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The 2D Motion of Perfect Fluid with a Free Surface

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The 3D continuous subalgebra is used to searching new partially invariant solution of incompressible perfect fluid equations. It can be interpreted as a non-stationary motion of a plane layer with one free surface. The velocity field and pressure are determined in analytical form by using Lagrangian coordinates.

Keywords: perfect fluid, partially invariant solution, non-stationary motion, free surfaces.

Governing flow equations and main results

The Euler equations for 2D motions of a perfect fluid are recorded by

$$\begin{gathered} u_t + uu_x + vu_y + \frac{1}{\rho}\,p_x = 0, \quad u_x + v_y = 0, \\ v_t + uv_x + vv_y + \frac{1}{\rho}\,p_y = 0, \end{gathered} \tag{1}$$

where ρ is the constant fluid density, u and v are the velocity components in the x and y directions, respectively, p is the pressure. The group of point transformations admitted by the system (1) is computed in [1]. Corresponding this group basic continuous Lie algebra includes the three parametrical subalgebra $\langle \partial_x, t\partial_u + \partial_x, \partial_p \rangle$. It has the invariants t, y, v and partly invariant solution of (1) rang two and defect two necessary to seek in the form $u = u(x, y, t)$, $v = v(y, t)$, $p = p(x, y, t)$. From continuity equation $u_x + v_y = 0$ we obtain the relations

$$u(x, y, t) = u_1(y, t)x + u_2(y, t), \quad u_1(y, t) + v_y(y, t) = 0. \tag{2}$$

Impulse equations (1) are equivalent to the following

$$\begin{gathered} u_{1t} + vu_{1y} + u_1^2 = f(t), \quad \frac{1}{\rho}\,p = l(y, t) - f(t)\frac{x^2}{2}\,, \\ l_y = -v_t - vv_y, \quad u_{2t} + vu_{2y} + u_1u_2 = 0 \end{gathered} \tag{3}$$

with arbitrary function $f(t)$.

Let us introduce the Lagrangian coordinates (η, t) by the solving Cauchy problem

$$\frac{dy}{dt} = v(y, t), \quad y\big|_{t=0} = \eta. \tag{4}$$

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We introduce the following denotations

$$\mathring{u}_1(\eta,t)=u_1(y(\eta,t),t),\quad \mathring{u}_2(\eta,t)=u_2(y(\eta,t),t),\quad \mathring{v}(\eta,t)=v(y(\eta,t),t),$$

where $y(\eta,t)$ is a solution of (4). Then the first equations (3) can be reduced to Riccati equation

$$\mathring{u}_{1t}+\mathring{u}_1^{\,2}=f(t).$$

It has general solution

$$\mathring{u}_1(\eta,t)=\frac{\partial}{\partial t}\left\{\ln\left[g(t)\left(1+u_{10}(\eta)\int_0^t\frac{1}{g^2(t)}\,dt\right)\right]\right\}. \tag{5}$$

Here $g(t)$ is the solution of the Cauchy problem

$$g''-f(t)g=0,\quad g(0)=1,\quad g'(0)=0, \tag{6}$$

and $u_{10}(\eta)$ is the initial value of function $u_1(y,t)$.

The another functions can be found by the formulae

$$y(\eta,t)=\frac{1}{g(t)}\int_0^\eta\left[1+u_{10}(\eta)\int_0^t\frac{1}{g^2(t)}\,dt\right]^{-1}d\eta; \tag{7}$$

$$\mathring{v}(\eta,t)=-\int_0^\eta\mathring{u}_1(\eta,t)\exp\left[-\int_0^t\mathring{u}_1(\eta,t)\,dt\right]d\eta; \tag{8}$$

$$\mathring{u}_2(\eta,t)=u_{20}(\eta)\exp\left[-\int_0^t\mathring{u}_1(\eta,t)\,dt\right]; \tag{9}$$

$$\mathring{l}(\eta,t)=l_1(t)-\int_0^\eta\mathring{v}_t(\eta,t)\exp\left[-\int_0^t\mathring{u}_1(\eta,t)\,dt\right]d\eta \tag{10}$$

with arbitrary function $l_1(t)$. So, all unknowns can be determined in analytical form.

Now we show that this solution can be interpreted as an unsteady motion in a strip with one free boundary, see Fig. 1.

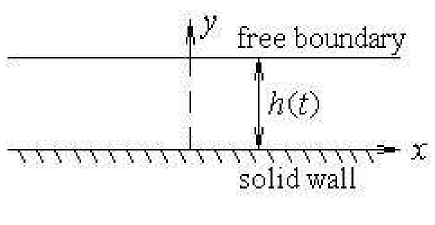

Fig. 1 Geometry of the motion

Really, at the initial time liquid fills the strip of thickness $y=h_0=\text{const}$. The line $y=0$ is a rigid wall. Initial velocity field has the form $u_0(x,y)=u_{10}(y)x+u_{20}(y)$, $v_0(y)=-\int_0^y u_{10}(y)\,dy$, $v_0(0)=0$. The upper line $y=h_0$ is a free boundary and at the initial time the pressure $p(h_0,0)$ coincides with outer pressure $p_{\text{out}}=p_{10}+p_{00}x^2/2$. For all $t>0$ the strip motion is described by the formulae are found above, where $p_{\rm out} = p_1(t) - p_0(t)x^2/2$ must be given, so $f(t) = p_0(t)$, $f(0) = p_{00}$. The evolution of the free boundary is defined as

$$h(t) = \frac{1}{g(t)} \int_0^{h_0} \left[1 + u_{10}(\eta) \int_0^t \frac{1}{g^2(t)}\, dt\right]^{-1} d\eta. \tag{11}$$

Let us consider two simple cases of the solution (5)–(11), when $u_{10} = a = \text{const}$ or $u_{10} = b\eta$, $b = \text{const}$. For the first case the exact solution can be written in Eulerian coordinates as

$$\begin{gathered} u(x, y, t) = \frac{\partial}{\partial t} \ln G(t)x + \frac{1}{G(t)}\, u_{20}(G(t)y), \\ v(y, t) = -\frac{\partial}{\partial t} \ln G(t)y, \\ \frac{1}{\rho}\, p(x, y, t) = l_1(t) + \frac{\partial^2}{\partial t^2} \ln G(t)\frac{y^2}{2} - f(t)\frac{x^2}{2}\,, \end{gathered} \tag{12}$$

where

$$G(t) = g(t)\left[1 + a \int_0^t \frac{1}{g^2(t)}\, dt\right]. \tag{13}$$

The equation of the free boundary is

$$y = h(t) = \frac{h_0}{G(t)}\,. \tag{14}$$

If we take $g(t) = \cos\omega t$, $f(t) = -\omega^2$ ($g(t) = \operatorname{ch}\omega t$, $f(t) = \omega^2$), $\omega = \text{const}$, then the solution exists up to the time $t_* = \pi/2\omega$ (exists for all time). The solution has to be periodic one if $g(t) = 2 - \cos\omega t$, $f(t) = \omega^2 \cos\omega t(2 - \cos\omega t)^{-1}$.

For the second case the formulae have a more complicated shapes and we give here only equation of the free boundary, namely,

$$y = h(t) = \frac{1}{bg(t) \int_0^t g^{-2}(t)\, dt} \ln\left[1 + bh_0 \int_0^t g^{-2}(t)\, dt\right]. \tag{15}$$

Remark 1. In well-known [2] solutions are sought of the shape $\psi(x, y, t) = F(y, t)x + G(y, t)$ for stream function ($u_1 = \psi_y$, $v_1 = -\psi_x$). The unknowns satisfy the eq's

$$\begin{gathered} F_{ty} + (F_y)^2 - FF_{yy} = f_1(t), \quad G_{ty} + F_yG_y - FG_{yy} = f_2(t), \\ G = \int U\, dy - hF + h'_t y, \quad h''_{tt} - f_1(t)h = f_2(t). \end{gathered}$$

Some particular solution are presented in handbook, see [2, table 13.9, p. 944]. But in this paper we have found exact solution in analytical form.

The problem has a stationary solution. Indeed, the function $v(y)$ satisfies the eq'n $vv_{yy} - v_y^2 = -f_0 = \text{const}$ with general solution

a) $v = \sqrt{\dfrac{f_0}{|C_1|}} \sin\left[\sqrt{|C_1|}\,(C_2 \pm y)\right]$, $f_0 > 0$, $C_1 < 0$;

b) $v = \pm\sqrt{f_0}\,(C_2 \pm y)$, $f_0 > 0$, $C_1 = 0$;

c) $v = \sqrt{\dfrac{|f_0|}{C_1}} \operatorname{ch}\left[\sqrt{C_1}\,(C_2 \pm y)\right]$, $f_0 < 0$, $C_1 > 0$;

d) $v = \sqrt{\dfrac{f_0}{C_1}} \operatorname{sh}\left[\sqrt{C_1}\,(C_2 \pm y)\right]$, $f_0 > 0$, $C_1 > 0$.

However, the only case a) has a physical meaning. Really, let us take $h_0 = \pi/\sqrt{|C_1|}$, then we obtain formulae

$$u = -\sqrt{f_0}\, x \cos\frac{\pi y}{h_0}\,, \quad v = \frac{h_0\sqrt{f_0}}{\pi}\,\sin\frac{\pi y}{h_0}\,,$$

$$\frac{1}{\rho}\,p = l_0 - \frac{f_0}{2}\left(x^2 + \frac{h_0^2}{\pi^2}\sin^2\frac{\pi y}{h_0}\right), \quad l_0 = \text{const} > 0,$$

which describe the flow in a strip $0 < y < h_0$, $|x| < \infty$, with rigid walls $y = 0, h_0$. The solution obtained is the periodical with respect variable y.

Conclusion

The partially invariant solution of the perfect fluid equations is investigates. This new solution describes the unsteady motion with a free surface. As was shown by examples the solution may has collapse in finite time or to be periodical one. Not that this phenomenon depends on pressure gradient $f(t)$. There has been previous works devoted exact solutions of perfect fluid motions with a free surfaces [3].

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Двумерное движение идеальной жидкости со свободной поверхностью

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Непрерывная трёхмерная подалгебра используется для нахождения нового частично инвариантного решения уравнений идеальной несжимаемой жидкости. Оно интерпретируется как нестационарное движение плоского слоя со свободной поверхностью. При этом поля скоростей и давлений определяются (с помощью переменных Лагранжа) в аналитическом виде.

Ключевые слова: идеальная жидкость, частично инвариантное решение, нестационарное движение, свободная поверхность.