Journal of Siberian Federal University. Mathematics & Physics 2014, 7(4), 515–529

УДК 517.55

Some Examples of Finding the Sums of Multiple Series

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Received 06.05.2014, received in revised form 19.07.2014, accepted 26.09.2014

A method of finding residue integrals for some systems of non-algebraic equations are presented. Such integrals are connected to the power sums of roots for the system of equations. It is shown how the obtained results can be used for calculating sums of multidimensional series.

Keywords: residue integral, power sum, multiple series.

Introduction

A method based on multidimensional residue theory for the elimination n unknowns from a system of n non-linear algebraic equations (in the characteristic zero setting) was proposed by L.A. Aizenberg [1]. Its further developments were implemented in [2–4]. The algorithmic method (inspired by the Aizenberg and Yuzhakov strategy) introduced by M. Elkadi and A. Yger [5]. The basic idea of the method is to find certain residue integrals connected to the power sums of roots of a given system of equations (in the positive powers) avoiding finding the roots, and to apply then the recurrent Newton formulas. This method is less time-consuming and does not increase the multiplicity of the roots in comparison with the classical method.

The set of roots of a system of n non-algebraic equations in n variables is in general infinite. Moreover, multi Newton sums (with exponents in $\mathbb{N}^n$) of the roots of such systems lead usually to divergent series. In the present work, we attach residue integrals to specific systems of n non-linear equations, compute such residue integrals, and deduce from this computation (provided such series do converge) the values of the sums of multi-Newton series (with exponents in $(-\mathbb{N}^*)^n$) formed with the roots of such non-linear systems which do not belong to the union of coordinate planes.

In the papers [6–10] a class of systems of equations containing entire or meromorphic functions was considered. In [11] a computer algebra algorithm that computes the corresponding residue integrals and applies to them the recurrent Newton formulas is presented.

Our goal is to generalize statements from the papers [6–10] to a another class of systems of non-algebraic equations; to obtain formulas for calculation of residue integrals, to give connection with power sums and to give the corresponding computer algebra algorithm.

In [6, 7], the following system of functions was considered:

$$f_1(z),\ f_2(z), \dots, f_n(z),$$

where $z = (z_1, z_2, \dots, z_n)$. Each $f_j(z)$ is analytic in the neighborhood of $0 \in \mathbb{C}^n$ and has the form

$$f_j(z) = z^{\beta^j} + Q_j(z), \quad j = 1, 2, \dots, n,$$

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where $\beta^j = (\beta^j_1, \beta^j_2, \ldots, \beta^j_n)$ is a multi-index with integer nonnegative coordinates, $z^{\beta^j} = z_1^{\beta_1^j} \cdot z_2^{\beta_2^j} \cdots z_n^{\beta_n^j}$, and $\|\beta^j\| = \beta^j_1 + \beta^j_2 + \ldots + \beta^j_n = k_j$, $j = 1, 2, \ldots, n$. Functions Q_j are expanded in a neighborhood of zero into an absolutely and uniformly converging Taylor series of the form

$$Q_j(z) = \sum_{\|\alpha\| > k_j} a^j_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \geqslant 0$, $\alpha_j \in \mathbb{Z}$, and $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$.

The formulas for calculation of residue integrals

$$J_\beta = \frac{1}{(2\pi i)^n} \int_{\gamma(r)} \frac{1}{z^{\beta+U}} \cdot \frac{df}{f}$$

in terms of coefficients of $Q_j(z)$ were obtained.

Then such systems was considered in [8,11]. One received multidinensional Newton formulas for such systems.

In the papers [9,10] was considered the class of systems in which functions

$$f_j(z) = (z^{\beta^j} + Q_j(z))e^{P_j}, \quad j = 1, 2, \ldots, n, \tag{1}$$

and in the paper [10] is given computer realization of considerable method.

Here we consider the system [12] in the case when the monomials z^{β^j} in the system (1) are replaced with products of linear functions.

1. Residue integrals

We consider a system of functions $f_1)(z),\ f_2(z), \ldots, f_n(z)$ and a system of equations

$$\begin{cases} f_1(z) = [(1 - a_{11}z_1)^{m_{11}} \cdot \ldots \cdot (1 - a_{1n}z_n)^{m_{1n}} + Q_1(z)]\, e^{P_1(z)} = 0, \\ f_2(z) = [(1 - a_{21}z_1)^{m_{21}} \cdot \ldots \cdot (1 - a_{2n}z_n)^{m_{2n}} + Q_2(z)]\, e^{P_2(z)} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(z) = [(1 - a_{n1}z_1)^{m_{n1}} \cdot \ldots \cdot (1 - a_{nn}z_n)^{m_{nn}} + Q_n(z)]\, e^{P_n(z)} = 0, \end{cases} \tag{2}$$

where m_{ij} are natural numbers, a_{ij} are complex numbers, different for each fixed j, $P_i(z)$, and $Q_i(z)$ are entire functions.

Denote by $q_i(z_1, \ldots, z_n)$ expression of the form

$$q_i(z_1, \ldots, z_n) = (1 - a_{i1}z_1)^{m_{i1}} \cdot \ldots \cdot (1 - a_{in}z_n)^{m_{in}}, \quad i = 1, \ldots n. \tag{3}$$

Then, our system can be rewritten as

$$f_i(z_1, \ldots, z_n) = [q_i(z_1, \ldots, z_n) + Q_i(z_1, \ldots, z_n)]\, e^{P_i(z_1, \ldots, z_n)},\ i = 1, 2, \ldots, n. \tag{4}$$

For each i we define a function

$$h_i(z) = \begin{cases} q_i(z), & \text{если } a_{ij} \neq 0, \quad \text{для всех} \quad j; \\ q_i(z) \cdot \dfrac{1}{z_{j_1}} \cdot \ldots \cdot \dfrac{1}{z_{j_k}}, & \text{если } a_{ij_1} = \ldots = a_{ij_k} = 0. \end{cases} \tag{5}$$

The system of equations

$$h_i(z) = 0, \quad i = 1, 2, \ldots, n \tag{6}$$

has $n!$ isolated roots in $\overline{\mathbb{C}}^n$ ($\overline{\mathbb{C}}^n$ is a theory of functions space). Let $J = (j_1, \ldots, j_n)$ be multi-index is a permutation of $(1, \ldots, n)$. Then the roots of (6) are

$$a_J = \begin{cases} (1/a_{1j_1}, \ldots, 1/a_{nj_n}), & \text{if all } a_{kj_k} \neq 0, \quad k = 1, \ldots, n; \\ (1/a_{1j_1}, \ldots, \infty_{[i_1]}, \ldots, \infty_{[i_k]}, \ldots, 1/a_{nj_n}), & \text{if } a_{i_1 j_{i_1}} = \ldots = a_{i_k j_{i_k}} = 0, \end{cases}$$

where $k, j = 1, \ldots, n$.

Denote by Γ_h the cycles

$$\Gamma_h = \{z \in \mathbb{C}^n : |h_i| = r_i, \quad r_i > 0, \ i = \overline{1, n}\}. \tag{7}$$

For the case when all $a_{k,j_k} \neq 0$ we define a cycle Γ_{h,a_J} by

$$\begin{cases} |1 - a_{1j_1} z_1| = r_1, \\ |1 - a_{2j_2} z_2| = r_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ |1 - a_{nj_n} z_n| = r_n. \end{cases} \tag{8}$$

If $a_{i_1 j_{i_1}} = \ldots = a_{i_k j_{i_k}} = 0$ for some $i_1, \ldots, i_k$ then Γ_{h,a_J} is defined by

$$\begin{cases} |1 - a_{1j_1} z_1| = r_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \left|\dfrac{1}{z_{i_1}}\right| = r_{i_1}, \\ \ldots\ldots\ldots\ldots\ldots \\ \left|\dfrac{1}{z_{i_k}}\right| = r_{i_k}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ |1 - a_{nj_n} z_n| = r_n, \end{cases} \tag{9}$$

Lemma 1. *For sufficiently small r_i a global cycle Γ_h has connected components (local cycles) in the neighborhoods of the roots a_J. Moreover, Γ_h is homologous to the sum of the local cycles Γ_{h,a_J}.*

Consider the system of equations

$$F_i(z,t) = (q_i(z) + t \cdot Q_i(z)) e^{P_i(z)} = 0 \quad i = 1, 2, \ldots, n, \tag{10}$$

depending on the real parameter $t \geqslant 0$.

Let $r_1, \ldots r_n > 0$ be the fixed real numbers. Then, for sufficiently small $t > 0$, the inequalities

$$\left|q_i(z)\right| > \left|t \cdot Q_i(z)\right|, \ i = 1, \ldots, n.$$

hold on the cycles

$$\Gamma_h = \{z \in \mathbb{C}^n \colon |h_i| = r_i, \ i = 1, \ldots, n\}$$

because the cycles Γ_h are compact.

By $J_\gamma(t)$ we denote the residue integral

$$\begin{aligned} J_\gamma(t) &= \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_h} \frac{1}{z^{\gamma+I}} \cdot \frac{dF}{F} = \\ &= \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_h} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{dF_1}{F_1} \wedge \frac{dF_2}{F_2} \wedge \ldots \wedge \frac{dF_n}{F_n}, \end{aligned} \tag{11}$$

where $\gamma = (\gamma_1, \dots \gamma_n)$ is multi-index.

Denote
$$G_i(z,t) = q_i(z) + t \cdot Q_i(z), \quad i = 1, 2, \dots, n.$$
Let I be a multi-index of the length n, consisting of s ones and $n-s$ zeros ($s = 0, \dots, n$). Denote by Δ_I Jacobian of the system of functions such that to each "one" on the j-th place in I there corresponds j-th row of the derivatives $(\partial G_j/\partial z_i)$, $1 \leqslant i \leqslant n$ in Δ_I; and, to each "zero" on the k-th place in I there corresponds k-th row of the derivatives $(\partial P_k/\partial z_i)$, $1 \leqslant i \leqslant n$ in Δ_I.

Theorem 1 ([12]). *Under the assumptions made for the functions F_i defined by (10) the following formulas for $J_\gamma(t)$ as convergent series are valid:*
$$J_\gamma(t) = \sum_J \sum_I \sum_{\alpha^s} (-t)^{||\alpha^s||}(-1)^{s(J)} \frac{1}{\beta(\alpha^s, J)!} \cdot \frac{\partial^{||\beta^s||}}{\partial z^{\beta^s}} \left[\frac{\Delta_I(t)}{z_1^{\gamma_1+1} \cdot \ldots \cdot z_n^{\gamma_n+1}} \cdot \frac{Q^{\alpha^s}(I)}{q^{\alpha^s+I}(I,J)} \right]_{z=a_J},$$
where $(-1)^{s(J)} = 1$, if J is even permutation, and $(-1)^{s(J)} = -1$, if J is odd permutation, α^s is multi-index of order s, i_l is a number of l-the unit of I, $q^{\alpha^s+I}(I,J) = q_1^{\alpha_1^s+1}[j_1] \cdot \ldots \cdot q_s^{\alpha_s^s+1}[j_n]$, and $q_p[j_p]$ is product of all $(1-a_{p1}z_1)^{m_{p1}} \cdot \ldots \cdot (1-a_{pn}z_n)^{m_{pn}}$ besides $(1-a_{p_{jp}}z_{jp})^{m_{pj_p}}$, $Q^{\alpha^s}(I) = Q_{i_1}^{\alpha_1^s} \cdot \ldots \cdot Q_{i_s}^{\alpha_s^s}$,
$$\beta(\alpha^s, J) = (m_{1j_1} \cdot (\alpha_{j_1}^s + 1) - 1, \dots, m_{sj_n} \cdot (\alpha_{j_n}^s + 1) - 1),$$
$$\beta(\alpha^s, J)! = \prod_p (m_{pj_p} \cdot (\alpha_{j_p}^s + 1) - 1)!,$$
$$\frac{\partial^{||\beta||}}{\partial z^\beta} = \frac{\partial^{m_{1j_1}\cdot(\alpha_{j_1}^s+1)-1+\ldots+m_{sj_n}\cdot(\alpha_{j_n}^s+1)-1}}{\partial z_1^{m_{1j_1}\cdot(\alpha_{j_1}^s+1)-1} \cdot \ldots \cdot \partial z_n^{m_{sj_n}\cdot(\alpha_{j_n}^s+1)-1}}.$$

2. Residue integrals and power sums

Under certain restrictions on Q_i and P_i the considered residue integrals are connected to the power sums of roots of the system (2).

Suppose that $Q_i(z)$ are polynomials:
$$Q_i(z) = z_1 \cdots z_n \sum_{|\alpha\| \geqslant 0} C_\alpha^i z^\alpha \quad i = 1, 2, \dots, n, \tag{12}$$
where α is a multi-index, $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$, and $\deg_{z_j} Q_i \leqslant m_{ij}$, $i, j = 1, \dots, n$ for all non-zero a_{ij}. If $a_{ij} = 0$ then there is no restriction on $\deg_{z_j} Q_i$.

Functions P_j $(j = 1, 2, \dots, n)$ are the polynomials
$$P_j(z) = \sum_{0 \leqslant \|\eta\| \leqslant p_j} b_\eta^j z^\eta, \tag{13}$$
where $\eta = (\eta_1, \dots, \eta_n)$ is a multi-index.

Assuming that all $w_j \neq 0$, we substitute $z_j = \dfrac{1}{w_j}$, $j = 1, \dots, n$ in the functions
$$F_i(z,t) = (q_i(z) + t \cdot Q_i(z))\, e^{P_i(z)}, \quad i = 1, 2, \dots, n.$$
Consequently, for $i = 1, \dots, n$ we get
$$F_i\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}, t\right) = \left(q_i\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right) + t \cdot Q_i\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)\right) e^{P_i\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)}.$$

And finally we arrive at

$$F_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n},t\right)=$$
$$=\left(\left(1-a_{i1}\frac{1}{w_1}\right)^{m_{i1}}\cdot\ldots\cdot\left(1-a_{in}\frac{1}{w_n}\right)^{m_{in}}+t\cdot Q_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)\right)e^{P_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)}=$$
$$=\left(\left(\frac{1}{w_1}\right)^{m_{i1}}\cdot\ldots\cdot\left(\frac{1}{w_n}\right)^{m_{in}}\cdot(w_1-a_{i1})^{m_{i1}}\cdot\ldots\cdot(w_n-a_{in})^{m_{in}}+\right. \tag{14}$$
$$\left.+t\cdot Q_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)\right)e^{P_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)}=$$
$$=\left(\frac{1}{w_1}\right)^{m_{i1}}\cdot\ldots\cdot\left(\frac{1}{w_n}\right)^{m_{in}}\cdot\left(\widetilde{q}_i(w)+t\cdot\widetilde{Q}_i(w)\right)e^{P_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)},$$

where $\widetilde{q}_i$ are the functions

$$\widetilde{q}_i=(w_1-a_{i1})^{m_{i1}}\cdot\ldots\cdot(w_n-a_{in})^{m_{in}},$$

and $\widetilde{Q}_i$ are the polynomials

$$\widetilde{Q}_i=w_1^{m_{i1}}\cdot\ldots\cdot w_n^{m_{in}}\cdot Q_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right).$$

In the above calculations it is not important whether a_{ij} vanish or not. Indeed, assume that in $F_i(z,t)=\left(q_i(z)+t\cdot Q_i(z)\right)e^{P(z)}$, $i=1,\ldots,n$, some $a_{ij}=0$ vanishes. If, for instance, $a_{11}=0$, then after substitution $z_j=\dfrac{1}{w_j}$, $j=1,\ldots,n$, the function F_1 takes the form

$$F_1\left(\frac{1}{w_1},\ldots,\frac{1}{w_n},t\right)=\left(q_1\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)+t\cdot Q_1\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)\right)e^{P_1\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)},\quad i=1,2,\ldots n.$$

Then

$$F_1\left(\frac{1}{w_1},\ldots,\frac{1}{w_n},t\right)=\left(\left(1-a_{12}\frac{1}{w_1}\right)^{m_{12}}\cdot\ldots\cdot\left(1-a_{1n}\frac{1}{w_n}\right)^{m_{1n}}+\right.$$
$$\left.+t\cdot Q_1\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)\right)e^{P_1\left(\frac{1}{w_1},\ldots\frac{1}{w_n}\right)}=$$
$$=\left(\left(\frac{1}{w_1}\right)^{\deg_{w_1}Q_1}\cdot\ldots\cdot\left(\frac{1}{w_n}\right)^{m_{1n}}\cdot(w_1)^{\deg_{w_1}Q_1}\cdot\ldots\cdot(w_n-a_{1n})^{m_{1n}}+\right.$$
$$\left.+t\cdot Q_i\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)\right)e^{P_1\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)}=$$
$$=\left(\frac{1}{w_1}\right)^{\deg_{w_1}Q_1}\cdot\ldots\cdot\left(\frac{1}{w_n}\right)^{m_{1n}}\cdot\left(\widetilde{q}_1(w)+t\cdot\widetilde{Q}_1(w)\right)e^{P_1\left(\frac{1}{w_1},\ldots\frac{1}{w_n}\right)},$$

where $\widetilde{q}_1$ is the function $\widetilde{q}_1=(w_1)^{\deg_{w_1}Q_1}\cdot\ldots\cdot(w_n-a_{1n})^{m_{1n}}$, and $\widetilde{Q}_1$ are polynomials of the form $\widetilde{Q}_1=w_1^{\deg_{w_1}Q_1}\cdot\ldots\cdot w_n^{m_{1n}}\cdot Q_1\left(\dfrac{1}{w_1},\ldots\dfrac{1}{w_n}\right)$. That is, one can take $m_{11}=\deg_{w_1}Q_1$.

From (12) we derive that

$$\deg_{w_j}\widetilde{Q}_1<m_{1j},\ j=1,\ldots,n.$$

Denote

$$\widetilde{G_i}(w,t) = \widetilde{q}_i(w) + t\cdot \widetilde{Q}_i(w), \quad i = 1,2,\dots,n. \tag{15}$$

When $0 \leqslant t \leqslant 1$, the system (15) has a finite number of roots in $\mathbb{C}^n$, depending on parameter t, and has no infinite roots in $\overline{\mathbb{C}^n}$ (see [13]).

Sufficiently close to zero t on the cycle

$$\widetilde{\Gamma}_h = \{w \in \mathbb{C}^n : \left| h_i\left(\frac{1}{w_1},\dots \frac{1}{w_n}\right)\right| = \varepsilon_i, \quad i = 1,2,\dots,n\},$$

compactness of the cycle implies

$$\left| q_i\left(\frac{1}{w_1},\dots \frac{1}{w_n}\right)\right| > \left| t\cdot Q_i\left(\frac{1}{w_1},\dots \frac{1}{w_n}\right)\right|, \quad i = 1,2,\dots,n.$$

Therefore $\widetilde{\Gamma}_h$ is homologous to the sum of the cycles $\widetilde{\Gamma}_{h,a_J}$

$$\begin{cases} \left|1 - a_{1i_1}\frac{1}{w_1}\right| = \varepsilon_1, \\ \left|1 - a_{2i_2}\frac{1}{w_2}\right| = \varepsilon_2, \\ \dots \\ \left|1 - a_{ni_n}\frac{1}{w_n}\right| = \varepsilon_n. \end{cases} \tag{16}$$

obtained from the cycles Γ_{h,a_J} by the substitution $z_j = \dfrac{1}{w_j}$.

The equation

$$\left|1 - a_{ji_j}\frac{1}{w_j}\right| = \varepsilon$$

defines a circle. Indeed, let us first rewrite it in the form

$$\left|1 - a_{ji_j}\frac{1}{w_j}\right| = \varepsilon, \quad \text{then} \quad |w_j - a_{ji_j}| = \varepsilon|w_j|.$$

Thus

$$|w_j - a_{ji_j}|^2 = \varepsilon^2|w_j|^2, \quad \text{then} \quad (1-\varepsilon^2)\left|w_j - \frac{a_{ji_j}}{1-\varepsilon^2}\right|^2 = \frac{\varepsilon^2\cdot|a_{ji_j}|^2}{(1-\varepsilon^2)},$$

$$\left|w_j - \frac{a_{ji_j}}{1-\varepsilon^2}\right|^2 = \frac{\varepsilon^2\cdot|a_{ji_j}|^2}{(1-\varepsilon^2)^2}, \quad j = 1,\dots,n.$$

For sufficiently small ε the point a_{ji_j} lies inside this circle, and therefore $\widetilde{\Gamma}_{h,a_J}$ is homologous to the cycle

$$\begin{cases} |w_1 - a_{1j_1}| = \varepsilon_1, \\ |w_2 - a_{2j_2}| = \varepsilon_2, \\ \dots\dots\dots\dots\dots\dots \\ |w_n - a_{nj_n}| = \varepsilon_n. \end{cases}$$

Here some a_{ij} can vanish.

Lemma 2 ([12])**.** *Let P_j be defined by (13), and the inequality*

$$l^1 + \dots + l^n \leqslant \gamma \tag{17}$$

holds for a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$*, where* $l^j = (l_1^j, \dots, l_n^j)$ *and* l_i^j *is a degree of* P_i *in* z_j *for* $i, j = 1, \dots, n$ *(i.e.* n *scalar inequalities* $l_i^1 + \dots + l_i^n \leqslant \gamma_i$ *hold).*

Then

$$J_\gamma(t) = \frac{(-1)^n}{(2\pi\sqrt{-1})^n} \int_{\widetilde{\Gamma}_h} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot \frac{d\widetilde{G}_1}{\widetilde{G}_1} \wedge \frac{d\widetilde{G}_2}{\widetilde{G}_2} \wedge \dots \wedge \frac{d\widetilde{G}_n}{\widetilde{G}_n}. \tag{18}$$

(Inequality (17) means that it holds coordinatewise).

Lemma 3 ([12]). *Let* $\widetilde{\Delta} = \widetilde{\Delta}(w, t)$ *be the Jacobian of the system* $\widetilde{G}_1(w, t), \dots, \widetilde{G}_n(w, t)$ *из (15). Then*

$$J_\gamma(t) = \sum_{K \in \Re} (-t)^{||K||+n} \sum_J (-1)^{s(J)} \frac{1}{\beta(K, J)!} \cdot \frac{\partial^{||\beta||}}{\partial w^\beta} \left[\widetilde{\Delta} \cdot w_1^{\gamma_1+1} \cdot \dots \cdot w_n^{\gamma_n+1} \cdot \frac{\widetilde{Q}^K}{\widetilde{q}^{K+I}(J)} \right]_{w=a_J},$$

where $Q^K = Q_1^{k_1} \cdot \dots \cdot Q_n^{k_n}$*, and*

$$\Re = \{K = (k_1, \dots, k_n) \colon \text{there exists } i \text{ such that } \|K\| < \gamma_i + 2,\ i = 1, \dots, n\}.$$

All the notations here are as in Theorem 1.

Denote by $z^{(j)}(t) = (z_{j1}(t), \dots, z_{jn}(t))$, $j = 1, \dots, p$ the zeros of the system (2) with the functions tQ_i, where Q_i are defined by (12) and do not lie on coordinate subspaces. Since w_j do not lie on coordinate subspaces, then $z_{jm} = \dfrac{1}{w_{jm}}$, $m = 1, \dots, n$ and therefore we have finite number of zeros. Consequently $p \leqslant s$.

Theorem 2 ([12]). *The following equality holds:*

$$\sum_{j=1}^{p} \frac{1}{z_{j1}(t)^{\gamma_1+1} \cdot z_{j2}(t)^{\gamma_2+1} \cdots z_{jn}(t)^{\gamma_n+1}}$$

$$= \sum_{K \in \Re} (-t)^{||K||+n} \sum_J (-1)^{s(J)} \frac{1}{\beta(K, J)!} \cdot \frac{\partial^{||\beta||}}{\partial w^\beta} \left[\widetilde{\Delta}(t) \cdot w_1^{\gamma_1+1} \cdot \dots \cdot w_n^{\gamma_n+1} \cdot \frac{\widetilde{Q}^K}{\widetilde{q}^{K+I}(J)} \right]_{w=a_J}.$$

Thus, the power sum of (zeros of (15)) is a polynomial on t, and therefore, the equality in Theorem 2 also holds for $t = 1$.

Denote $\sigma_{\gamma+I} = \sum\limits_{j=1}^{p} \dfrac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdots z_{jn}^{\gamma_n+1}}$, where $z^{(j)} = (z_{j1}, \dots, z_{jn}) = (z_{j1}(1), \dots, z_{jn}(1))$, $j = 1, \dots, n$.

Theorem 3 ([12]). *For the system (2) with functions* f_j *defined by (4) and* Q_i *defined by (12) the following formulas are valid:*

$$\sigma_{\gamma+I} = \sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdots z_{jn}^{\gamma_n+1}}$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\|K\| \geqslant 0} (-1)^{\|K\|+n} \sum_J (-1)^{s(J)} \int\limits_{\widetilde{\Gamma}_{h,a_J}} \widetilde{\Delta} \cdot w_1^{\gamma_1+1} \dots w_n^{\gamma_n+1} \cdot \frac{\widetilde{Q}_1^{k_1} \cdot \dots \cdot \widetilde{Q}_n^{k_n}}{\widetilde{q}_1^{k_1+1} \cdot \dots \cdot \widetilde{q}_n^{k_n+1}} dz$$

$$= \sum_{K \in \Re} (-1)^{||K||+n} \sum_J (-1)^{s(J)} \frac{1}{\beta(K, J)!} \cdot \frac{\partial^{||\beta||}}{\partial w^\beta} \left[\widetilde{\Delta} \cdot w_1^{\gamma_1+1} \cdot \dots \cdot w_n^{\gamma_n+1} \cdot \frac{\widetilde{Q}^K}{\widetilde{q}^{K+I}(J)} \right]_{w=a_J},$$

where $z^{(j)} = z^{(j)}(1)$.

3. Examples

Example 1.1. Consider the system of equations in two complex variables

$$\begin{cases} f_1(z_1,z_2) = (1 + a_1z_1 - a_2z_2)e^{(c_1z_1+c_2z_2)} = 0, \\ f_2(z_1,z_2) = (1 - b_1z_1 + b_2z_2)e^{(d_1z_1+d_2z_2)} = 0. \end{cases} \tag{19}$$

Jacobian $\Delta = a_1b_2 - a_2b_1$ different from zero.

The root of system (19) is $z_1 = -\dfrac{a_2+b_2}{\Delta}$, $z_2 = -\dfrac{a_1+b_1}{\Delta}$. Here we suppose, that the root not lie on the coordinate planes. Therefore $a_1 + b_1 \neq 0$, $a_2 + b_2 \neq 0$, then

$$\sigma_{\gamma+I} = \frac{(-1)^{\gamma_1+\gamma_2}\cdot\Delta^{\gamma_1+\gamma_2+2}}{(a_1+b_1)^{\gamma_2+1}(a_2+b_2)^{\gamma_1+1}}.$$

In particular,

$$\sigma_{(2,2)} = \frac{\Delta^4}{(a_1+b_1)^2(a_2+b_2)^2}.$$

We make the change of variables $z_1 = \dfrac{1}{w_1}$ и $z_2 = \dfrac{1}{w_2}$. System will go into

$$\begin{cases} \widetilde{f}_1 = w_1w_2 + a_1w_2 - a_2w_1 = (w_1+a_1)(w_2-a_2) + a_1a_2 = 0, \\ \widetilde{f}_2 = w_1w_2 - b_1w_2 + b_2w_1 = (w_1-b_1)(w_2+b_2) + b_1b_2 = 0, \end{cases} \tag{20}$$

its Jacobian is $\widetilde{\Delta} = (w_2 - a_2)(w_1 - b_1) - (w_1 + a_1)(w_2 + b_2)$.

Now Theorem 3 implies

$$J_\gamma = \sigma_{\gamma+I} =$$

$$= \sum_{K\in\Re}\frac{1}{(2\pi i)^2}\int\limits_{\widetilde{\Gamma}_{q,a_J}}\frac{w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdot(a_1a_2)^{k_1}(b_1b_2)^{k_2}\cdot\widetilde{\Delta}}{(w_1+a_1)^{k_1+1}\cdot(w_2-a_2)^{k_1+1}\cdot(w_1-b_1)^{k_2+1}\cdot(w_2+b_2)^{k_2+1}}dw_1\wedge dw_2, \tag{21}$$

where $\Re = \{\gamma|\quad \exists i : \gamma_i + 2 > k_1 + k_2, \quad i = 1, 2\}$, a $\widetilde{\Gamma}_{q,a_J}$ are cycles of the form $\{|w_1 + a_1| = r_{11}, |w_2 + b_2| = r_{22}\}$, taken with positive orientation and $\{|w_2 - a_2| = r_{12}, |w_1 - b_1| = r_{21}\}$, taken with negative orientation.

Calculate these integrals, in particular, we have

$$J_{(1,1)} = a_1^2b_2^2 + a_2^2b_1^2 - \frac{2a_1a_2^2b_1^2}{a_1+b_1} - \frac{2a_1^2a_2b_2^2}{a_2+b_2} - \frac{2a_1^2b_1b_2^2}{a_1+b_1} - \frac{2a_2^2b_1^2b_2}{a_2+b_2} +$$
$$+\frac{2a_1^2b_1^2b_2^2}{(a_1+b_1)^2} + \frac{2a_2^2b_1^2b_2^2}{(a_2+b_2)^2} + \frac{2a_1^2a_2^2b_1^2}{(a_1+b_1)^2} + \frac{2a_1^2a_2^2b_2^2}{(a_2+b_2)^2} + \frac{2a_1^2a_2b_1^2b_2}{(a_1+b_1)^2} + \frac{2a_1a_2^2b_1b_2^2}{(a_2+b_2)^2}.$$

Therefore

$$\frac{\Delta^4}{(a_1+b_1)^2(a_2+b_2)^2} = a_1^2b_2^2 + a_2^2b_1^2 - \frac{2a_1a_2^2b_1^2}{a_1+b_1} - \frac{2a_1^2a_2b_2^2}{a_2+b_2} - \frac{2a_1^2b_1b_2^2}{a_1+b_1} - \frac{2a_2^2b_1^2b_2}{a_2+b_2} + \tag{22}$$
$$+\frac{2a_1^2b_1^2b_2^2}{(a_1+b_1)^2} + \frac{2a_2^2b_1^2b_2^2}{(a_2+b_2)^2} + \frac{2a_1^2a_2^2b_1^2}{(a_1+b_1)^2} + \frac{2a_1^2a_2^2b_2^2}{(a_2+b_2)^2} + \frac{2a_1^2a_2b_1^2b_2}{(a_1+b_1)^2} + \frac{2a_1a_2^2b_1b_2^2}{(a_2+b_2)^2}.$$

Example 1.2. Recall the expansion of Γ–function an infinite product:

$$\frac{1}{\Gamma(1-z)} = e^{-\gamma z}\prod_{k=1}^{\infty}\left(1 - \frac{z}{k}\right)e^{\frac{z}{k}},$$

where γ is Euler constant.

Consider the system of equations

$$\begin{cases} f_1(z_1,z_2) = \dfrac{e^{\gamma(-a_1z_1+a_2z_2)}}{\Gamma(1-(-a_1z_1+a_2z_2))} = \prod\limits_{k=1}^{\infty}\left(1-\dfrac{-a_1z_1+a_2z_2}{k}\right)e^{\frac{-a_1z_1+a_2z_2}{k}} = 0, \\ f_2(z_1,z_2) = \dfrac{e^{\gamma(b_1z_1-b_2z_2)}}{\Gamma(1-(b_1z_1-b_2z_2))} = \prod\limits_{s=1}^{\infty}\left(1-\dfrac{b_1z_1-b_2z_2}{s}\right)e^{\frac{b_1z_1-b_2z_2}{s}} = 0. \end{cases} \tag{23}$$

Each function is expanded into an infinite product of functions from the system of type (20). The roots of the system (23) are the points

$$\left(\frac{a_2s+b_2k}{a_1b_2-a_2b_1}, \frac{a_1s+b_1k}{a_1b_2-a_2b_1}\right).$$

In our case $a_1b_2 \neq a_2b_1$.

Therefore

$$\sigma(2,2) = J_{(1,1)} = \sum_{k,s=1}^{\infty}\frac{(a_1b_2-a_2b_1)^4}{(a_1s+b_1k)^2(a_2s+b_2k)^2}.$$

This series converges when $\dfrac{k^2}{s^2} \neq \dfrac{a_1}{b_1}$ and $\dfrac{k^2}{s^2} \neq \dfrac{a_2}{b_2}$.

Thus

$$\sigma(2,2) = J_{(1,1)} =$$

$$= \sum_{k,s=1}^{\infty}\frac{a_1^2b_2^2+a_2^2b_1^2}{k^2s^2} - \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2b_2^2}{k^2s(a_2s+b_2k)} - \sum_{k,s=1}^{\infty}\frac{2a_1a_2^2b_1^2}{k^2s(a_1s+b_1k)} - \sum_{k,s=1}^{\infty}\frac{2a_2^2b_1^2b_2}{ks^2(a_2s+b_2k)} -$$

$$- \sum_{k,s=1}^{\infty}\frac{2a_1^2b_1b_2^2}{ks^2(a_1s+b_1k)} + \sum_{k,s=1}^{\infty}\frac{2a_2^2b_1^2b_2^2}{s^2(a_2s+b_2k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1^2b_1^2b_2^2}{s^2(a_1s+b_1k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2^2b_2^2}{k^2(a_2s+b_2k)^2} +$$

$$+ \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2^2b_1^2}{k^2(a_1s+b_1k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1a_2^2b_1b_2^2}{ks(a_2s+b_2k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2b_1^2b_2}{ks(a_1s+b_1k)^2}.$$

Therefore from (22) we have, that

$$\sum_{k,s=1}^{\infty}\frac{(a_1b_2-a_2b_1)^4}{\pi^4(a_1s+b_1k)^2(a_2s+b_2k)^2} = \sum_{k,s=1}^{\infty}\frac{a_1^2b_2^2+a_2^2b_1^2}{k^2s^2} -$$

$$- \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2b_2^2}{k^2s(a_2s+b_2k)} - \sum_{k,s=1}^{\infty}\frac{2a_1a_2^2b_1^2}{k^2s(a_1s+b_1k)} - \sum_{k,s=1}^{\infty}\frac{2a_2^2b_1^2b_2}{ks^2(a_2s+b_2k)} - \sum_{k,s=1}^{\infty}\frac{2a_1^2b_1b_2^2}{ks^2(a_1s+b_1k)} +$$

$$+ \sum_{k,s=1}^{\infty}\frac{2a_2^2b_1^2b_2^2}{s^2(a_2s+b_2k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1^2b_1^2b_2^2}{s^2(a_1s+b_1k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2^2b_2^2}{k^2(a_2s+b_2k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2^2b_1^2}{k^2(a_1s+b_1k)^2} +$$

$$+ \sum_{k,s=1}^{\infty}\frac{2a_1a_2^2b_1b_2^2}{ks(a_2s+b_2k)^2} + \sum_{k,s=1}^{\infty}\frac{2a_1^2a_2b_1^2b_2}{ks(a_1s+b_1k)^2}.$$

Use the identity [14, Ch. 5, Item 5.1. no. 2,12]

$$\sum_{k=0}^{\infty}\frac{1}{(k+a)^n} = \frac{(-1)^n}{(n-1)!}\psi^{(n-1)}(a),$$

$$\sum_{k=1}^{\infty} \frac{1}{k(kn+m)} = \frac{1}{m}\left[\psi\left(\frac{m}{n}+1\right)+C\right],$$

where $\psi(t) = \dfrac{\Gamma'(t)}{\Gamma(t)}$.

We obtain

$$\sum_{k,s=1}^{\infty} \frac{1}{k^2(s+ak)^2} = -\sum_{k=1}^{\infty}\frac{1}{a^2k^4} + \sum_{k=1}^{\infty}\frac{1}{k^2}\psi'(ak),$$

$$\sum_{k,s=1}^{\infty} \frac{1}{s^2k(ak+bs)} = \sum_{s=1}^{\infty}\frac{1}{bs^3}\left[\psi\left(\frac{bs}{a}+1\right)+C\right] = \sum_{s=1}^{\infty}\frac{C}{bs^3} + \sum_{s=1}^{\infty}\frac{1}{bs^3}\left[\psi\left(\frac{bs}{a}\right)+\frac{a}{bs}\right],$$

$$\sum_{k,s=1}^{\infty} \frac{1}{ks(k+as)^2} = \sum_{k,s=1}^{\infty}\frac{1}{as^2k(k+as)} - \sum_{k=1}^{\infty}\frac{1}{as^2(k+as)^2} = \sum_{s=1}^{\infty}\frac{1}{a^2s^3}\left[\psi(as+1)+C\right]+$$

$$+\sum_{s=1}^{\infty}\frac{1}{a^3s^4} - \sum_{s=1}^{\infty}\frac{1}{as^2}\psi'(as).$$

Transform the expression

$$\sum_{k,s=1}^{\infty} \frac{(a_1b_2-a_2b_1)^4}{\pi^4(a_1s+b_1k)^2(a_2s+b_2k)^2} = (a_1^2b_2^2+a_2^2b_1^2)\sum_{k,s=1}^{\infty}\frac{1}{k^2s^2}+$$

$$+\left(\frac{4a_2b_1^3b_2}{a_1}+\frac{4a_1b_1b_2^3}{a_2}-8a_1^2a_2^2-8b_1^2b_2^2\right)\sum_{k=1}^{\infty}\frac{1}{k^4} - (2a_1^2a_2b_2+2a_1a_2^2b_1)\sum_{k=1}^{\infty}\frac{C}{k^3}-$$

$$-2a_1^2a_2b_2\sum_{k=1}^{\infty}\frac{\psi\left(\frac{b_2k}{a_2}\right)}{k^3} - 2a_1a_2^2b_1\sum_{k=1}^{\infty}\frac{\psi\left(\frac{b_1k}{a_1}\right)}{k^3}+$$

$$+(2a_1b_1b_2^2-2a_2b_1^2b_2)\sum_{s=1}^{\infty}\frac{\psi\left(\frac{a_2s}{b_2}\right)}{s^3} + (2a_2b_1^2b_2-2a_1b_1b_2^2)\sum_{s=1}^{\infty}\frac{\psi\left(\frac{a_1s}{b_1}\right)}{s^3}+$$

$$+2a_1^2b_2^2\sum_{k=1}^{\infty}\frac{\psi'\left(\frac{b_2k}{a_2}\right)}{k^2} + 2a_2^2b_1^2\sum_{k=1}^{\infty}\frac{\psi'\left(\frac{b_1k}{a_1}\right)}{k^2}+$$

$$+(2a_2^2b_1^2-2a_1a_2b_1b_2)\sum_{s=1}^{\infty}\frac{\psi'\left(\frac{a_2s}{b_2}\right)}{s^2} + (2a_1^2b_2^2-2a_1a_2b_1b_2)\sum_{s=1}^{\infty}\frac{\psi'\left(\frac{a_1s}{b_1}\right)}{s^2}$$

Consider the expression

$$\sum_{k=1}^{\infty}\frac{\psi(tk)}{k^3}.$$

Differentiate its by t. We have

$$\left(\sum_{k=1}^{\infty}\frac{\psi(tk)}{k^3}\right)_t' = \sum_{k=1}^{\infty}\frac{\psi'(tk)}{k^2}.$$

Therefore, our double series expressed in terms of one-dimensional series of the same type.

Example 2.1. Consider the system of equations in three complex variables

$$\begin{cases} f_1(z_1,z_2,z_3) = 1 - a_1z_1 - a_2z_2 - a_3z_3 + a_1a_2z_1z_2 + a_1a_3z_1z_3 + a_2a_3z_2z_3 = \\ = (1-a_1z_1)(1-a_2z_2)(1-a_3z_3) + a_1a_2a_3z_1z_2z_3 = 0, \\ f_2(z_1,z_2,z_3) = 1 - b_1z_1 - b_2z_2 - b_3z_3 + b_1b_2z_1z_2 + b_1b_3z_1z_3 + b_2b_3z_2z_3 = \\ = (1-b_1z_1)(1-b_2z_2)(1-b_3z_3) + b_1b_2b_3z_1z_2z_3 = 0, \\ f_3(z_1,z_2,z_3) = 1 - c_1z_1 - c_2z_2 - c_3z_3 + c_1c_2z_1z_2 + c_1c_3z_1z_3 + c_2c_3z_2z_3 = \\ = (1-c_1z_1)(1-c_2z_2)(1-c_3z_3) + c_1c_2c_3z_1z_2z_3 = 0. \end{cases} \tag{24}$$

The roots of system (24) are $(z_{j1}, z_{j2}, z_{j3}), j = 1,2,3$.

We make the change of variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$ and $z_3 = \dfrac{1}{w_3}$. Our system transforms into

$$\begin{cases} \widetilde{f}_1 = w_1w_2w_3 - a_1w_2w_3 - a_2w_1w_3 - a_3w_1w_2 + a_1a_2w_3 + a_1a_3w_2 + a_2a_3w_1 = \\ = (w_1-a_1)(w_2-a_2)(w_3-a_3) + a_1a_2a_3 = 0, \\ \widetilde{f}_2 = w_1w_2w_3 - b_1w_2w_3 - b_2w_1w_3 - b_3w_1w_2 + b_1b_2w_3 + b_1b_3w_2 + b_2b_3w_1 = \\ = (w_1-b_1)(w_2-b_2)(w_3-b_3) + b_1b_2b_3 = 0, \\ \widetilde{f}_3 = w_1w_2w_3 - c_1w_2w_3 - c_2w_1w_3 - c_3w_1w_2 + c_1c_2w_3 + c_1c_3w_2 + c_2c_3w_1 = \\ = (w_1-c_1)(w_2-c_2)(w_3-c_3) + c_1c_2c_3 = 0, \end{cases} \tag{25}$$

where $\widetilde{\Delta}$ is Jacobian of system (25)

$$\widetilde{\Delta} = (w_2-a_2)(w_3-a_3)[(w_1-b_1)(w_3-b_3)(w_1-c_1)(w_2-c_2) - (w_1-b_1)(w_2-b_2)(w_1-c_1)(w_3-c_3)] -$$

$$-(w_1-a_1)(w_3-a_3)[(w_2-b_2)(w_3-b_3)(w_1-c_1)(w_2-c_2) - (w_1-b_1)(w_2-b_2)(w_2-c_2)(w_3-c_3)] +$$

$$+(w_1-a_1)(w_2-a_2)[(w_2-b_2)(w_3-b_3)(w_1-c_1)(w_3-c_3) - (w_1-b_1)(w_3-b_3)(w_2-c_2)(w_3-c_3)].$$

Now Theorem 3 implies

$$J_{(0,0,0)} = \sigma_{(1,1,1)} = \sum_{k_1+k_2+k_3<2} \frac{1}{(2\pi i)^2} \int\limits_{\widetilde{\Gamma}_{q,a_J}} \frac{w_1w_2w_3 \cdot (a_1a_2a_3)^{k_1}(b_1b_2b_3)^{k_2}(c_1c_2c_3)^{k_3} \cdot \widetilde{\Delta}}{(w_1-a_1)^{k_1+1}(w_2-a_2)^{k_1+1}(w_3-a_3)^{k_1+1}} \times$$
$$\times \frac{dw_1 \wedge dw_2 \wedge dw_3}{(w_1-b_1)^{k_2+1}(w_2-b_2)^{k_2+1}(w_3-b_3)^{k_2+1} \cdot (w_1-c_1)^{k_3+1}(w_2-c_2)^{k_3+1}(w_3-c_3)^{k_3+1}},$$

where $\widetilde{\Gamma}_{q,a_J}$ are cycles of the form $\{|w_1-a_1| = r_{11}, |w_2-b_2| = r_{22}, |w_3-c_3| = r_{33}\}$; $\{|w_3-a_3| = r_{13}, |w_1-b_1| = r_{21}, |w_2-c_2| = r_{32}\}$; $\{|w_2-a_2| = r_{12}, |w_3-b_3| = r_{23}, |w_1-c_1| = r_{31}\}$ taken with positive orientation and $\{|w_1-a_1| = r_{11}, |w_3-b_3| = r_{23}, |w_2-c_2| = r_{32}\}$; $\{|w_2-a_2| = r_{12}, |w_1-b_1| = r_{21}, |w_3-c_3| = r_{33}\}$; $\{|w_3-a_3| = r_{13}, |w_2-b_2| = r_{22}, |w_1-c_1| = r_{31}\}$ taken with negative orientation.

Calculate these integrals. We obtain

$$\begin{aligned} J_{(0,0,0)} = \sigma_{(1,1,1)} = a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 + a_3b_2c_1 + \\ + \frac{a_3c_1c_2c_3}{a_3-c_3}\cdot\left[\frac{b_1}{b_1-c_1}+\frac{b_2}{b_2-c_2}\right] + \frac{a_1b_1b_2b_3}{a_1-b_1}\cdot\left[\frac{c_3}{c_3-b_3}+\frac{c_2}{c_2-b_2}\right] + \\ + \frac{a_2b_1b_2b_3}{a_2-b_2}\cdot\left[\frac{c_3}{c_3-b_3}+\frac{c_1}{c_1-b_1}\right] + \frac{a_3b_1b_2b_3}{a_3-b_3}\cdot\left[\frac{c_2}{c_2-b_2}+\frac{c_1}{c_1-b_1}\right] + \\ + \frac{a_1c_1}{a_1-c_1}\cdot\left[\frac{b_2c_2c_3}{b_2-c_2}+\frac{b_3c_2c_3}{b_3-c_3}+\frac{a_2a_3b_2}{a_2-b_2}+\frac{a_2a_3b_3}{a_3-b_3}\right] + \\ + \frac{a_2c_2}{a_2-c_2}\cdot\left[\frac{b_1c_1c_3}{b_1-c_1}+\frac{b_3c_1c_3}{b_3-c_3}+\frac{a_1a_3b_3}{a_3-b_3}+\frac{a_1a_3b_1}{a_1-b_1}\right]. \end{aligned} \tag{26}$$

Example 2.2. Consider the system of equations

$$\begin{cases} f_1(z_1,z_2,z_3) = \dfrac{\sin\sqrt{a_1z_1+a_2z_2+a_3z_3-a_1a_2z_1z_2-a_1a_3z_1z_3-a_3a_3z_2z_3}}{\sqrt{a_1z_1+a_2z_2+a_3z_3-a_1a_2z_1z_2-a_1a_3z_1z_3-a_3a_3z_2z_3}} = \\ = \prod\limits_{k=1}^{\infty}\left(1-\dfrac{a_1z_1+a_2z_2+a_3z_3-a_1a_2z_1z_2-a_1a_3z_1z_3-a_3a_3z_2z_3}{k^2\pi^2}\right) = 0, \\ f_2(z_1,z_2,z_3) = \dfrac{\sin\sqrt{b_1z_1+b_2z_2+b_3z_3-b_1b_2z_1z_2-b_1b_3z_1z_3-b_3b_3z_2z_3}}{\sqrt{b_1z_1+b_2z_2+b_3z_3-b_1b_2z_1z_2-b_1b_3z_1z_3-b_3b_3z_2z_3}} = \\ = \prod\limits_{s=1}^{\infty}\left(1-\dfrac{b_1z_1+b_2z_2+b_3z_3-b_1b_2z_1z_2-b_1b_3z_1z_3-b_3b_3z_2z_3}{s^2\pi^2}\right) = 0, \\ f_3(z_1,z_2,z_3) = \dfrac{\sin\sqrt{c_1z_1+c_2z_2+c_3z_3-c_1c_2z_1z_2-c_1c_3z_1z_3-c_3c_3z_2z_3}}{\sqrt{c_1z_1+c_2z_2+c_3z_3-c_1c_2z_1z_2-c_1c_3z_1z_3-c_3c_3z_2z_3}} = \\ = \prod\limits_{m=1}^{\infty}\left(1-\dfrac{c_1z_1+c_2z_2+c_3z_3-c_1c_2z_1z_2-c_1c_3z_1z_3-c_3c_3z_2z_3}{m^2\pi^2}\right) = 0. \end{cases} \tag{27}$$

Each function is expanded into an infinite product of functions from the sytems of the type (25). Transform Formula (26). We obtain

$$\begin{aligned} J_{(0,0,0)} = \sum_{k,s,m=1}^{\infty}\frac{a_1b_2c_3+a_1b_3c_2+a_2b_1c_3+a_2b_3c_1+a_3b_1c_2+a_3b_2c_1}{\pi^6k^2s^2m^2} + \\ + \sum_{k,s,m=1}^{\infty}\frac{a_3c_1c_2c_3}{\pi^6m^2(a_3m^2-c_3k^2)}\cdot\left[\frac{b_1}{b_1m^2-c_1s^2}+\frac{b_2}{b_2m^2-c_2s^2}\right] + \\ + \sum_{k,s,m=1}^{\infty}\frac{a_1b_1b_2b_3}{\pi^6s^2(a_1s^2-b_1k^2)}\cdot\left[\frac{c_3}{c_3s^2-b_3m^2}+\frac{c_2}{c_2s^2-b_2m^2}\right] + \\ + \sum_{k,s,m=1}^{\infty}\frac{a_2b_1b_2b_3}{\pi^6s^2(a_2s^2-b_2k^2)}\cdot\left[\frac{c_3}{c_3s^2-b_3m^2}+\frac{c_1}{c_1s^2-b_1m^2}\right] + \\ + \sum_{k,s,m=1}^{\infty}\frac{a_3b_1b_2b_3}{\pi^6s^2(a_3s^2-b_3k^2)}\cdot\left[\frac{c_2}{c_2s^2-b_2m^2}+\frac{c_1}{c_1s^2-b_1m^2}\right] + \\ + \sum_{k,s,m=1}^{\infty}\frac{a_1c_1}{\pi^6(a_1m^2-c_1k^2)}\cdot\left[\frac{b_2c_2c_3}{m^2(b_2m^2-c_2s^2)}+\frac{b_3c_2c_3}{m^2(b_3m^2-c_3s^2)}\right] + \\ + \sum_{k,s,m=1}^{\infty}\frac{a_1c_1}{\pi^6(a_1m^2-c_1k^2)}\cdot\left[\frac{a_2a_3b_2}{k^2(a_2s^2-b_2k^2)}+\frac{a_2a_3b_3}{k^2(a_3s^2-b_3k^2)}\right] + \end{aligned}$$

$$+\sum_{k,s,m=1}^{\infty}\frac{a_2c_2}{\pi^6(a_2m^2-c_2k^2)}\cdot\left[\frac{b_1c_1c_3}{m^2(b_1m^2-c_1s^2)}+\frac{b_3c_1c_3}{m^2(b_3m^2-c_3s^2)}\right]+$$

$$+\sum_{k,s,m=1}^{\infty}\frac{a_2c_2}{\pi^6(a_2m^2-c_2k^2)}\cdot\left[\frac{a_1a_3b_3}{k^2(a_3s^2-b_3k^2)}+\frac{a_1a_3b_1}{k^2(a_1s^2-b_1k^2)}\right].$$

Second member of identity has the form $\sum\limits_{k,s,m=1}^{\infty}\frac{1}{\pi^6s^2(as^2-bk^2)(cs^2-dm^2)}$.

Use identity [14, Ch. 5, Item 5.1.25, no. 4] (if $a>0$) $\sum\limits_{k=0}^{\infty}\frac{1}{(k^2+a^2)}=\frac{1}{2a^2}+\frac{\pi\operatorname{cth}(\pi a)}{2a}$. Then we have

$$\sum_{k,s,m=1}^{\infty}\frac{1}{\pi^6s^2(as^2-bk^2)(cs^2-dm^2)}=\sum_{s=1}^{\infty}\frac{1}{\pi^6bds^2}\cdot\left[\frac{-1}{2(-a/b)s^2}+\frac{\pi\operatorname{cth}(\pi\sqrt{-a/b}s)}{2\sqrt{-a/b}s}\right]\times$$

$$\times\left[\frac{-1}{2(-c/d)s^2}+\frac{\pi\operatorname{cth}(\pi\sqrt{-c/d}s)}{2\sqrt{-c/d}s}\right]=$$

$$=\sum_{s=1}^{\infty}\frac{1}{4\pi^6acs^6}+\sum_{s=1}^{\infty}\frac{\operatorname{cth}(\pi\sqrt{-c/d}s)}{4\pi^5\sqrt{-cd}as^5}+\sum_{s=1}^{\infty}\frac{\operatorname{cth}(\pi\sqrt{-a/b}s)}{4\pi^5\sqrt{-ab}cs^5}+\sum_{s=1}^{\infty}\frac{\operatorname{cth}(\pi\sqrt{-a/b}s)\cdot\operatorname{cth}(\pi\sqrt{-c/d}s)}{4\pi^4\sqrt{abcd}s^4}.$$

Let ${}_2\Phi_1(e^{2t},e^{2t};e^{4t},x)$ be a basic hypergeometric series (see,for example, [14, p. 793]).

Consider known formula [14, Ch. 5, Item 5.2.18, no. 13]

$$\sum_{s=1}^{\infty}\frac{x^{s-1}}{e^{2ts}-1}=\frac{1}{x}\sum_{s=1}^{\infty}\frac{x^s}{e^{2ts}-1}=\frac{1}{x}\cdot\frac{x}{e^{4t}-1}\cdot{}_2\Phi_1(e^{2t},e^{2t};e^{4t},x)=\frac{1}{e^{4t}-1}{}_2\Phi_1(e^{2t},e^{2t};e^{4t},x).$$

Therefore

$$\sum_{s=1}^{\infty}\frac{\operatorname{cth}(ts)}{s^5}=\sum_{s=1}^{\infty}\frac{1}{s^5}+2\sum_{s=1}^{\infty}\frac{1}{s^5(e^{2ts}-1)}=$$

$$=\sum_{s=1}^{\infty}\frac{1}{s^5}+2\frac{1}{e^{4t}-1}\int_0^1\frac{1}{y}\,dy\int_0^y\frac{1}{x}\,dx\int_0^x\frac{1}{w}\,dw\int\_0^w\frac{1}{v}\,dv\int_0^v{}_2\Phi_1(e^{2t},e^{2t};e^{4t},u)\,du.$$

Integrating by parts, easy to show, that

$$\int_0^1\frac{1}{y}\,dy\int_0^y\frac{1}{x}\,dx\int_0^x\frac{1}{w}\,dw\int_0^w\frac{1}{v}\,dv\int_0^v{}_2\Phi_1(e^{2t},e^{2t};e^{4t},u)\,du=\frac{1}{4}\int_0^1\ln^4y\cdot{}_2\Phi_1(e^{2t},e^{2t};e^{4t},y)\,dy.$$

Therefore

$$\sum_{s=1}^{\infty}\frac{\operatorname{cth}(ts)}{s^5}=\zeta(5)+\frac{1}{2(e^{4t}-1)}\int_0^1\ln^4y\cdot{}_2\Phi_1(e^{2t},e^{2t};e^{4t},y)\,dy.$$

Simplify the expression

$$\sum_{s=1}^{\infty}\frac{\operatorname{cth}(as)\cdot\operatorname{cth}(bs)}{s^4}=\sum_{s=1}^{\infty}\frac{1}{s^4}\cdot\frac{1+e^{-2as}}{1-e^{-2as}}\cdot\frac{1+e^{-2bs}}{1-e^{-2bs}}=$$
$$=\sum_{s=1}^{\infty}\frac{1}{s^4}\cdot\left[1+\frac{2}{e^{2as}-1}+\frac{2}{e^{2bs}-1}+\frac{4}{(e^{2as}-1)(e^{2bs}-1)}\right]. \tag{28}$$

Now consider the expression $\dfrac{4}{(e^{2as}-1)(e^{2bs}-1)}$. Substituting $\dfrac{a}{b}=2$, we have

$$\frac{4}{(e^{2as}-1)(e^{2bs}-1)}=\frac{4}{(e^{2bs}-1)^2(e^{2bs}+1)}=\frac{1}{2(e^{2bs}-1)^2}-\frac{1}{4(e^{2bs}-1)}+\frac{1}{4(e^{2bs}+1)}.$$

We note that

$$\frac{\partial}{\partial b}\left[\frac{1}{e^{2bs}-1}\right]=-\frac{2se^{2bs}}{(e^{2bs}-1)^2}=-\frac{2s}{(e^{2bs}-1)}-\frac{2s}{(e^{2bs}-1)^2}.$$

Thus

$$\frac{1}{(e^{2bs}-1)^2}=-\frac{1}{e^{2bs}-1}-\frac{1}{2s}\cdot\frac{\partial}{\partial b}\left[\frac{1}{e^{2bs}-1}\right].$$

Therefore,

$$\frac{\partial}{\partial b}\left[\sum_{s=1}^{\infty}\frac{1}{2s^5}\cdot\frac{1}{e^{2bs}-1}\right]=\frac{1}{s^4(e^{2bs}-1)^2}+\frac{1}{s^4(e^{2bs}-1)}.$$

Of the formulas obtained above the sum (28) is equal to $\dfrac{1}{8}\int_0^1 \ln^4 y\cdot\dfrac{\partial}{\partial b}\left[{}_2\Phi_1(e^{2b},e^{2b};e^{4b},y)\right]dy$. Therefore

$$\sum_{k,s,m=1}^{\infty}\frac{1}{\pi^6 s^2(as^2-bk^2)(cs^2-dm^2)}=\frac{1}{3780ac}+$$

$$+\frac{1}{4\pi^5\sqrt{-cda}}\cdot\left(\zeta(5)+\frac{1}{2(e^{4\pi\sqrt{-c/d}}-1)}\int_0^1 \ln^4 y\cdot{}_2\Phi_1(e^{2\pi\sqrt{-c/d}},e^{2\pi\sqrt{-c/d}};e^{4\pi\sqrt{-c/d}},y)\,dy\right)+$$

$$+\frac{1}{4\pi^5\sqrt{-abc}}\cdot\left(\zeta(5)+\frac{1}{2(e^{4\pi\sqrt{-a/b}}-1)}\int_0^1 \ln^4 y\cdot{}_2\Phi_1(e^{2\pi\sqrt{-a/b}},e^{2\pi\sqrt{-a/b}};e^{4\pi\sqrt{-a/b}},y)\,dy\right)+$$

$$+\frac{1}{360\sqrt{abcd}}-$$

$$-\frac{1}{4\pi^4\sqrt{abcd}}\cdot\frac{1}{(e^{4\pi\sqrt{-a/b}}-1)}\int_0^1 \ln^3 y\cdot{}_2\Phi_1(e^{2\pi\sqrt{-a/b}},e^{2\pi\sqrt{-a/b}};e^{4\pi\sqrt{-a/b}},y)\,dy$$

$$-\frac{1}{4\pi^4\sqrt{abcd}}\cdot\frac{1}{(e^{4\pi\sqrt{-c/d}}-1)}\int_0^1 \ln^3 y\cdot{}_2\Phi_1(e^{2\pi\sqrt{-c/d}},e^{2\pi\sqrt{-c/d}};e^{4\pi\sqrt{-c/d}},y)\,dy+$$

$$+\frac{3}{32\pi^4\sqrt{abcd}}\cdot\frac{1}{(e^{4\pi\sqrt{-c/d}}-1)}\int_0^1 \ln^3 y\cdot{}_2\Phi_1(e^{2\pi\sqrt{-c/d}},e^{2\pi\sqrt{-c/d}};e^{4\pi\sqrt{-c/d}},y)\,dy-$$

$$-\frac{1}{8\pi^4\sqrt{abcd}}\cdot\int_0^1 \ln^4 y\cdot\frac{\partial}{\partial\pi\sqrt{-c/d}}\left[\frac{1}{e^{4\pi\sqrt{-c/d}}-1}{}_2\Phi_1(e^{2\pi\sqrt{-c/d}},e^{2\pi\sqrt{-c/d}};e^{4\pi\sqrt{-c/d}},y)\right]dy-$$

$$-\frac{1}{32\pi^4\sqrt{abcd}}-\frac{1}{64\pi^4\sqrt{abcd}}\cdot\int_0^1 \ln^3 y\cdot{}_2\Phi_1(e^{2\pi\sqrt{-c/d}},-1;-e^{2\pi\sqrt{-c/d}},y)\,dy,$$

substituting given $\sqrt{\dfrac{ad}{bc}}=2$.

Thus we have, that $\sigma_{(1,1,1)}$ calculated in terms of well-known expression.

This work was supported by the Russian Foundation for Basic Research, 12-01-00007-a.

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Некоторые примеры нахождения сумм кратных рядов

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Рассмотрен метод нахождения вычетных интегралов для определенных систем неалгебраических уравнений. Такие интегралы связаны со степенными суммами корней системы уравнений. Показано, как эти результаты можно применить к нахождению сумм кратных рядов.

Ключевые слова: вычетный интеграл, степенная сумма, кратные ряды.