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## A Note on a Distance Function in Bergman Type Analytic Function Spaces of Several Variables

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*New sharp estimate concerning distance function in certain Bergman-type spaces of analytic functions on tube domains over symmetric cones is obtained. This is the first result of this type for tube domains over symmetric cones. New similar results in analytic mixed norm spaces on products of tube domains over symmetric cones will also be provided.*

*Keywords: distance estimates, tube domains, Bergman spaces.*

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### 1. Introduction and preliminaries

In this note we obtain a sharp distance estimate in spaces of analytic functions in tube domains over symmetric cones.

This line of investigation can be considered as continuation of previous papers (see, for example, [1, 2] and [3] and references there).

This new results are contained in the second and third section of this note. We remark that for the first time in literature we consider this type extremal problem related with distance estimates in spaces of analytic functions on tube domains over symmetric cones.

The first section contains known required preliminaries on analysis on symmetric cones.

In one dimensional tubular domain which is upperhalfspace  $C_+$  (see, for example, [4]) our theorem is not new and it was obtained recently in [5].

Moreover arguments we provided below in proof are similar to those we have in one dimension and the base of proof is again the so-called Bergman reproducing formula, but in tubular domain over symmetric cone (see, for example, [4] for this integral representation).

We shortly remind the history of this problem.

Recently various papers appeared where arguments which can be seen in [6] were extended in various directions (see, for example, [1–3]).

In particular in mentioned papers various new results on distances for analytic function spaces in higher dimension (unit ball and polydisk ) were obtained. Namely new results for large scales of analytic mixed norm spaces in higher dimension were proved.

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Later several new sharp results for harmonic functions of several variables in the unit ball and upperhalfplane of Euclidean space were also obtained (see, for example, [1] and references there).

We mention separately [5] and [7] where the case of higher dimension was considered in special cases of analytic spaces on subframe and new similar results in the context of bounded strictly pseudoconvex domains with smooth boundary were also provided.

The classical Bergman representation formula in various forms and in various domains serves as a base in all these papers in proofs of main results.

We would like to note also recently Wen Xu (see [8]) repeating arguments of Ruhan Zhao in the unit ball obtained results on distances from Bloch functions to some Mobius invariant function spaces in one and higher dimension in a relatively direct way.

Probably for the first time in literature these extremal problems connected with distances in analytic spaces appeared before in [9] and in [10], where this problem was formulated probably for the first time and certain concrete cases connected with spaces of bounded analytic functions in the unit disk were considered.

These results much later were mentioned in [11]. Some other results on distance problems in BMOA spaces can be found also in [12].

Various other extremal problems in analytic function spaces also were considered before in various papers (see, for example, [13–16]).

In those papers other results related to this topic and some applications of certain extremal problems can be found also.

The motivation of this problem related with distance function is to find a concrete formula which will help to calculate this function more concretely via the well-known Bergman kernel.

The goal of this note to develop further some ideas from our recent mentioned papers and to present a new sharp theorem in tubular domain over symmetric cones.

For formulation of our results we will need various standard definitions from the theory of tubular domains over symmetric cones (see, for example, [4, 17–19]).

Let  $T_\Omega = V + i\Omega$  be the tube domain over an irreducible symmetric cone  $\Omega$  in the complexification  $V^{\mathbb{C}}$  of an  $n$ -dimensional Euclidean space  $V$ . Let  $\mathcal{H}(T_\Omega)$  be the space of all holomorphic functions on  $T_\Omega$ . Following the notation of [17] and [4] we denote the rank of the cone  $\Omega$  by  $r$  and by  $\Delta$  the determinant function on  $V$ .

Letting  $V = \mathbb{R}^n$ , we have as an example of a symmetric cone on  $\mathbb{R}^n$  the Lorentz cone  $\Lambda_n$  which is a rank 2 cone defined for  $n \geq 3$  by

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1^2 - \dots - y_n^2 > 0, \quad y_1 > 0\}.$$

The determinant function in this case is given by the Lorentz form

$$\Delta(y) = y_1^2 - \dots - y_n^2$$

(see, for example, [4]).

Let us introduce some convenient notations regarding multi-indexes.

If  $t = (t_1, \dots, t_r)$ , then  $t^* = (t_r, \dots, t_1)$  and, for  $a \in \mathbb{R}$ ,  $t + a = (t_1 + a, \dots, t_r + a)$ . Also, if  $t, k \in \mathbb{R}^r$ , then  $t < k$  means  $t_j < k_j$  for all  $1 \leq j \leq r$ .

We are going to use the following multi-index

$$g_0 = \left( (j-1) \frac{d}{2} \right)_{1 \leq j \leq r}, \quad \text{where} \quad (r-1) \frac{d}{2} = \frac{n}{r} - 1.$$

For  $\tau \in \mathbb{R}_+$  and the associated determinant function  $\Delta(x)$  we set

$$A_\tau^\infty(T_\Omega) = \left\{ F \in \mathcal{H}(T_\Omega) : \|F\|_{A_\tau^\infty} = \sup_{x+iy \in T_\Omega} |F(x+iy)| \Delta^\tau(y) < \infty \right\},$$

This is a Banach space. We denote by  $\Delta_s$  the generalized power function (see for this function, for example, [4, 17]).

For  $1 \leq p, q < +\infty$  and  $\nu \in \mathbb{R}$ , and  $\nu > \frac{n}{r} - 1$  we denote by  $A_\nu^{p,q}(T_\Omega)$  the mixed-norm weighted Bergman space consisting of analytic functions  $F$  in  $T_\Omega$  such that

$$\|F\|_{A_\nu^{p,q}} = \left( \int_\Omega \left( \int_V |F(x+iy)|^p dx \right)^{q/p} \Delta^\nu(y) \frac{dy}{\Delta(y)^{n/r}} \right)^{1/q} < \infty.$$

This is a Banach space. For various properties of such functions we refer the reader to [4].

Replacing above A by L we will get as usual the corresponding larger space of all measurable functions in tube over symmetric cone with the same quazinorm (see [17, 18]).

It is known the  $A_\nu^{p,q}(T_\Omega)$  space is nontrivial if and only if  $\nu > \frac{n}{r} - 1$ , (see, for example, [4, 19]).

When  $p = q$  we write (see, for example, [4])

$$A_\nu^{p,q}(T_\Omega) = A_\nu^p(T_\Omega).$$

This is the classical weighted Bergman space with usual modification when  $p = \infty$ .

The (weighted) Bergman projection  $P_\nu$  is the orthogonal projection from the Hilbert space  $L_\nu^2(T_\Omega)$  onto its closed subspace  $A_\nu^2(T_\Omega)$  and it is given by the following integral formula (see [4]).

$$P_\nu f(z) = C_\nu \int_{T_\Omega} B_\nu(z, w) f(w) dV_\nu(w), \tag{1}$$

where

$$B_\nu(z, w) = \Delta^{-(\nu + \frac{n}{r})}((z - \bar{w})/i)$$

is the Bergman reproducing kernel for

$$A_\nu^2(T_\Omega)$$

(see [4, 17]).

Here we used the usual notation

$$dV_\nu(w) = \Delta^{\nu - \frac{n}{r}}(v) dudv,$$

(see [4]).

Below and here we use constantly the following notations  $w = u + iv \in T_\Omega$  and also  $z = x + iy \in T_\Omega$ .

Hence for any analytic function from  $A_\nu^2(T_\Omega)$  the following integral formula is valid (see also [4])

$$f(z) = C_\nu \int_{T_\Omega} B_\nu(z, w) f(w) dV_\nu(w). \tag{2}$$

In this case sometimes below we say simply that the  $f$  function allows Bergman representation via Bergman kernel with  $\nu$  index.

Note these assertions have direct copies in simpler cases of analytic function spaces in unit disk, polydisk, unit ball, upperhalfspace  $C_+$  and in spaces of harmonic functions in the unit ball

or upperhalfspace of Euclidean space  $R^n$ . These classical facts are well-known and can be found, for example, in [20] and in some items from references there.

Above and throughout the paper we write  $C$  (sometimes with lower or upper indexes) to denote positive constants which might be different each time we see them (and even in a chain of inequalities), but are independent of the functions or variables being discussed.

In this paper we will also need a pointwise estimate for the Bergman projection of functions in  $L^{p,q}(T_\Omega)$ , defined by integral formula (1), when this projection makes sense. Note such estimates in simpler cases of unit disk, unit ball and polydisk are well-known, (see, for example, [20]). Let us first recall the following known basic integrability properties for the determinant function, which appeared already above in definitions.

By  $\Delta_\beta(x)$  we denote determinant function (see [4]).

**Lemma 1.** *Let  $\alpha \in \mathbb{C}^r$  and  $y \in \Omega$ .*

1. *The integral*

$$J_\alpha(y) = \int_{\mathbb{R}^n} \left| \Delta_{-\alpha} \left( \frac{x + iy}{i} \right) \right| dx$$

*converges if and only if  $\operatorname{Re} \alpha > g_0^* + \frac{n}{r}$ . In that case*

$$J_\alpha(y) = C_\alpha |\Delta_{-\alpha+n/r}(y)|.$$

2. *For any multi-indices  $s$  and  $\beta$  from  $C^r$  and  $t \in \Omega$  the function*

$$y \mapsto \Delta_\beta(y + t) \Delta_s(y)$$

*belongs to  $L^1(\Omega, \frac{dy}{\Delta^{n/r}(y)})$  if and only if  $\operatorname{Re} s > g_0$  and  $\operatorname{Re}(s + \beta) < -g_0^*$ . In that case we have*

$$\int_{\Omega} \Delta_\beta(y + t) \Delta_s(y) \frac{dy}{\Delta^{n/r}(y)} = C_{\beta,s} \Delta_{s+\beta}(t).$$

We refer to Corollary 2.18 and Corollary 2.19 of [19] for the proof of the above lemma or [4]. We will need only "one-dimensional" version of this estimate.

Indeed as a corollary of one dimensional versions of these estimates (see, for example, [18] Theorem 3.9) we obtain the following vital estimate (3) which we will use in proof of our main result. Note also it is a complete analogue of the well-known Forelli-Rudin estimate for Bergman kernel in analytic spaces in the unit ball.

$$\int_{T_\Omega} \Delta^\beta(y) |B_{\alpha+\beta+\frac{n}{r}}(z, w)| dV(z) \leq C \Delta^{-\alpha}(v), \tag{3}$$

$\beta > -1, \alpha > \frac{n}{r} - 1, z = x + iy, w = u + iv$  (see [18]).

Let  $\tau$  be the set of all triples  $(p, q, \nu)$  such that  $1 \leq p, q < \infty, \nu > \frac{n}{r} - 1$ .

The following vital pointwise estimate can be found, for example, in [4].

**Lemma 2.** *Suppose  $(p, q, \nu) \in \tau$ . Then we have*

$$|P_\nu f(z)| \leq c_{p,q,r,\nu,n} \Delta^{-\frac{\nu}{q} - \frac{n}{rp}}(\operatorname{Im} z) \|f\|_{A_{p,q}^\nu}.$$

*Proof.* This is a consequence of the lemma 1 and Hölder's inequality (see [4]). □

## 2. New estimates for distances in analytic function spaces in tube domains over symmetric cones.

In this paper we restrict ourselves to  $\Omega$  irreducible symmetric cone in the Euclidean vector space  $R^n$  of dimension  $n$ , endowed with an inner product for which the cone  $\Omega$  is self dual. We denote by  $T_\Omega = R^n + i\Omega$  the corresponding tube domain in  $C^n$ .

This section is devoted to formulation and proof of the main result of this paper. As previously in case of analytic functions in unit disk, polydisk, unit ball, and upperhalfspace  $C_+$  and in case of spaces of harmonic functions in Euclidean space (see, for example, [1–3, 5–7]) the role of the Bergman representation formula is crucial in these issues and our proof is heavily based on it. A variant of Bergman representation formula is available also in Bergman-type analytic function spaces in tubular domains over symmetric cones and this known fact (see [4, 17–19]), which is crucial also in various problems in analytic function spaces in tubular domains (see [4] and various references there) is also used in our proof below.

The following result can be found in [18] (section 4).

For all  $1 < p < \infty$  and  $1 < q < \infty$  and for all  $\frac{n}{r} \leq p_1$ , where  $\frac{1}{p_1} + \frac{1}{p} = 1$  and  $\frac{n}{r} - 1 < \nu$  and for all functions  $f$  from  $A_\nu^{p,q}$  and for all  $\frac{n}{r} - 1 < \alpha$  the Bergman representation formula with  $\alpha$  index or with the Bergman kernel  $B_\alpha(z, w)$  is valid.

We remark this result is a particular case of a more general assertion for analytic mixed norm  $A_\nu^{p,q}$  classes (see, for example, [18]), which (after some analysis of our proof below) means that our main result admits also some extensions, even to mixed norm spaces which we defined above. This will be discussed in our next paper which is in preparation.

We will also need for our proofs the following important fact on integral representation (see, for example, [21]). Let  $\nu > \frac{n}{r} - 1$ ,  $\alpha > \frac{n}{r} - 1$ , then for all functions from  $A_\alpha^\infty$  the Bergman integral representation with Bergman kernel  $B_{\alpha+\nu}(z, w)$  (with  $\alpha + \nu$  index) is valid.

We also note that by lemma 2 we have

$$|f(x + iy)|\Delta^{\frac{n}{rp} + \frac{\nu}{q}}(y) \leq c_{p,q,r,\nu} \|f\|_{A_\nu^{p,q}}, \quad x + iy \in T_\Omega, \quad (p, q, \nu) \in \tau.$$

Hence we have a continuous embedding  $A_\nu^p \hookrightarrow A_{\frac{n}{rp} + \frac{\nu}{p}}^\infty$  for  $(p, p, \nu) \in \tau$  and hence we have a problem to find sharp estimates

$$\text{dist}_{A_{\frac{n}{rp} + \frac{\nu}{p}}^\infty}(f, A_\nu^p)$$

where  $f \in A_{\frac{n}{rp} + \frac{\nu}{p}}^\infty$ .

This problem is solved in our next theorem below, which is the main result of this note. Let us set, for  $f \in \mathcal{H}(T_\Omega)$ ,  $s \in \mathbb{R}$  and  $\epsilon > 0$ :

$$V_{\epsilon,s}(f) = \{x + iy \in T_\Omega : |f(x + iy)|\Delta^s(y) \geq \epsilon\}$$

Let also  $w = u + iv \in T_\Omega$ ,  $z = x + iy \in T_\Omega$ . We denote by  $N_1$  and by  $N_2$  two sets—the first one is  $V_{\epsilon,s}(f)$ , the other one is the set of all those points, which are in tubular domain  $T_\Omega$ , but not in  $N_1$ .

**Theorem 1.** *Let  $1 < p < \infty$ ,  $\nu > p\left(\frac{n}{r} - 1\right)$ ,  $\beta > t + \frac{n}{r} - 1$ ,  $t = \frac{1}{p}\left(\nu + \frac{n}{r}\right)$ . Set, for  $f \in A_{\frac{n}{rp} + \frac{\nu}{p}}^\infty$ :*

$$l_1(f) = \text{dist}_{A_{\frac{n}{rp} + \frac{\nu}{p}}^\infty}(f, A_\nu^p),$$

$$l_2(f) = \inf \left\{ \epsilon > 0 : \int_{T_\Omega} \left( \int_{V_{\epsilon,t}(f)} \frac{\Delta^{\beta-t-\frac{n}{r}}(v) dudv}{\Delta^{\beta+\frac{n}{r}}((z-\bar{w})/i)} \right)^p \Delta^{\nu-\frac{n}{r}}(y) dx dy < \infty \right\}.$$

Then there is a positive large enough number  $\beta_0$  depending on parameters involved, so that for all  $\beta > \beta_0$  we have  $l_1(f) \asymp l_2(f)$ .

*Proof.* Let  $\nu > \frac{n}{r} - 1$ ,  $\tau > \frac{n}{r} - 1$ , then as we indicated above for all functions from  $A_\tau^\infty$  the integral representations of Bergman with Bergman kernel

$$B_{(\tau+\nu)}(z, w)$$

is valid.

We denote below the double integral which appeared in formulation of our theorem by  $G(f)$  and we will show first that  $l_1(f) \leq C l_2(f)$ . We assume now that  $l_2(f)$  is finite.

We use the Bergman representation formula which we provided above, namely (2), and using conditions on parameters we have the following equalities.

First obviously we have by remark from which we started this proof that for large enough  $\beta$

$$f(z) = C_\beta \int_{T_\Omega} B_\beta(z, w) f(w) dV_\beta(w) = f_1(z) + f_2(z),$$

$$f_1(z) = C_\beta \int_{N_2} B_\beta(z, w) f(w) dV_\beta(w),$$

$$f_2(z) = C_\beta \int_{N_1} B_\beta(z, w) f(w) dV_\beta(w).$$

Then we estimate both functions separately using lemmas provided above. Using definitions of  $N_1$  and  $N_2$  and using estimate (3) we will have

$$f_1 \in A_{\frac{n}{rp} + \frac{\nu}{p}}^\infty$$

and

$$f_2 \in A_\nu^p.$$

We will prove both estimates below. First we easily note the last inclusion follows directly from the fact that  $l_2$  is finite. Indeed we have

$$\int_{T_\Omega} |f_2(z)|^p \Delta^{\nu-\frac{n}{r}}(y) dx dy \leq C \int_{T_\Omega} (C_\beta \int_{N_1} |B_\beta(z, w)| |f(w)| dV_\beta(w))^p \Delta^{\nu-\frac{n}{r}}(y) dx dy.$$

Turning to the first estimate it can be easily seen that the norm of  $f_1$  can be estimated from above by  $C\epsilon$ , for some positive constant  $C$ , since we have immediately

$$|f_1(z)| \Delta^{\frac{n}{rp} + \frac{\nu}{p}}(y) \leq C (\Delta^{\frac{n}{rp} + \frac{\nu}{p}}(y)) (C_\beta \int_{N_2} |B_\beta(z, w)| |f(w)| dV_\beta(w))$$

and hence

$$\sup_{N_2} |f_1(w)| \Delta^t(v) \leq C\epsilon.$$

Note the last estimate follows directly from definition of  $N_2$  set and the following inequality which follows from (3) (see also [18]).

$$\int_{T_\Omega} \Delta^{-t}(y) |B_\beta(z, w)| dV_\beta(z) \leq C \Delta^{-t}(v),$$

$z = x + iy$ ,  $w = u + iv$ , for all  $\beta$ , so that  $\beta > \beta_0$ , for some large enough fixed  $\beta_0$ , which depends on  $n, r, \nu, p$  and for  $t = \left(\frac{1}{p}\right) \left(\nu + \frac{n}{r}\right)$  and  $\nu > p \left(\frac{n}{r} - 1\right)$  (see [18] Theorem 3.9). This gives immediately one part of our theorem. Indeed, we have obviously

$$l_1 \leq C_2 \|f - f_2\|_{A_t^\infty} = C_3 \|f_1\|_{A_t^\infty} \leq C_4 \epsilon$$

It remains to prove that  $l_2 \leq l_1$ . Let us assume  $l_1 < l_2$ . Then there are two numbers  $\epsilon$  and  $\epsilon_1$ , both positive such that there exists  $f_{\epsilon_1}$ , so that this function is in  $A_\nu^p$  and  $\epsilon > \epsilon_1$  and also the following conditions holds.

$$\|f - f_{\epsilon_1}\|_{A_t^\infty} \leq \epsilon_1$$

and  $G(f) = \infty$ , where  $G$  is a double integral in formulation of theorem in  $l_2$  (see for similar arguments, in one dimension, for example, [5]).

Next from

$$\|f - f_{\epsilon_1}\|_{A_t^\infty} \leq \epsilon_1$$

we have the following two estimates, the second one is a direct corollary of first one. First we have for  $z = x + iy$

$$(\epsilon - \epsilon_1) \tau_{V_{\epsilon,t}}(z) \Delta^{-t}(y) \leq C |f_{\epsilon_1}(z)|,$$

where  $\tau_{V_{\epsilon,t}}(z)$  is a characteristic function of  $V = V_{\epsilon,t}(f)$  set we defined above.

And from last estimate we have directly multiplying both sides by Bergman kernel  $B_\beta(z, w)$  and integrating by tube  $T_\Omega$  both sides with measure  $dV_\beta$ .

$$G(f) \leq C \int_{T_\Omega} (L(f_{\epsilon_1}))^p \Delta^{\nu - \frac{n}{r}}(y) dy dx,$$

where

$$L = L(f_{\epsilon_1}, z)$$

and

$$L(f_{\epsilon_1}, z) = \int_{T_\Omega} |f_{\epsilon_1}(w)| |B_\beta(z, w)| dV_\beta(w).$$

Put  $\beta + \frac{n}{r} = k_1 + k_2$ , where  $k_1 = \beta - \frac{n}{r} - \mu$ ,  $k_2 = \mu + 2\frac{n}{r} \left(\frac{1}{p} + \frac{1}{p_1}\right)$ .

By classical Holder inequality with  $p$  and  $p_1$ ,  $p^{-1} + p_1^{-1} = 1$  we have

$$L^p \leq C I_1 I_2,$$

where

$$I_1(f) = \int_{T_\Omega} |f_1(z)|^p |\Delta^s((z - \bar{w})/i)| \Delta^{(\beta - \frac{n}{r})p}(y) dx dy,$$

$$I_2^{\frac{p}{p_1}} = \int_{T_\Omega} |\Delta^v((z - \bar{w})/i)| dx dy,$$

where  $f_1 = f_{\epsilon_1}$  and

$$s = \mu p - 2\frac{n}{r} - \beta p + p\frac{n}{r},$$

$$v = -2\frac{n}{r} - \mu p_1.$$

Note from here  $I_2$  can be calculated directly using (3). We have  $I_2 \leq C\Delta^{-\mu p}(v)$ ,  $\mu > \frac{n}{rp_1} - \frac{1}{p_1}$ . It remains to integrate both sides of the upper estimate for  $L$  by  $T_\Omega$ . Choosing then  $\mu$ , parameter, so that the estimate (3) can be used again after changing the order of integration and making some minor final calculations with indexes we will get what we need. Note here we have to use also the fact

$$\nu > p \left( \frac{n}{r} - 1 \right)$$

which was given in formulation of our theorem.

Indeed we will have finally,

$$\int_{T_\Omega} \left( \int_{T_\Omega} |f_{\epsilon_1}(z)| B_\beta(z, w) |dV_\beta(z)| \right)^p \Delta^{\nu - \frac{n}{r}}(v) dV(w) \leq C \|f_{\epsilon_1}\|_{A_\nu^p}^p.$$

$$G(f) \leq C \|f_{\epsilon_1}\|_{A_\nu^p},$$

but we also have

$$f_{\epsilon_1} \in A_\nu^p.$$

This will give a contradiction with the equality  $G(f) = \infty$ . We proved the estimate which we wanted to show from the start. The proof of our theorem is now complete.

Note finally the main theorem can be easily reformulated in terms of analytic Besov spaces. For this we have to use known embeddings connecting analytic Besov and Bergman spaces (see, for example [19, 21]).

We define now the same analytic function spaces we considered in this paper in product domains (products of tubular domains over symmetric cones  $T_\Omega \times T_\Omega$ ). We define first the space of all analytic function spaces in  $T_\Omega^m$  as  $H(T_\Omega^m)$  is the space of all analytic  $f(z_1, \dots, z_m)$  analytic by each variable separately.

Let also

$$A_\nu^p(T_\Omega^m) = \left\{ f \in H(T_\Omega^m) : \underbrace{\int_{T_\Omega} \dots \int_{T_\Omega}}_m |f(z_1, \dots, z_m)|^p \left( \prod_{j=1}^m \Delta(\text{Im } z_j) \right)^{\nu - \frac{n}{r}} du_j dv_j < \infty \right\},$$

where  $1 \leq p < \infty$ ,  $\nu > \frac{n}{r} - 1$ ,

$$A_t^\infty(T_\Omega^m) = \left\{ f \in H(T_\Omega^m) : \sup_{z \in T_\Omega^m} |f(z_1, \dots, z_m)| \prod_{j=1}^m \Delta^t(\text{Im } z_j) < \infty \right\}, \quad t \in \mathbb{R}.$$

Spaces of functions on product domains (not only analytic) were under attention during last several decades (see, for example [22–24] and various references there).

We note  $A_\nu^p$  and  $A_t^\infty$  are both Banach spaces for values of parameters we defined them.

Repeating arguments we provided during the proof of main theorem of this paper we arrive at following extension of that theorem to the case of product domains:

**Theorem 2.** *Let  $m \geq 1$ ,  $1 \leq p < \infty$ ,  $\nu > p \left( \frac{n}{r} - 1 \right)$ ,  $\beta_0 > t + \frac{n}{r} - 1$ ,  $t = \frac{1}{p} \left( \nu + \frac{n}{r} \right)$ ;  $f \in A_t^\infty(T_\Omega^m)$ . Then for all  $\beta > \beta_0$*

$$\text{dist}_{A_t^\infty(T_\Omega^m)}(f, A_\nu^p(T_\Omega^m)) \asymp$$



$$\asymp \inf \left\{ \varepsilon > 0 : \int_{T_\Omega} \dots \int_{T_\Omega} \left( \int_{V_{\varepsilon,t}(f)} \frac{\prod_{j=1}^m \Delta^{\beta-t-\frac{n}{r}}(\operatorname{Im} z_j)}{\prod_{j=1}^m \Delta^{\beta+\frac{n}{r}(\frac{z_j-w_j}{i})}} du_j dv_j \right)^p \times \right. \\ \left. \times \prod_{j=1}^m [\Delta^{\nu-\frac{n}{r}}(\operatorname{Im} y_j)] dx_j dy_j < \infty \right\},$$

where

$$V_{\varepsilon,t}(f) = \left\{ z \in T_\Omega^m : |f(z_1, \dots, z_m)| \prod_{j=1}^m \Delta^t(\operatorname{Im} z_j) \geq \varepsilon \right\}.$$

We omit the proof since it is based mainly on simple procedure of adding variables.

And we provide one more extension to so called mixed norm spaces on product domains  $T_\Omega^m$ .

### 3. Extremal problems in mixed norm spaces in product of tubular domains over symmetric cones

We define mixed norm spaces on product of tubular domains

$$A_{\vec{\nu}}^{\vec{p}}(T_\Omega^m) = A_{\nu_1, \dots, \nu_m}^{p_1, \dots, p_m}(T_\Omega^m) = \\ = \left\{ f \in H(T_\Omega^m) : \left( \int_{T_\Omega} \dots \left( \int_{T_\Omega} |f(z_1, \dots, z_m)|_1^p \Delta^{\nu_1-\frac{n}{r}}(\operatorname{Im} z_1) dv(z_1) \right)^{p_2/p_1} \dots \Delta^{\nu_m-\frac{n}{r}}(\operatorname{Im} z_m) \right)^{1/p_m} < \infty \right\}$$

where  $p_i \geq 1$ ,  $i = 1, \dots, m$ ,  $\nu_j > \frac{n}{r} - 1$ ,  $j = 1, \dots, m$ . These are Banach spaces.

In  $R^n$  such spaces studied in [25]. See also [24, 26] for similar analytic spaces.

We will need for our theorem the following lemmas. Proofs are the same as in [24] for unit disk case, we omit details referring to [24].

**Lemma 3.** Let  $\alpha_j > \frac{n}{r} - 1$ ,  $j = 1, \dots, m$ ,  $1 \leq p_i < \infty$ ,  $i = 1, \dots, m$ . Then

$$\sup_{z \in T_\Omega^m} |f(z)| \left[ \Delta^{(\frac{2n}{r} + \alpha)/p}(\operatorname{Im} z) \right] \leq c \|f\|_{A_\alpha^{\vec{p}}(T_\Omega^m)};$$

**Lemma 4.** Let  $\nu_j > \nu_0$ ,  $j = 1, \dots, m$  for some fixed large enough  $\nu_0$ . Then we have the following embeddings

$$A_\alpha^{\vec{p}}(T_\Omega^m) \subset A_{\nu_0}^1(T_\Omega^m) \subset A_t^\infty(T_\Omega^m)$$

where  $\nu_0 = \nu_0(\alpha_1, \dots, \alpha_m, p)$ ,  $1 \leq p_j < \infty$ ,  $j = 1, \dots, m$ ,  $t = \nu_0 + \frac{n}{r}$ .

Based on these lemma 3 and 4 we formulate the following theorem for  $A_\alpha^{\vec{p}}$  spaces.

**Theorem 3.** Let  $1 \leq p_j < \infty$ ,  $\alpha_j > \frac{n}{r} - 1$ ,  $j = 1, \dots, m$ ,  $f \in A_t^\infty(T_\Omega^m)$ ,  $t_j = \nu_j + \frac{n}{r}$ ,  $j = 1, \dots, m$ ,  $\nu_j > \nu_0$ ,  $\beta > \beta_0$  for some fixed large enough  $\beta_0$  and  $\nu_0$ . Then we have the estimate

$$\operatorname{dist}_{A_t^\infty(T_\Omega^m)}(f, A_\alpha^{\vec{p}}) \geq \inf \left\{ \varepsilon > 0 : \int_{T_\Omega} \dots \int_{T_\Omega} \left( \int_{V_{\varepsilon,t}(f)} \frac{\prod_{j=1}^m \Delta^{\beta-t-\frac{n}{r}}(\operatorname{Im} z_j)}{\prod_{j=1}^m \Delta^{\beta+\frac{n}{r}(\frac{z_j-w_j}{i})}} du_j dv_j \right) \times \right.$$

$$\left. \times \prod_{j=1}^m \Delta^{\nu_j - \frac{n}{r}}(\operatorname{Im} y_j) dx_j dy_j < \infty \right\}.$$

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## References

- [1] M.Arsenovic, R.Shamoyan, On some extremal problems in spaces of harmonic functions, *ROMAI J.*, **7**(2011), 13–34.
- [2] R.Shamoyan, O.Mihic, On new estimates for distances in analytic function spaces in the unit disc, polydisc and unit ball, *Bollet. de la Asoc. Matematica Venezolana*, **42**(2010), no. 2, 89–103.
- [3] R.Shamoyan, O.Mihic, On new estimates for distances in analytic function spaces in higher dimension, *Siberian Electronic Mathematical Reports*, **6**(2009), 514–517.
- [4] D.Bekolle, A.Bonami, G.Garrigos, C.Nana, M.Peloso, F. Ricci, Lecture notes on Bergman projectors in tube domain ver cones, an analytic and geometric viewpoint, *Proceeding of the International Workshop on Classical Analysis*, Yaounde, 2001.
- [5] R.Shamoyan, M.Arsenovic, Some remarks on extremal problems in weighted Bergman spaces of analytic functions, *Communication of the Korean Math. Society*, **27**(2012), no. 4, 753–762.
- [6] R.Zhao, Distance from Bloch functions to some Möbius invariant spaces, *Ann. Acad. Sci. Fenn.*, **33**(2008), 303–313.
- [7] R.Shamoyan, M.Radnia, Some new estimates for distances in analytic function spaces of several complex variables and double Bergman representation formula, *J. of Nonlinear Sci. and Appl.*, **3**(2010), no. 1, 48–54.
- [8] W.Xu, Distances from Bloch functions to some Mobious invariant function spaces, *J. of Function Spaces and Appl.*, **7**(2009), 91–104.
- [9] J.M.Anderson, J.Chunie, Ch.Pommerenke, On Bloch functions and normal functions, *J. Reine. Angew. Math.*, **270**(1974), 12–37.
- [10] J.M.Anderson, Bloch functions — the basic theory, *Operators and Function Theory, Lancaster, Reidel, Dordrecht*, **153**(1985), 1–17.
- [11] J.Xiao, Geometric  $Q_p$  functions, *Frontiers in Mathematics*, Birkhauser-Verlag, 2006.
- [12] P.Ghatage, D.Zheng, Analytic functions of bounded mean oscillation and the Bloch space, *Integ. Equat. Oper. Theory*, **17**(1993), 501–515.
- [13] L.Ahlfors, Bounded analytic functions, *Duke Math. J.*, **14**(1947), 1–14.
- [14] W.Rudin, Analytic functions of Hardy class, *Trans. Amer. Math. Soc.*, **78**(1955), 46–66.

- [15] D.Khavinson, M. Stessin, Certain linear extremal problems in Bergman spaces of analytic functions, *Indiana Univ. Math. J.*, **46**(1997), no. 3, 933–974.
- [16] S.Khavinson, On an extremal problem in the theory of analytic function, *Russ. Math. Survey*, **32**(1949), no. 4, 158–159.
- [17] J.Faraut, A.Koranyi, Analysis on symmetric cones, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1994.
- [18] B.F.Sehba, Bergman type operators in tube domains over symmetric cones, *Proc. Edinburg. Math. Society*, **52**(2009), no. 2, 529–544.
- [19] D.Debertol, Besov spaces and boundedness of weighted Bergman projections over symmetric tube domains, Dottorato di Ricerca in Matematica, Universita di Genova, Politecnico di Torino, 2003, 21–72.
- [20] P.Duren, A.Schuster, Bergman spaces, *Mathematical Surveys and Monographs*, v. 100, AMS, RI, 2004, 251–256.
- [21] D.Bekolle, A.Bonami, G.Garrigos, F.Ricci, B.Sehba, Analytic Besov spaces and Hardy type inequalities in tube domains over symmetric cones, *J. Reine. Angew. Math.*, **647**(2010), 25–56.
- [22] R.F.Shamoyan, E.V.Povprits, Multifunctional analytic spaces on products of Bounded strictly pseudoconvex domains and embedding theorems, *Kragujevac J. of Math.*, **37**(2013), no. 2, 221–244.
- [23] P.Jakobczak, The boundary regularity of the solution of the df equation in products of strictly pseudoconvex domains, *Pacific J. of Math.*, **121**(1986), no. 2, 371–386.
- [24] F.A.Shamoyan, O.V.Yaroslavtseva, Continuous projections, duality, and the diagonal mapping in weighted spaces of holomorphic functions with mixed norm, *Zap. Nauchn. Sem. POMI*, **247**(1997), 268–275, (in Russian).
- [25] A.Benedek, R.Panzone, The space  $L^p$ , with mixed norm, *Duke Math. J.*, **28**(1961), no. 3, 301–324.
- [26] S.M.Nikolskiy, Approximation of functions of several variables and embedding theorems, Springer-Verlag, Berlin, 1975.

## О функции дистанции в пространствах типа Бергмана аналитических функций нескольких переменных

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*Получена новая точная оценка, относящаяся к функции расстояния в некоторых пространствах аналитических функций типа Бергмана в трубчатых областях над симметрическими конусами. Это первые результаты такого рода в трубчатых областях над симметрическими конусами.*

*Ключевые слова: оценки расстояний, трубчатая область.*