

УДК 519.41/47

Some Minimal Conditions in Certain Extremely Large Classes of Groups

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Received 10.10.2014, received in revised form 10.11.2014, accepted 26.12.2014

Let \mathfrak{L} (respectively \mathfrak{T}) be the minimal local in the sense of D. Robinson class of groups, containing the class of weakly graded (respectively primitive graded) groups and closed with respect to forming subgroups and series. In the present paper, we completely describe: the \mathfrak{L} -groups with the minimal conditions for non-abelian subgroups and for non-abelian non-normal subgroups; the \mathfrak{T} -groups with the minimal conditions for (all) subgroups and for non-normal subgroups. By the way, we establish that every \overline{IH} -group, belonging to \mathfrak{L} , is solvable.

Keywords: local classes of groups; minimal conditions; non-abelian, Chernikov, Artinian, Dedekind, \overline{IH} -groups; weakly, locally, binary, primitive graded groups.

1. Introduction. Some preliminary data

In the present paper, the author continues his investigations [1–5]. Remind that the class \mathfrak{X} of groups is called local (in our sense), if it includes every group that has a local system of subgroups belonging to \mathfrak{X} or, in the other words, a local system of \mathfrak{X} -subgroups (see [1]). Further, introduce the definition.

Definition. The class \mathfrak{X} of groups will be called local in the sense of D. Robinson, if it includes every group G such that for any finite set F of elements of G , there exists some \mathfrak{X} -subgroup S of G , containing F .

(In connection with this definition, see [6, p. 93].)

The following useful elementary lemmas hold.

Lemma 1. *Assume that some local class \mathfrak{X} of groups is closed with respect to forming subgroups. Then \mathfrak{X} is local in the sense of D. Robinson.*

Proof. Let G , F and S be from Definition. Since $\langle F \rangle \subseteq S \in \mathfrak{X}$, $\langle F \rangle \in \mathfrak{X}$. Thus all finitely generated subgroups of G form its local system of \mathfrak{X} -subgroups. Consequently, $G \in \mathfrak{X}$. \square

Remind that the class \mathfrak{X} of groups is closed with respect to forming series, if every group, having a series with \mathfrak{X} -factors, belongs to \mathfrak{X} . Sometimes a series is also called a generalized normal system (S.N. Chernikov).

Lemma 2. *Let \mathfrak{X} and \mathfrak{Y} be respectively the minimal local and the minimal local in the sense of D. Robinson classes of groups, containing some fixed class \mathfrak{B} of groups and closed with respect to forming subgroups and series (generalized normal systems). Then $\mathfrak{X} = \mathfrak{Y}$.*

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Proof. Indeed, \mathfrak{X} (respectively \mathfrak{Y}) is the intersection of all local (respectively all local in the sense of D.Robinson) classes of groups, containing \mathfrak{V} and closed with respect to forming subgroups and series. Since all local in the sense of D.Robinson classes of group are local, obviously, $\mathfrak{X} \subseteq \mathfrak{Y}$. But in view of Lemma 1, \mathfrak{X} is local in the sense of D. Robinson. Thus $\mathfrak{X} = \mathfrak{Y}$. Lemma is proven. \square

Remind that the group G is called weakly graded, if for $g, h \in G$, the subgroup $\langle g, g^h \rangle$ possesses a subgroup of finite index $\neq 1$ whenever $|\langle g \rangle| = \infty$, or g is a p -element $\neq 1$ with some odd prime p and also $[g^p, h] = 1$ and the subgroup $\langle g, h \rangle$ is periodic (N. S. Chernikov [2, P. 22]). The class of weakly graded groups is very large and includes, for instance, the classes of binary graded, locally graded, binary finite, locally finite, locally solvable groups; the classes of linear groups, 2-groups, periodic Shunkov groups; all Kurosh-Chernikov classes of groups.

Remind that the group G is called primitive graded, if for $g, h \in G$, the subgroups $\langle g, g^h \rangle$ possesses a subgroup of finite index $\neq 1$ whenever g is a p -element $\neq 1$ with some odd p and also $[g^p, h] = 1$ and $\langle g, h \rangle$ is periodic (N. S. Chernikov [1]). The class of weakly graded groups is obviously a subclass of the class of primitive graded groups. Further, every Ol'shanskiy's infinite simple torsion-free group with cyclic proper subgroups (see, for instance, [7] or [8, Theorem 28.3]) is primitive graded but is not weakly graded. Thus, it is a proper subclass.

Also remind: an infinite non-abelian group with normal infinite non-abelian subgroups is called an \overline{IH} -group (S. N. Chernikov, see, for instance, [9]). S. N. Chernikov has obtained a lot of principal results on \overline{IH} -groups (see, for instance, [9, Chapter 6]).

Remind that by definition the group satisfies the minimal condition for some subgroups, if it has not a descending infinite chain of these subgroups. Below $\min - \overline{ab}$ and $\min - \overline{ab\bar{n}}$ are the minimal conditions for non-abelian and non-abelian non-normal subgroups respectively; see, for instance, [2, p. 23]. (Clearly, abelian groups satisfy $\min - \overline{ab}$). Also $\min - \bar{n}$ is the minimal condition for non-normal subgroups (see, for instance, [1], [2, p. 23]). The groups, satisfying the minimal condition for (all) subgroups, are called Artinian.

Remind that a group, in which all subgroups are normal, is called Dedekind. The Dedekind groups are exactly the abelian groups and the groups $G = Q \times T \times R$ with a quaternion group Q , an elementary abelian 2-group T and an abelian group R with all elements of odd orders (R. Baer's Theorem, see [10]).

2. The main results

The following very general theorems are the main results of the present paper.

Theorem 1. *Let \mathfrak{L} be the minimal local in the sense of D. Robinson class of groups, containing the class \mathfrak{L}_\circ of weakly graded groups and closed with respect to forming subgroups, series (and, at the same time, to forming subcartesian products). The non-abelian group $G \in \mathfrak{L}$ satisfies $\min - \overline{ab}$ (respectively $\min - \overline{ab\bar{n}}$) iff it is Chernikov (respectively a Chernikov group or a solvable group with normal non-abelian subgroups).*

Theorem 2. *Let \mathfrak{T} be the minimal local in the sense of D. Robinson class of groups, containing the class \mathfrak{L}_0 of primitive graded groups and closed with respect to forming subgroups, series (and, at the same time, to forming subcartesian products). The group $G \in \mathfrak{T}$ is Artinian (respectively satisfies $\min - \bar{n}$) iff it is Chernikov (respectively Chernikov or Dedekind).*

Theorem 3. *Let \mathfrak{L} be the same as in Theorem 1, G be an \overline{IH} -group and R be the intersection of all infinite non-abelian subgroups of G . The group G is solvable iff $R \in \mathfrak{L}$.*

The Ol'shanskiy's Examples of infinite simple groups G , in which every proper non-identity subgroup has a prime order (see, for instance, [11]) show: in Theorems 1–3, the conditions: " $G \in \mathfrak{L}$ ", " $G \in \mathfrak{T}$ " and " $R \in \mathfrak{L}$ " are essential (may not be rejected). Also each Ol'shanskiy's infinite simple torsion-free group with cyclic proper subgroups (see, for instance [7] or [8, Theorem 28.3]) is primitive graded, satisfies $\min - \overline{ab}$, $\min - \overline{abn}$ and is not Chernikov or solvable. Thus, it is not possible to replace in Theorem 1 the condition: " $G \in \mathfrak{L}$ ", by the condition: " $G \in \mathfrak{T}$ ".

3. Proofs of the main results

Proof of Theorem 1. First, by virtue of Lemma 2, \mathfrak{L} is the minimal local (in our) sense class of groups, containing the class \mathfrak{L}_0 and closed with respect to forming subgroups and series. Then in consequence of Lemma 1.37 [6], \mathfrak{L} is closed with respect to forming subcartesian products. For ordinals $\alpha \neq 0$ define by induction: if for some ordinal β , $\alpha = \beta + 1$, then \mathfrak{L}_α is the class of all groups with a local system of subgroups, possessing a series with \mathfrak{L}_β -factors; otherwise $\mathfrak{L}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{L}_\beta$. The class \mathfrak{L}_0 is, of course, closed with respect to forming subgroups. Assume that for some $\alpha > 0$, all \mathfrak{L}_β , $\beta < \alpha$, are closed with respect to forming subgroups. Then it is easy to see: \mathfrak{L}_α is also closed with respect to forming subgroups. Thus, all classes \mathfrak{L}_α and, at the same time, their union $\mathfrak{Q} = \bigcup_\alpha \mathfrak{L}_\alpha$ are closed with respect to forming subgroups. Let a group F have some series with \mathfrak{Q} -factors. For each factor of the series, take some ordinal α such that \mathfrak{L}_α contains it. For the set (of cardinality $\leq |F|$) of all taken ordinals, there exists some ordinal ζ such that $\alpha < \zeta$ whenever α belongs to this set. Then all factors of the series belong to \mathfrak{L}_ζ . Therefore $F \in \mathfrak{L}_{\zeta+1}$, (i.e. $F \in \mathfrak{Q}$). Now let a group F have a local system of subgroups $K \in \mathfrak{Q}$. For each K , take some ordinal α such that \mathfrak{L}_α contains K . For the set of all taken ordinals, there exists some ordinal ζ such that $\alpha < \zeta$ whenever α belongs to this set. Then all K belong to \mathfrak{L}_ζ . Therefore $F \in \mathfrak{L}_{\zeta+1}$, (i.e. $F \in \mathfrak{Q}$).

Since $\mathfrak{L}_0 \subseteq \mathfrak{Q} \subseteq \mathfrak{L}$ and \mathfrak{Q} is closed with respect to forming subgroups, series, and also \mathfrak{Q} is local, we have: $\mathfrak{Q} = \mathfrak{L}$.

Suppose that there exist some non-abelian non-Chernikov groups $G \in \mathfrak{L}$ with $\min - \overline{ab}$. Let η be minimal among all ordinals ι , for which \mathfrak{L}_ι includes such G . In view of Theorem B [12], $\eta > 0$. It is easy to see: for some ordinal λ , $\eta = \lambda + 1$. Therefore such G from \mathfrak{L}_η possesses a local system of subgroups S having a series with \mathfrak{L}_λ -factors. Every factor, clearly, satisfies $\min - \overline{ab}$ and so is Chernikov or abelian. Obviously, a series has a refinement with finite factors. Then by virtue of S.N. Chernikov's is Theorem 6.1 [9], S is Chernikov or abelian. Since G is not abelian, all its non-abelian subgroups S form a local system. Consequently, G is locally finite. Then in view of Shunkov's Theorem [13], G must be Chernikov, which is a contradiction.

Suppose that there exist non-abelian non-Chernikov groups $G \in \mathfrak{L}$ with $\min - \overline{abn}$, which are not solvable with normal non-abelian subgroups. Let η be as above. In view of Theorem 1 [2], $\eta > 0$. It is easy to see: for some ordinal λ , $\eta = \lambda + 1$. Therefore such G from \mathfrak{L}_η possesses a local system of subgroups S having a series with \mathfrak{L}_λ -factors. Every factor, clearly, satisfies $\min - \overline{abn}$ and so is Chernikov or solvable. Therefore, obviously, a series has a refinement with finite factors. Further, any finitely generated non-identity subgroup M of G belongs to some S . Consequently M has a series with finite factors. Therefore obviously M has a subgroup of finite index $\neq 1$. Thus, G is locally graded. Then in view of Corollary 8 [2], G is a Chernikov group or a solvable group with normal non-abelian subgroups, which is a contradiction. Theorem is proven. \square

Proof of Theorem 2. First, by virtue of Lemma 2, \mathfrak{T} is the minimal local (in our sense)

class of groups, containing \mathfrak{L}_0 and closed with respect to forming subgroups and series. Then in consequence of Lemma 1.37 [??], \mathfrak{T} is closed with respect to forming subcartesian products. The class \mathfrak{L}_0 is, of course, closed with respect to forming subgroups. Let \mathfrak{L}_α , $\alpha > 0$, and \mathfrak{Q} be the same as in the proof of Theorem 1. Then: \mathfrak{L}_α and \mathfrak{Q} are closed with respect to forming subgroups, \mathfrak{Q} is closed with respect to forming series and \mathfrak{Q} is local, and $\mathfrak{Q} = \mathfrak{T}$ (see the proof of Theorem 1).

Suppose that there exist Artinian non-Chernikov groups $G \in \mathfrak{T}$. Let η be minimal among all ordinal ι , for which \mathfrak{L}_ι includes such G . Every Artinian primitive graded group is periodic and, at the same time, weakly graded. In view of Theorem B [12], Artinian weakly graded groups are Chernikov. Thus, $\eta > 0$. It is easy to see: for some ordinal λ , $\eta = \lambda + 1$. Therefore the Artinian non-Chernikov group G from \mathfrak{L}_η possesses a local system of some subgroup S having a series with \mathfrak{L}_λ -factors. Since S is Artinian, the series is ascending. Every its factor is Artinian and so Chernikov. Then in consequence of O. Yu. Schmidt's Theorem (see, for instance, [6, Theorem 1.45]), S is locally finite. Consequently, G is locally finite. Therefore in view of Shunkov-Kegel-Wehrfritz Theorem [14, 15], G is Chernikov, which is a contradiction.

Let H be any group with $\min - \bar{n}$. If H is Artinian, then H' is Artinian too. Assume that H is non-Artinian. It is easy to see: H contains some non-Artinian subgroup B such that every non-Artinian subgroup of B is normal in H ; B has some descending series $B = B_0 \supset B_1 \supset B_2 \supset \dots \supset B_\gamma = \bigcap_{\alpha < \gamma} B_\alpha \supseteq 1$ such that B_γ is Artinian. Obviously all B/B_α , $\alpha < \gamma$, are Dedekind. Consequently, in view of R. Baer's Theorem [10], $|(B/B_\alpha)'| \leq 2$, $\alpha < \gamma$. Therefore $|(B/B_\gamma)'| \leq 2$. Put $(B/B_\gamma)' = D/B_\gamma$. Then D is Artinian. Therefore B/D is abelian non-Artinian. Since H/D does not satisfy the minimal condition for abelian subgroups and obviously satisfies $\min - \bar{n}$, it is Dedekind (see Corollary 2 [16]). Put $(H/D)' = T/D$. In view of R. Baer's Theorem [10], $|T/D| \leq 2$. Since D is Artinian, T is Artinian too. Therefore, because of $H' \subseteq T$, H' is Artinian.

Now let G be a \mathfrak{T} -group with $\min - \bar{n}$. Since G' is an Artinian \mathfrak{T} -group, it is Chernikov (see above). Therefore in consequence of Theorem 1 [16], G is Chernikov or Dedekind. Theorem is proven. \square

Proof of Theorem 3. First, for every infinite non-abelian subgroup N of G , each subgroup $H \supseteq N$ of G is normal in it. Consequently, G/N is Dedekind. Therefore in view of R. Baer's Theorem [10], $(G/N)'' = 1$. Consequently, $G'' \subseteq R$.

Now assume that $R \in \mathfrak{L}$. Since each proper subgroup of R is finite or abelian, R satisfies $\min - \bar{ab}$. Therefore in view of Theorem 1, R is abelian or Chernikov. In the first case, $G''' = 1$. In the second case, R contains some characteristic abelian subgroup T with $|R : T| < \infty$. Clearly, $(C_G(R/T) \cap R)/T = Z(R/T)$. Consequently $(C_G(R/T) \cap R)'' = 1$. Since $(G/R)'' = 1$, we have: $(C_G(R/T)/C_G(R/T) \cap R)'' = 1$. Thus, $C_G(R/T)$ is solvable of derived length ≤ 4 . Further, $|G : C_G(R/T)| < \infty$. Let V be a maximal normal solvable subgroup of G containing $C_G(R/T)$. If $V \neq G$, then G/V is non-abelian. At the same time, G/V contains some non-abelian subgroup M/V with abelian proper subgroups. In view of Miller-Moreno Theorem [17], M/V is solvable. Then M is infinite non-abelian solvable. Therefore M is normal in G , which is a contradiction. Theorem is proven. \square

The following proposition was in fact established in the proofs of Theorems 1 and 2.

Proposition. *Let \mathfrak{V} be some class of groups, closed with respect to forming subgroups, and \mathfrak{K} be the minimal local in the sense of D. Robinson class of groups, containing \mathfrak{V} and closed with respect to forming subgroups, series (and, at the same time, to forming subcartesian products). Then:*

- (i) *if every Artinian \mathfrak{V} -group is Chernikov, then every Artinian \mathfrak{K} -group is Chernikov;*

- (ii) if every \mathfrak{V} -group with $\min - \overline{ab}$ is Chernikov or abelian, then every \mathfrak{K} -group with $\min - \overline{ab}$ is Chernikov or abelian;
- (iii) if every \mathfrak{V} -group with $\min - \overline{n}$ is Chernikov or Dedekind, then every \mathfrak{K} -group with $\min - \overline{n}$ is Chernikov or Dedekind;
- (iv) if every \mathfrak{V} -group with $\min - \overline{ab\overline{n}}$ is a Chernikov or abelian group, or a non-abelian solvable group with normal non-abelian subgroups, then every \mathfrak{K} -group with $\min - \overline{ab\overline{n}}$ is the same.

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Некоторые условия минимальности в некоторых экстремально больших классах групп

Николай С. Черников

Пусть \mathfrak{L} (соответственно, \mathfrak{T}) — минимальный локальный в смысле Д. Робинсона класс групп, содержащий класс слабоступенчатых (соответственно, примитивно ступенчатых) групп и замкнутый относительно образования подгрупп и обобщенных нормальных систем. В настоящей работе мы полностью описываем: \mathfrak{L} -группы с условиями минимальности для неабелевых подгрупп и для неабелевых ненормальных подгрупп; \mathfrak{T} -группы с условиями минимальности для (всех) подгрупп и для ненормальных подгрупп. Попутно мы устанавливаем, что любая \overline{HN} -группа, принадлежащая к \mathfrak{L} , разрешима.

Ключевые слова: локальные классы групп; условия минимальности; неабелевы, черниковские, артиновы, дедекиндовы, \overline{HN} -группы; слабо, локально, бинарно, примитивно ступенчатые группы.