* $\gamma_\sigma = \partial D_1 \setminus \Delta_\sigma$ *is an arc of regularity for the series* (1) *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (6) *interpolating the coefficients* f_n *such that the entire function*

$$\varphi(\zeta) := \phi(\zeta) \frac{\prod_{j=1}^{p} a_j{}^{a_j \zeta}}{\prod_{k+1}^{q} |c_k|^{c_k \zeta}}$$

satisfies the conditions

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \overline{\lim_{\theta \to 0}} \frac{h_\varphi(\theta)}{|\theta|} + \frac{\pi}{2} l \leqslant \sigma.$$

1. Proof of Theorem 2

To begin with, we prove theorem 2 in the case when all c_k are positive, i.e. $l = 0$. Then the statement is the following.

The sum of the series (1) *extends analytically to the open sector* $\mathbb{C} \setminus \Delta_\sigma$ *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (6) *interpolating the coefficients* f_n *such that the indicator function of*

$$\varphi(\zeta) := \phi(\zeta) \frac{\prod_{j=1}^{p} a_j{}^{a_j \zeta}}{\prod_{k+1}^{q} c_k{}^{c_k \zeta}} \tag{8}$$

satisfies the conditions

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \max\left\{h_\varphi\left(-\frac{\pi}{2}\right), h_\varphi\left(\frac{\pi}{2}\right)\right\} \leqslant \sigma. \tag{9}$$

The indicator of an entire function of exponential type has the following property [7]: *if* $h_\varphi(0) = 0$ *then for* $\alpha \in (0, \pi)$

$$h_\varphi(\theta) \leqslant c_\alpha |\sin\theta| \quad \textit{for all} \quad |\theta| \leqslant \alpha,$$

$$c_\alpha = \frac{1}{\sin\alpha} \max\{h_\varphi(\alpha), h_\varphi(-\alpha)\}.$$

Let φ be an entire function of the form (8) satisfying the conditions (9). Show that the series (1) extends to the open sector $\mathbb{C} \setminus \Delta_\sigma$. It follows from the definition of an indicator that

$$|\varphi(re^{i\theta})| \leqslant e^{h_\varphi(\theta) r + o(r)} \ \text{ for } \ \theta \in \mathbb{R},$$

where $o(r)$ is infinitesimally small compared to r as $r \to \infty$.
Taking into account the property of indicator function stated above, we get

$$|\varphi(re^{i\theta})| \leqslant e^{\sigma |\sin\theta| r + o(r)} \ \text{ for } \ |\theta| \leqslant \frac{\pi}{2}.$$

Since $\varphi(\zeta)$ has the form (8), we obtain the inequality

$$|\phi(re^{i\theta})|\frac{\prod_{j=1}^{p}|a_j^{a_j re^{i\theta}}|}{\prod_{k=1}^{q}|c_k^{c_k re^{i\theta}}|}\leqslant e^{\sigma|\sin\theta|r+o(r)} \text{ for } |\theta|\leqslant\frac{\pi}{2},$$

which in terms of $\zeta=\xi+i\eta$ is written as

$$|\phi(\zeta)|\leqslant\left(\frac{\prod_{j=1}^{p}|a_j^{a_j\zeta}|}{\prod_{k=1}^{q}|c_k^{c_k\zeta}|}\right)^{-1}e^{\sigma|\eta|+o(|\zeta|)} \text{ for } \zeta\in\Delta_{\frac{\pi}{2}}. \tag{10}$$

We need the following estimate.

Lemma 1. *For all* $\zeta\in\Delta_{\frac{\pi}{2}}$

$$\left|\frac{\prod_{j=1}^{p}\Gamma(a_j\zeta+b_j)}{\prod_{k=1}^{q}\Gamma(c_k\zeta+d_k)}\right|\leqslant\frac{\prod_{j=1}^{p}|a_j^{a_j\zeta}|}{\prod_{k=1}^{q}|c_k^{c_k\zeta}|}e^{o(|\zeta|)}. \tag{11}$$

Proof. It is easy to see that for $|\zeta|\to\infty$ one has

$$|a\zeta|^{a\xi}\left(1-\frac{|b|}{|a\zeta|}\right)^{|a\zeta|}e^{-a\eta\arg(\zeta)}\leqslant|a\zeta+b|^{a\zeta}\leqslant|a\zeta|^{a\xi}(1+\frac{|b|}{|a\zeta|})^{|a\zeta|}e^{-a\eta\arg(\zeta)}.$$

This fact together with Stirling's formula gives

$$\frac{\prod_{j=1}^{p}|\Gamma(a_j\zeta+b_j)|}{\prod_{k=1}^{q}|\Gamma(c_k\zeta+d_k)|}\sim\frac{\prod_{j=1}^{p}|(a_j\zeta+b_j)^{(a_j\zeta+b_j)}e^{-(a_j\zeta+b_j)}(2\pi(a_j\zeta+b_j))^{\frac{1}{2}}|}{\prod_{k=1}^{q}|(c_k\zeta+d_k)^{(c_k\zeta+d_k)}e^{-(c_k\zeta+d_k)}(2\pi(c_k\zeta+d_k))^{\frac{1}{2}}|}\leqslant$$

$$\leqslant\frac{\prod_{j=1}^{p}|a_j\zeta|^{a_j\xi}(1+\frac{|b_j|}{|a_j\zeta|})^{|a_j\zeta|}e^{-a_j\eta\arg(\zeta)}|(a_j\zeta+b_j)^{b_j}e^{-(a_j\zeta+b_j)}(2\pi(a_j\zeta+b_j))^{\frac{1}{2}}|}{\prod_{k=1}^{q}|c_k\zeta|^{c_k\xi}(1-\frac{|d_k|}{|c_k\zeta|})^{c_k\zeta}e^{-c_k\eta\arg(\zeta)}|(c_k\zeta+d_k)^{d_k}e^{-(c_k\zeta+d_k)}(2\pi(c_k\zeta+d_k))^{\frac{1}{2}}|}\leqslant$$

$$\leqslant\frac{\prod_{j=1}^{p}|a_j^{a_j\zeta}|}{\prod_{k=1}^{q}|c_k^{c_k\zeta}|}\left|\zeta^{\zeta(\sum_{j=1}^{p}a_j-\sum_{k=1}^{q}c_k)}\right|\left|e^{-\zeta(\sum_{j=1}^{p}a_j-\sum_{k=1}^{q}c_k)}\right|\times$$

$$\times\frac{\prod_{j=1}^{p}(1+\frac{|b_j|}{|a_j\zeta|})^{a_j\xi}e^{-a_j\eta\arg(\zeta)}}{\prod_{k=1}^{q}(1+\frac{|d_k|}{|c_k\zeta|})^{c_k\xi}e^{-c_k\eta\arg(\zeta)}}\times\frac{\prod_{j=1}^{p}|a_j\zeta+b_j|^{b_j}e^{-b_j}|2\pi(a_j\zeta+b_j)|^{\frac{1}{2}}}{\prod_{k=1}^{q}|c_k\zeta+d_k|^{d_k}e^{-d_k}|2\pi(c_k\zeta+d_k)|^{\frac{1}{2}}}.$$

In view of (7), this inequality after some simplifications turns into

$$\left|\frac{\prod_{j=1}^{p}\Gamma(a_j\zeta+b_j)}{\prod_{k=1}^{q}\Gamma(c_k\zeta+d_k)}\right|\leqslant\frac{\prod_{j=1}^{p}|a_j^{a_j\zeta}|}{\prod_{k=1}^{q}|c_k^{c_k\zeta}|}|A\zeta+B|^{C}$$

where A,B and C are some constants.

Since $|A\zeta+B|^C=e^{\ln|A\zeta+B|^C}$ and

$$\lim_{|\zeta|\to\infty}\frac{\ln|A\zeta+B|^C}{|\zeta|}=0,$$

we get $|A\zeta+B|^C=e^{o(|\zeta|)}$ as $\zeta\to\infty$, i.e. the lemma's statement. □

It follows form (10) and (11) that for a meromorphic function $\psi(\zeta)$ defined by (6) we have

$$|\psi(\zeta)| \leqslant e^{\sigma|\eta|+o(|\zeta|)} \quad \text{for} \quad \zeta \in \Delta_{\frac{\pi}{2}}. \tag{12}$$

Consider the following function

$$g(\zeta, z) = \frac{z^{\zeta}}{e^{2\pi i\zeta} - 1}$$

of two complex variables $\zeta = \xi + i\eta,\ z = x + iy$. It is meromorphic in $\zeta \in \mathbb{C}$ and holomorphic in $z \in \mathbb{C} \setminus \mathbb{R}_+$.

Denote $D^* := \cup_{m\in\mathbb{Z}} D_{1/4}(m)$.
Notice that there exists a constant $c > 0$ such that

$$|e^{2\pi i\zeta} - 1| > \frac{e^{\pi(|\eta|-\eta)}}{c} \quad \text{for} \quad \zeta \in \mathbb{C} \setminus D^*.$$

From this we get the estimate

$$|g(\zeta, z)| < ce^{\xi \log|z| - (\pi - |\pi - \arg z|)|\eta|}$$

for $\zeta \in \mathbb{C} \setminus D^*$ and $z \in \mathbb{C} \setminus \mathbb{R}_+$. Using (12) for $\zeta \in \Delta_{\frac{\pi}{2}} \setminus D^*$ and $z \in \mathbb{C} \setminus \mathbb{R}_+$, we see that

$$|\psi(\zeta)||g(\zeta, z)| < ce^{\xi \log|z| - (\pi - \sigma - |\pi - \arg z|)|\eta| + o(|\zeta|)}. \tag{13}$$

For $\zeta \in (\Delta_{\frac{\pi}{2}} \setminus D^*)$ and $z \in \mathbb{C} \setminus \Delta_{\sigma+\delta}$ there is the following bound

$$|\psi(\zeta)||g(\zeta, z)| < ce^{\xi \log|z| - \delta|\eta| + o(|\zeta|)}.$$

Consider the integral

$$I_m = \int\limits_{\partial G_m} \psi(\zeta) g(\zeta, z) d\zeta,$$

over the oriented boundary of G_m that consists of the segments (see Fig. 1)

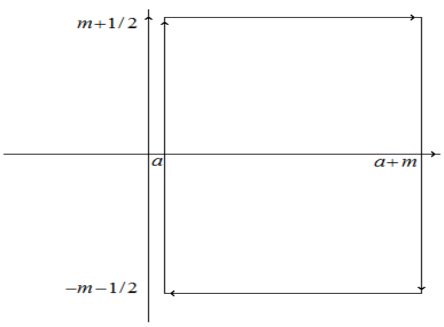

Fig. 1.

$$\Gamma_m^1 = \left[a - i\left(m + \frac{1}{2}\right), a + i\left(m + \frac{1}{2}\right)\right],$$
$$\Gamma_m^2 = \left[a + i\left(m + \frac{1}{2}\right), a + m + i\left(m + \frac{1}{2}\right)\right],$$
$$\Gamma_m^3 = \left[a + m + i\left(m + \frac{1}{2}\right), a + m - i\left(m + \frac{1}{2}\right)\right],$$
$$\Gamma_m^4 = \left[a + m - i\left(m + \frac{1}{2}\right), a - i\left(m + \frac{1}{2}\right)\right]$$

where $\frac{1}{4} < a < \frac{3}{4}$.

The integral I_m is the sum of four integrals $I_m^1, I_m^2, I_m^3, I_m^4$ over $\Gamma_m^1, \Gamma_m^2, \Gamma_m^3, \Gamma_m^4$ respectively. For $\zeta \in \Delta_{\frac{\pi}{2}} \setminus D^*$ и $z \in \mathbb{C} \setminus \Delta_{\sigma+\delta}$ there hold the following estimates

$$I_m^2 = \int\limits_{\Gamma_m^2} |\psi(\zeta)g(\zeta, z)||d\zeta| \leqslant ce^{-\delta(m+\frac{1}{2})} \int\limits_{a}^{a+m} e^{\xi \ln|z| + o(|\zeta|)} d\xi,$$

$$I_m^3 = \int\limits_{\Gamma_m^3} |\psi(\zeta)g(\zeta, z)||d\zeta| \leqslant ce^{(a+m)\ln|z| + o(m)} \int\limits_{-i(m+\frac{1}{2})}^{i(m+\frac{1}{2})} d\eta,$$

$$I_m^4 = \int\limits_{\Gamma_m^4} |\psi(\zeta)g(\zeta, z)||d\zeta| \leqslant ce^{-\delta(m+\frac{1}{2})} \int\limits_{a+m}^{a} e^{\xi \ln|z| + o(|\zeta|)} d\xi.$$

We see that for $z \in D_1 \setminus \Delta_{\sigma+\delta}$ the integrals I_m^2, I_m^3, I_m^4 tend to 0 as $m \to \infty$.

Thus,

$$\lim_{m\to\infty} I_m = \lim_{m\to\infty} \int\limits_{\partial G_m} \psi(\zeta)g(\zeta, z)d\zeta = \lim_{m\to\infty} \int\limits_{\Gamma_m^1} \psi(\zeta)g(\zeta, z)d\zeta = \lim_{m\to\infty} I_m^1.$$

In the domain G_m, the integrand has simple poles in real integer points and finitely many poles in points $\frac{-\nu - b_j}{a_j} \in G_m$ $\nu = 0, 1, ...$ (recall that a_j, b_j are parameters in the definition (6) of $\psi(\zeta)$).

The residue theorem yields

$$\int\limits_{\partial G_m} \varphi(\zeta)g(\zeta, z)d\zeta = \sum_{n=1}^{m} \varphi(n)z^n + P(z),$$

where $P(z)$ is a polynomial.

Consider the integral

$$I = \int\limits_{a-i\infty}^{a+i\infty} \varphi(\zeta)g(\zeta, z)d\zeta.$$

For $\zeta = a + i\eta$ and $z \in \mathbb{C} \setminus \Delta_{\sigma+\delta}$ we have

$$|\varphi(\zeta)||g(\zeta, z)| < ce^{a \ln|z| - \delta|\eta| + o(|\zeta|)}.$$

It follows from this inequality that the integral I converges absolutely and uniformly on any compact subset $K \subset \mathbb{C} \setminus \Delta_{\sigma+\delta}$, and defines a holomorphic function on the set of interior points of K. For $z \in D_1 \setminus \Delta_{\sigma+\delta}$

$$\int\limits_{\Gamma_m^1} \varphi(\zeta) g(\zeta, z) d\zeta \to \int\limits_{a-i\infty}^{a+i\infty} \varphi(\zeta) g(\zeta, z) d\zeta \quad \text{as} \quad m \to \infty.$$

Since $I_m \to I$ as $m \to \infty$, $I(z) = f(z) + P(z)$ for $z \in D_1 \cap K^o$. This means that $f(z)$ extends analytically to K^o. Because K is an arbitrary compact set in $\mathbb{C} \setminus \Delta_{\sigma+\delta}$ for any small δ, the function $f(z)$ extends to the open sector $\mathbb{C} \setminus \Delta_\sigma$. Thus, the theorem is proved if all c_k are positive.

Prove now the theorem in the case when c_k may be negative. Without loss of generality we may assume that only c_q among c_k is negative, i.e. $\dfrac{l}{2} = -c_q$. Then

$$\psi(\zeta) = \phi(\zeta) \frac{\prod_{j=1}^{p} \Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q-1} \Gamma(c_k\zeta + d_k)\Gamma\left(-\frac{l}{2}\zeta + d\right)},$$

$$\varphi(\zeta) := \phi(\zeta) \frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta}}{\prod_{k+1}^{q} |c_k|^{c_k\zeta}} = \phi(\zeta) \frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta} \left(\frac{l}{2}\right)^{\frac{l}{2}\zeta}}{\prod_{k+1}^{q-1} {c_k}^{c_k\zeta}}.$$

According to the condition of the theorem

$$\max\left\{h_\varphi\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}l, h_\varphi\left(\frac{\pi}{2}\right) + \frac{\pi}{2}l\right\} \leqslant \sigma.$$

Note that the function $\psi(\zeta)$ may be rewritten in the form (6) such that all c_k are positive

$$\psi(\zeta) = \phi(\zeta) \frac{\prod_{j=1}^{p} \Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q-1} \Gamma(c_k\zeta + d_k)} \Gamma\left(1 + \frac{l}{2}\zeta + d\right) \sin\pi\left(-\frac{l}{2}\zeta - d\right).$$

Consider now the entire function

$$\tilde{\varphi}(\zeta) := \phi(\zeta) \sin\pi\left(-\frac{l}{2}\zeta - d\right) \frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta} \left(\frac{l}{2}\right)^{\frac{l}{2}\zeta}}{\prod_{k+1}^{q-1} {c_k}^{c_k\zeta}}.$$

Its indicator is bounded

$$h_{\tilde{\varphi}}(\theta) = \lim_{r\to\infty} \frac{1}{r} \ln\left|\phi(re^{i\theta}) \frac{\prod_{j=1}^{p} {a_j}^{a_j re^{i\theta}} \left(\frac{l}{2}\right)^{\frac{l}{2}re^{i\theta}}}{\prod_{k+1}^{q-1} {c_k}^{c_k re^{i\theta}}} \frac{e^{i\pi\frac{l}{2}re^{i\theta}} - e^{-i\pi\frac{l}{2}re^{i\theta}}}{2i}\right| \leqslant h_\varphi(\theta) + \pi\frac{l}{2}|\sin(\theta)|.$$

Thus

$$h_{\tilde{\varphi}}(0) = 0, \quad h_{\tilde{\varphi}}\left(\pm\frac{\pi}{2}\right) \leqslant \sigma.$$

The function $\tilde{\varphi}(\zeta)$ satisfies the conditions of (9), hence the sum of the series (1) extends analytically to the open sector $\mathbb{C} \setminus \Delta_\sigma$. Theorem 2 is proof.

The proof of Theorem 3 is largely similar to that of Theorem 2. Namely, from condition 2) of Theorem 3 it follows that for any $\alpha > 0$ there exists $\delta > 0$ such that $h_\varphi(\theta) \leqslant (\sigma + \delta)|\sin\theta|$ for $|\theta| \leqslant \alpha$. Consequently, the bounds (12) and (13) for the modulus of $\psi(\zeta)$ and $\psi(\zeta)g(\zeta, z)$ hold for $\zeta \in \Delta_\alpha$. The domains G and G_m become

$$G = D_1 \cup \Delta_\alpha^o \text{ and } G_m = \left\{\zeta = \xi + i\eta \in G : \xi \leqslant m + \frac{1}{2}\right\}$$

(see Fig. 2), i.e. $\partial G_m = \Gamma_m^1 \cup \Gamma_m^2$.

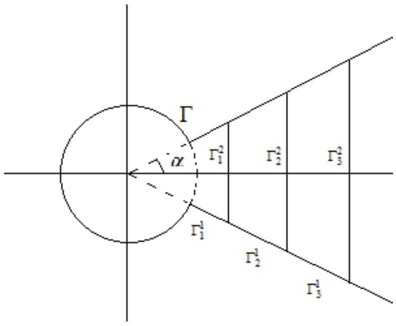

Fig. 2.

The integral I_m is then the sum I_m^1 and I_m^2 over Γ_m^1, and Γ_m^2, and for $z \in K \cap D_1^o$ the integral $I_m^2 \to 0$ as $m \to \infty$.

The integral I over ∂G converges for $\zeta \in \Delta_\alpha, z \in K$, (Fig. 3) where $K = D_{e^\varepsilon} \setminus (\Delta_{\sigma+2\delta}^o \cup D_{\frac{1}{2}})$, $\varepsilon = \dfrac{\delta \sin\alpha}{2}$.

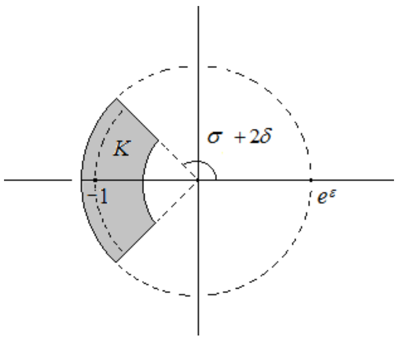

Fig. 3.

The rest of the proof is the same.

As for the proof Theorem 1, it is enough to note that the main estimates (12) and (13) hold for all $\zeta \in \mathbb{C}$. Therfore, by choosing appropriate contours of integrations we prove analytic continuation of the sum of the series to $\mathbb{C} \setminus (\partial D_1 \cap \Delta_\sigma)$.

2. Examples

Consider two examples clarifying why interpolation of the coefficients by meromorphic functions, and not entire, may be advantageous.

Example 1. Consider the series

$$f(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2}{3}n + \frac{1}{3}\right) 3^n}{\Gamma(n+1)\Gamma\left(-\frac{1}{3}n + \frac{4}{3}\right) 2^{\frac{2}{3}n}} z^n, \tag{14}$$

whose domain of convergence is the unit disk. Its coefficients

$$f_n = \frac{\Gamma(\frac{2}{3}n + \frac{1}{3})3^n}{\Gamma(n+1)\Gamma(-\frac{1}{3}n + \frac{4}{3})2^{\frac{2}{3}n}}$$

are given by the values of a meromorphic function of the form (6), namely,

$$\psi(\zeta) = \frac{3^\zeta}{2^{\frac{2}{3}\zeta}} \frac{\Gamma(\frac{2}{3}\zeta + \frac{1}{3})}{\Gamma(\zeta+1)\Gamma(-\frac{1}{3}\zeta + \frac{4}{3})}. \tag{15}$$

In this case the entire function from Theorem 2 is

$$\varphi(\zeta) = \frac{3^\zeta}{2^{\frac{2}{3}\zeta}} \frac{\left(\frac{2}{3}\right)^{\frac{2\zeta}{3}}}{\left(\frac{1}{3}\right)^{\frac{-\zeta}{3}}} \equiv 1. \tag{16}$$

Here $l = 1 + \dfrac{1}{3} - \dfrac{2}{3} = \dfrac{2}{3}$, $\quad h_\varphi(\theta) = 0$ and $\max\left\{h_\varphi\left(-\dfrac{\pi}{2}\right) - \dfrac{\pi}{3}, h_\varphi\left(\dfrac{\pi}{2}\right) + \dfrac{\pi}{3}\right\} \leqslant \dfrac{\pi}{3}$.

According to Theorem 2, the series (14) extends analytically to the open sector $\mathbb{C} \setminus \Delta_{\frac{\pi}{3}}$.

Note that the series (17) is the normalized series representing a branch of solution to the algebraic equation $y^3 - zy - 1 = 0$. This branch has poles in $e^{-i\frac{2}{3}\pi}$ and $e^{i\frac{2}{3}\pi}$ and extends to the sector $\mathbb{C} \setminus \Delta_{\frac{2}{3}\pi}$ [9].

It seems that an entire function interpolating the coefficients cannot be constructed so easily.

Example 2. Consider now the series

$$f(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{3} + \frac{1}{3}\right) 3^n}{\Gamma(n+1)\Gamma\left(\frac{-2n}{3} + \frac{4}{3}\right) 2^{\frac{2n}{3}}} z^n, \tag{17}$$

with the same domain of convergence the unit disk. Its coefficients are

$$f_n = \frac{\Gamma\left(\frac{n}{3} + \frac{1}{3}\right) 3^n}{\Gamma(n+1)\Gamma\left(\frac{-2n}{3} + \frac{4}{3}\right) 2^{\frac{2n}{3}}}.$$

They are interpolated by the following entire function

$$\varphi(z) = \frac{2\pi}{3^{\frac{1}{2}}} \frac{2^{-\frac{2}{3}z}}{\Gamma\left(\frac{z}{3} + \frac{2}{3}\right)\Gamma\left(\frac{z}{3} + 1\right)\Gamma\left(\frac{4}{3} - \frac{2z}{3}\right)}.$$

Indeed, in Gauss's multiplication formula

$$\Gamma(w)\Gamma\left(w + \frac{1}{m}\right) ...\Gamma\left(w + \frac{m-1}{m}\right) = m^{\frac{1}{2} - mw}(2\pi)^{\frac{m-1}{2}}\Gamma(mw)$$

let $m = 3$, $w = \dfrac{n}{3} + \dfrac{1}{3}$, then

$$\Gamma\left(\frac{n}{3}+\frac{1}{3}\right)\Gamma\left(\frac{n}{3}+\frac{2}{3}\right)\Gamma\left(\frac{n}{3}+1\right) = 3^{-\frac{1}{2}-n}2\pi\Gamma(n+1).$$

Express $\Gamma\left(\dfrac{n}{3}+\dfrac{1}{3}\right)$ through the other terms of this identity and substitute it into the expression for f_n, to see that $\varphi(n) = f_n\ n \in \mathbb{N}$.

Estimate $|\varphi(r)|$ by using Stirling's formula

$$|\varphi(r)| = \left|\frac{2}{3^{\frac{1}{2}}}\frac{2^{\frac{2}{3}r}\Gamma(\frac{2r}{3}-\frac{1}{3})sin(\pi\frac{2r-1}{3})}{\Gamma(\frac{r}{3}+\frac{2}{3})\Gamma(\frac{r}{3}+1)}\right| \sim$$

$$\sim \frac{2}{3^{\frac{1}{2}}}\frac{2^{\frac{2}{3}r}\left(2\pi\frac{2r-1}{3}\right)^{\frac{1}{2}}\left(\frac{2r}{3}-\frac{1}{3}\right)^{\frac{2r}{3}-\frac{1}{3}}e^{-\left(\frac{2r}{3}-\frac{1}{3}\right)}}{\left(2\pi\frac{r+2}{3}\right)^{\frac{1}{2}}\left(\frac{r}{3}+\frac{2}{3}\right)^{\frac{r}{3}+\frac{2}{3}}e^{-\left(\frac{r}{3}+\frac{2}{3}\right)}}\frac{\sin\left(\pi\frac{2r-1}{3}\right)}{\left(2\pi(\frac{r}{3}+1)\right)^{\frac{1}{2}}\left(\frac{r}{3}+1\right)^{\frac{r}{3}+1}e^{-\left(\frac{r}{3}+1\right)}} \leqslant Cr + e^{o(r)}.$$

It follows that

$$h_\varphi(0) = \varlimsup_{r\to\infty}\frac{\ln|\varphi(r)|}{r} \leqslant \varlimsup_{r\to\infty}\frac{\ln(Cr+e^{o(r)})}{r} \leqslant 0,$$

on the other hand

$$h_\varphi(0) \geqslant \varlimsup_{n\to\infty}\frac{\ln|\varphi(n)|}{n} = \varlimsup_{n\to\infty}\ln|f_n|^{\frac{1}{n}} = 0,$$

therefore $h_\varphi(0) = 0$.

In order to estimate $|\varphi(re^{i\frac{\pi}{2}})|$ and $|\varphi(re^{-i\frac{\pi}{2}})|$ we use the double-sided estimate for the Gamma-function (see [8])

$$c_1(|y|+1)^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|} \leqslant \Gamma(x+iy) \leqslant c_2(|y|+1)^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|},$$

where $x \in K \subset \mathbb{R}\setminus\{0,-1,-2,...\}$, K is compact. The constants c_1 and c_2 depend on the choice of K, $y \in \mathbb{R}$. Then

$$|\varphi(re^{\pm i\frac{\pi}{2}})| \leqslant C\frac{e^{\frac{\pi}{6}r}e^{\frac{\pi}{6}r}e^{\frac{2\pi}{6}r}}{c_1\left(\frac{r}{3}+1\right)^{\frac{2}{3}-\frac{1}{2}}c_1\left(\frac{r}{3}+1\right)^{1-\frac{1}{2}}c_1\left(\frac{2r}{3}+1\right)^{\frac{4}{3}-\frac{1}{2}}},$$

or

$$\ln|\varphi(re^{\pm i\frac{\pi}{2}})| \leqslant \frac{2\pi}{3}r + o(r).$$

Therefore

$$h_\varphi\left(\pm\frac{\pi}{2}\right) \leqslant \frac{2\pi}{3}.$$

It follows from Arakelian's Theorem [4] that the series (17) extends to the open sector $\mathbb{C}\setminus\Delta_{\frac{2}{3}\pi}$.

On the other hand, the coefficients of the series (17) are interpolated by the meromorphic function

$$\psi(\zeta) = \frac{3^\zeta}{2^{\frac{2}{3}\zeta}}\frac{\Gamma\left(\frac{1}{3}\zeta+\frac{1}{3}\right)}{\Gamma(\zeta+1)\Gamma\left(-\frac{2}{3}\zeta+\frac{4}{3}\right)}.$$

The entire function of Theorem 2 is

$$\Phi(\zeta) = \frac{3^\zeta}{2^{\frac{2}{3}\zeta}}\frac{3^{-\frac{1}{3\zeta}}}{\left(-\frac{2}{3}\right)^{-\frac{2}{3}\zeta}} \equiv 1,$$

and $l = 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}$, $h_\varphi(\theta) = 0$ and $\max\left\{h_\varphi\left(-\frac{\pi}{2}\right) - \frac{2\pi}{3}, h_\varphi\left(\frac{\pi}{2}\right) + \frac{2\pi}{3}\right\} \leqslant \frac{2\pi}{3}$.

Therefore, by Theorem 2 the series (17) extends to the open sector $\mathbb{C} \setminus \Delta_{\frac{2}{3}\pi}$.

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Аналитическое продолжение степенных рядов путем интерполяции коэффициентов мероморфными функциями

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В работе исследуется вопросы об аналитическом продолжении степенного ряда через открытую дугу на границе круга сходимости. Ответ на такой вопрос дан в терминах мероморфной функции специального вида, интерполирующей коэффициенты ряда. Получены условия при которых сумма ряда аналитически продолжается в некоторую окрестность дуги в сектор, определенный дугой, во всю комплексную плоскость, кроме некоторой дуги.

Ключевые слова: степенные ряды, аналитическое продолжение, интерполирующая мероморфная функция, индикатор функция.