Some Examples of Finding the Sums of Multiple Series

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A method of finding residue integrals for some systems of non-algebraic equations are presented. Such integrals are connected to the power sums of roots for the system of equations. It is shown how the obtained results can be used for calculating sums of multidimensional series.

Keywords: residue integral, power sum, multiple series.

Introduction

A method based on multidimensional residue theory for the elimination $n$ unknowns from a system of $n$ non-linear algebraic equations (in the characteristic zero setting) was proposed by L.A. Aizenberg [1]. Its further developments were implemented in [2–4]. The algorithmic method (inspired by the Aizenberg and Yuzhakov strategy) introduced by M. Elkadi and A. Yger [5]. The basic idea of the method is to find certain residue integrals connected to the power sums of roots of a given system of equations (in the positive powers) avoiding finding the roots, and to apply then the recurrent Newton formulas. This method is less time-consuming and does not increase the multiplicity of the roots in comparison with the classical method.

The set of roots of a system of $n$ non-algebraic equations in $n$ variables is in general infinite. Moreover, multi Newton sums (with exponents in $\mathbb{N}^n$) of the roots of such systems lead usually to divergent series. In the present work, we attach residue integrals to specific systems of $n$ non-linear equations, compute such residue integrals, and deduce from this computation (provided such series do converge) the values of the sums of multi-Newton series (with exponents in $(-\mathbb{N}^n)^n$) formed with the roots of such non-linear systems which do not belong to the union of coordinate planes.

In the papers [6–10] a class of systems of equations containing entire or meromorphic functions was considered. In [11] a computer algebra algorithm that computes the corresponding residue integrals and applies to them the recurrent Newton formulas is presented.

Our goal is to generalize statements from the papers [6–10] to a another class of systems of non-algebraic equations; to obtain formulas for calculation of residue integrals, to give connection with power sums and to give the corresponding computer algebra algorithm.

In [6, 7], the following system of functions was considered:

$$f_1(z), f_2(z), \ldots, f_n(z),$$

where $z = (z_1, z_2, \ldots, z_n)$. Each $f_j(z)$ is analytic in the neighborhood of $0 \in \mathbb{C}^n$ and has the form

$$f_j(z) = z^{\beta_j} + Q_j(z), \quad j = 1, 2, \ldots, n,$$
where $\beta^j = (\beta_1^j, \beta_2^j, \ldots, \beta_n^j)$ is a multi-index with integer nonnegative coordinates, $z^{\beta^j} = z_1^{\beta_1^j} \cdot z_2^{\beta_2^j} \cdots z_n^{\beta_n^j}$, and $\|\beta^j\| = \beta_1^j + \beta_2^j + \ldots + \beta_n^j = k_j$, $j = 1, 2, \ldots, n$. Functions $Q_j$ are expanded in a neighborhood of zero into an absolutely and uniformly converging Taylor series of the form

$$Q_j(z) = \sum_{\|\alpha\| > k_j} a_\alpha^j z^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \geq 0$, $\alpha_j \in \mathbb{Z}$, and $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$.

The formulas for calculation of residue integrals

$$f_\beta = \frac{1}{(2\pi i)^n} \oint_{C(r)} \frac{1}{z^{\beta+\alpha}} \cdot \frac{df}{f}$$

in terms of coefficients of $Q_j(z)$ were obtained.

Then such systems was considered in [8, 11]. One received multidimensional Newton formulas for such systems.

In the papers [9, 10] was considered the class of systems in which functions

$$f_j(z) = (z^{\beta^j} + Q_j(z))e^{P_j}, \quad j = 1, 2, \ldots, n, \quad (1)$$

and in the paper [10] is given computer realization of considerable method.

Here we consider the system [12] in the case when the monomials $z^{\beta^j}$ in the system (1) are replaced with products of linear functions.

1. Residue integrals

We consider a system of functions $f_1(z)$, $f_2(z)$, \ldots, $f_n(z)$ and a system of equations

$$\begin{cases}
f_1(z) = [(1 - a_{11}z_1)^{m_{11}} \cdot \ldots \cdot (1 - a_{1n}z_n)^{m_{1n}} + Q_1(z)] e^{P_1(z)} = 0, \\
f_2(z) = [(1 - a_{21}z_1)^{m_{21}} \cdot \ldots \cdot (1 - a_{2n}z_n)^{m_{2n}} + Q_2(z)] e^{P_2(z)} = 0, \\
\vdots \\
f_n(z) = [(1 - a_{n1}z_1)^{m_{n1}} \cdot \ldots \cdot (1 - a_{nn}z_n)^{m_{nn}} + Q_n(z)] e^{P_n(z)} = 0,
\end{cases} \quad (2)$$

where $m_{ij}$ are natural numbers, $a_{ij}$ are complex numbers, different for each fixed $j$, $P_i(z)$, and $Q_i(z)$ are entire functions.

Denote by $q_i(z_1, \ldots, z_n)$ expression of the form

$$q_i(z_1, \ldots, z_n) = (1 - a_{1i}z_1)^{m_{1i}} \cdot \ldots \cdot (1 - a_{ni}z_n)^{m_{ni}}, \quad i = 1, \ldots, n. \quad (3)$$

Then, our system can be rewritten as

$$f_i(z_1, \ldots, z_n) = [q_i(z_1, \ldots, z_n) + Q_i(z_1, \ldots, z_n)] e^{P_i(z_1, \ldots, z_n)}, \quad i = 1, 2, \ldots, n. \quad (4)$$

For each $i$ we define a function

$$h_i(z) = \begin{cases}
q_i(z), & \text{если } a_{ij} \neq 0, \quad \text{для всех } j; \\
q_i(z) \cdot \frac{1}{z_{j_1}} \cdot \ldots \cdot \frac{1}{z_{j_k}}, & \text{если } a_{ij_1} = \ldots = a_{ij_k} = 0.
\end{cases} \quad (5)$$

The system of equations

$$h_i(z) = 0, \quad i = 1, 2, \ldots, n \quad (6)$$
has \( n! \) isolated roots in \( \mathbb{C}^n \) (\( \mathbb{C}^n \) is a theory of functions space). Let \( J = (j_1, \ldots, j_n) \) be multi-index is a permutation of \((1, \ldots, n)\). Then the roots of (6) are

\[
a_J = \begin{cases} 
(1/a_{i_1}, \ldots, 1/a_{n_{j_n}}), & \text{if all } a_{k,j_k} \neq 0, \quad k = 1, \ldots, n; \\
(1/a_{i_1}, \ldots, \infty_{[i_1]}, \ldots, \infty_{[i_k]}, \ldots, 1/a_{n_{j_n}}), & \text{if } a_{i_1,j_1} = \ldots = a_{i_k,j_k} = 0,
\end{cases}
\]

where \( k, j = 1, \ldots, n \).

Denote by \( \Gamma \) the cycles

\[
\Gamma_h = \{ z \in \mathbb{C}^n : |h_i| = r_i, \quad r_i > 0, \quad i = 1, \ldots, n \}.
\]

For the case when all \( a_{k,j_k} \neq 0 \) we define a cycle \( \Gamma_{h,a_J} \) by

\[
\begin{cases} 
|1 - a_{1,j_1} z_1| = r_1, \\
|1 - a_{2,j_2} z_2| = r_2, \\
|1 - \ldots| = r_n.
\end{cases}
\]

If \( a_{i_1,j_1} = \ldots = a_{i_k,j_k} = 0 \) for some \( i_1, \ldots, i_k \) then \( \Gamma_{h,a_J} \) is defined by

\[
\begin{cases} 
|1 - a_{1,j_1} z_1| = r_1, \\
\ldots| = r_i, \\
|1| = r_k, \\
\ldots| = r_n,
\end{cases}
\]

**Lemma 1.** For sufficiently small \( r_i \) a global cycle \( \Gamma_h \) has connected components (local cycles) in the neighborhoods of the roots \( a_J \). Moreover, \( \Gamma_h \) is homologous to the sum of the local cycles \( \Gamma_{h,a_J} \).

Consider the system of equations

\[
F_i(z, t) = (q_i(z) + t \cdot Q_i(z)) e^{P_i(z)} = 0 \quad i = 1, 2, \ldots, n,
\]

depending on the real parameter \( t \geq 0 \).

Let \( r_1, \ldots, r_n > 0 \) be the fixed real numbers. Then, for sufficiently small \( t > 0 \), the inequalities

\[
|q_i(z)| > |t \cdot Q_i(z)|, \quad i = 1, \ldots, n.
\]

hold on the cycles

\[
\Gamma_h = \{ z \in \mathbb{C}^n : |h_i| = r_i, \quad i = 1, \ldots, n \}
\]

because the cycles \( \Gamma_h \) are compact.

By \( J_\gamma(t) \) we denote the residue integral

\[
J_\gamma(t) = \frac{1}{(2\pi \sqrt{-1})^n} \int_{\Gamma_h} \frac{1}{z^{\gamma+1}} \frac{dF}{F} = \frac{1}{(2\pi \sqrt{-1})^n} \int_{\Gamma_h} \frac{dF_1}{F_1} \wedge \frac{dF_2}{F_2} \wedge \ldots \wedge \frac{dF_n}{F_n},
\]

\[
= \frac{1}{(2\pi \sqrt{-1})^n} \int_{\Gamma_h} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdot \ldots \cdot z_n^{\gamma_n+1}} \frac{dF_1}{F_1} \wedge \frac{dF_2}{F_2} \wedge \ldots \wedge \frac{dF_n}{F_n},
\]

\[
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\]
where \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is multi-index.

Denote
\[
G_i(z,t) = q_i(z) + t \cdot Q_i(z), \quad i = 1, 2, \ldots, n.
\]

Let \( I \) be a multi-index of the length \( n \), consisting of \( s \) ones and \( n - s \) zeros \( (s = 0, \ldots, n) \). Denote by \( \Delta_J \) Jacobian of the system of functions such that to each “one” on the \( j \)-th place in \( I \) there corresponds \( j \)-th row of the derivatives \( (\partial G_{ij}/\partial z_i) \), \( 1 \leq i \leq n \) in \( \Delta_J \); and, to each “zero” on the \( k \)-th place in \( I \) there corresponds \( k \)-th row of the derivatives \( (\partial P_{ij}/\partial z_i) \), \( 1 \leq i \leq n \) in \( \Delta_J \).

**Theorem 1** ([12]). Under the assumptions made for the functions \( F_i \) defined by (10) the following formulas for \( J_i(t) \) as convergent series are valid:

\[
J_i(t) = \sum_J \sum_I \sum_j (-1)^{||\alpha^s||} (-1)^{s(J)} \frac{1}{\beta(\alpha^s, J)!} \frac{\partial^{||\beta^s||}}{\partial z^\beta} \left[ \frac{\Delta_J(t)}{z_1^{\gamma_1+1} \cdots z_n^{\gamma_n+1}}, \frac{Q^{\alpha^s}(I)}{q^{\alpha^s+1}(I,J)} \right]_{z=a_J},
\]

where \((-1)^{s(J)} = 1 \), if \( J \) is even permutation, and \((-1)^{s(J)} = -1 \), if \( J \) is odd permutation, \( \alpha^s \) is multi-index of order \( s \), \( \beta \) is a multi-index, \( a_J \) is number of \( l \)-the unit of \( I \), \( q^{\alpha^s+1}(I,J) = q_1^{\alpha_1+1}[j_1] \cdots q_n^{\alpha_n+1}[j_n] \), and \( q_i[j_i] \) is product of all \((1-a_p z_i)^{m_{ij}} \cdots (1-a_p z_i)^{m_{ij}} \) besides \((1-a_{pi} z_i)^{m_{pi}}, Q^{\alpha^s}(I) = Q_i^{\alpha_i} \cdots Q_i^{\alpha_i^s} \), \( \beta(\alpha^s, J) = (m_{ij} \cdot (\alpha^s_i + 1) - 1, \ldots, m_{ij} \cdot (\alpha^s_n + 1) - 1), \beta(\alpha^s, J)! = \prod_p (m_{pj} \cdot (\alpha_n^s + 1) - 1)) \),

\[
\frac{\partial^{||\beta^s||}}{\partial z^\beta} = \frac{\partial^{m_{ij} \cdot (\alpha^s_i + 1) - 1 + \cdots + m_{ij} \cdot (\alpha^s_n + 1) - 1}}{\partial z_1^{m_{ij} \cdot (\alpha^s_i + 1) - 1} \cdots \partial z_n^{m_{ij} \cdot (\alpha^s_n + 1) - 1}}.
\]

2. Residue integrals and power sums

Under certain restrictions on \( Q_i \) and \( P_i \), the considered residue integrals are connected to the power sums of roots of the system (2).

Suppose that \( Q_i(z) \) are polynomials:
\[
Q_i(z) = z_1 \cdots z_n \sum_{\alpha \geq 0} C_{\alpha} z^\alpha \quad i = 1, 2, \ldots, n,
\]

where \( \alpha \) is a multi-index, \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \), and \( \operatorname{deg}_{z_j} Q_i \leq m_{ij}, i, j = 1, \ldots, n \) for all non-zero \( a_{ij} \). If \( a_{ij} = 0 \) then there is no restriction on \( \operatorname{deg}_{z_j} Q_i \).

Functions \( P_j \ (j = 1, 2, \ldots, n) \) are the polynomials
\[
P_j(z) = \sum_{0 \leq ||\eta|| \leq P_j} b_{\eta}^j z^\eta,
\]

where \( \eta = (\eta_1, \ldots, \eta_n) \) is a multi-index.

Assuming that all \( w_j \neq 0 \), we substitute \( z_j = \frac{1}{w_j}, j = 1, \ldots, n \) in the functions
\[
F_i(z,t) = (q_i(z) + t \cdot Q_i(z)) e^{P_i(z)}, \quad i = 1, 2, \ldots, n.
\]

Consequently, for \( i = 1, \ldots, n \) we get
\[
F_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}, t\right) = \left( q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) + t \cdot Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) \right) e^{P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right)}.
\]
And finally we arrive at

$$F_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}, t\right) =$$

$$= \left(1 - a_{11} \frac{1}{w_1} \right)^{m_{11}} \cdots \left(1 - a_{1n} \frac{1}{w_n} \right)^{m_{1n}} + t \cdot Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n} \right) e^{P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n} \right)} =$$

$$= \left(\frac{1}{w_1}\right)^{m_{11}} \cdots \left(\frac{1}{w_n}\right)^{m_{1n}} \left(w_1 - a_{11}\right)^{m_{11}} \cdots \left(w_n - a_{1n}\right)^{m_{1n}} +$$

$$+ t \cdot Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n} \right) e^{P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n} \right)} =$$

$$= \left(\frac{1}{w_1}\right)^{m_{11}} \cdots \left(\frac{1}{w_n}\right)^{m_{1n}} \left(\tilde{q}_i(w) + t \cdot \tilde{Q}_i(w)\right) e^{P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n} \right)},$$

where \(\tilde{q}_i \) are the functions

$$\tilde{q}_i = (w_1 - a_{11})^{m_{11}} \cdots (w_n - a_{1n})^{m_{1n}},$$

and \(\tilde{Q}_i \) are the polynomials

$$\tilde{Q}_i = w_1^{m_{11}} \cdots w_n^{m_{1n}} \cdot Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n} \right).$$

In the above calculations it is not important whether \(a_{ij} \) vanish or not. Indeed, assume that in \(F_i(z, t) = \left(q_i(z) + t \cdot Q_i(z)\right) e^{P_i(z)}, \ i = 1, \ldots, n, \) some \(a_{ij} = 0 \) vanishes. If, for instance, \(a_{11} = 0, \) then after substitution \(z_j = \frac{1}{w_j}, \ j = 1, \ldots, n, \) the function \(F_1 \) takes the form

$$F_1\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}, t\right) = q_1\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) + t \cdot Q_1\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) e^{P_1\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right)}, \ i = 1, 2, \ldots, n.$$

Then

$$F_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}, t\right) = \left(1 - a_{12} \frac{1}{w_1}\right)^{m_{12}} \cdots \left(1 - a_{1n} \frac{1}{w_n}\right)^{m_{1n}} +$$

$$+ t \cdot Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) e^{P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right)} =$$

$$= \left(\frac{1}{w_1}\right)^{d_{w_1}} Q_1 \cdots \left(\frac{1}{w_n}\right)^{d_{w_n}} \left(w_1\right)^{d_{w_1}} Q_1 \cdots \left(w_n - a_{1n}\right)^{m_{1n}} +$$

$$+ t \cdot Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) e^{P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right)} =$$

$$= \left(\frac{1}{w_1}\right)^{d_{w_1}} Q_1 \cdots \left(\frac{1}{w_n}\right)^{d_{w_n}} \left(\tilde{q}_1(w) + t \cdot \tilde{Q}_1(w)\right) e^{P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right)},$$

where \(\tilde{q}_1 \) is the function \(\tilde{q}_1 = (w_1)^{d_{w_1}} Q_1 \cdots (w_n - a_{1n})^{m_{1n}}, \) and \(\tilde{Q}_1 \) are polynomials of the form \(\tilde{Q}_1 = w_1^{d_{w_1}} Q_1 \cdots w_n^{m_{1n}} \cdot Q_1\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right). \) That is, one can take \(m_{11} = d_{w_1} Q_1. \)

From (12) we derive that

$$d_{w_j} \tilde{Q}_1 < m_{1j}, \ j = 1, \ldots, n.$$
Denote
\[
\tilde{G}_i(w, t) = \tilde{q}_i(w) + t \cdot \tilde{Q}_i(w), \quad i = 1, 2, \ldots, n. \tag{15}
\]
When \(0 \leq t \leq 1\), the system (15) has a finite number of roots in \(\mathbb{C}^n\), depending on parameter \(t\), and has no infinite roots in \(\mathbb{C}^n\) (see [13]).

Sufficiently close to zero \(t\) on the cycle
\[
\tilde{\Gamma}_h = \{w \in \mathbb{C}^n : h_i \left( \frac{1}{w_1}, \ldots, \frac{1}{w_n} \right) = \varepsilon, \quad i = 1, 2, \ldots, n\},
\]
compactness of the cycle implies
\[
\left| q_i \left( \frac{1}{w_1}, \ldots, \frac{1}{w_n} \right) \right| > \left| t \cdot Q_i \left( \frac{1}{w_1}, \ldots, \frac{1}{w_n} \right) \right|, \quad i = 1, 2, \ldots, n.
\]

Therefore \(\tilde{\Gamma}_h\) is homologous to the sum of the cycles \(\tilde{\Gamma}_{h,a_j}\)
\[
\begin{cases} 1 - a_{1j_1} \frac{1}{w_1} = \varepsilon_1, \\ 1 - a_{2j_2} \frac{1}{w_2} = \varepsilon_2, \\ \ldots, \\ 1 - a_{nj_n} \frac{1}{w_n} = \varepsilon_n. \tag{16} \end{cases}
\]

obtained from the cycles \(\Gamma_{h,a_j}\) by the substitution \(z_j = \frac{1}{w_j}\).

The equation
\[
1 - a_{jj_j} \frac{1}{w_j} = \varepsilon
\]
defines a circle. Indeed, let us first rewrite it in the form
\[
\left| 1 - a_{jj_j} \frac{1}{w_j} \right| = \varepsilon, \quad \text{then} \quad |w_j - a_{jj_j}| = \varepsilon |w_j|.
\]

Thus
\[
|w_j - a_{jj_j}|^2 = \varepsilon^2 |w_j|^2, \quad \text{then} \quad (1 - \varepsilon^2) \left( w_j - \frac{a_{jj_j}}{1 - \varepsilon^2} \right)^2 = \varepsilon^2 \left( \frac{|a_{jj_j}|^2}{1 - \varepsilon^2} \right)^2,
\]

\[
\left| w_j - \frac{a_{jj_j}}{1 - \varepsilon^2} \right|^2 = \varepsilon^2 \left( \frac{|a_{jj_j}|^2}{1 - \varepsilon^2} \right)^2, \quad j = 1, \ldots, n.
\]

For sufficiently small \(\varepsilon\) the point \(a_{jj_j}\) lies inside this circle, and therefore \(\tilde{\Gamma}_{h,a_j}\) is homologous to the cycle
\[
\begin{cases} |w_1 - a_{1j_1}| = \varepsilon_1, \\ |w_2 - a_{2j_2}| = \varepsilon_2, \\ \ldots, \\ |w_n - a_{nj_n}| = \varepsilon_n. \end{cases}
\]

Here some \(a_{jj_j}\) can vanish.

**Lemma 2** ([12]). Let \(P_j\) be defined by (13), and the inequality
\[
l^1 + \ldots + l^n \leq \gamma \tag{17}
\]
holds for a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$, where $V = (l_1, \ldots, l_k)$ and $l_i$ is a degree of $P_i$ in $z_j$ for $i, j = 1, \ldots, n$ (i.e. $n$ scalar inequalities $l_1^j + \ldots + l_n^j \leq \gamma_i$ hold).

Then

$$J_\gamma(t) = \frac{(-1)^n}{(2\pi \sqrt{-1})^n} \int_{\Gamma_h} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \frac{d\tilde{G}_1}{G_1} \wedge \frac{d\tilde{G}_2}{G_2} \wedge \ldots \wedge \frac{d\tilde{G}_n}{G_n}. \quad (18)$$

(Inequality (17) means that it holds coordinatewise).

**Lemma 3** ([12]). Let $\tilde{\Delta} = \tilde{\Delta}(w, t)$ be the Jacobian of the system $\tilde{G}_1(w, t), \ldots, \tilde{G}_n(w, t)$ w.r.t. (15).

Then

$$J_\gamma(t) = \sum_{K \in \mathbb{R}} (-t)^{||K||+n} \sum_J (-1)^s(J) \frac{1}{\beta(K, J)!} \left. \frac{\partial ||\beta||}{\partial w_j} \left[ \tilde{\Delta} \cdot w_1^{\gamma_1+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \tilde{Q}^K q^{K+I(J)} \right] \right|_{w=a_j},$$

where $Q^K = Q_1^{k_1} \cdot \ldots \cdot Q_n^{k_n}$, and

$$\mathbb{R} = \{ K = (k_1, \ldots, k_n) : \text{there exists } i \text{ such that } ||K|| < \gamma_i + 2, \ i = 1, \ldots, n \}. $$

All the notations here are as in Theorem 1.

Denote by $z^{(j)}(t) = (z_{j1}(t), \ldots, z_{jn}(t))$, $j = 1, \ldots, p$ the zeros of the system (2) with the functions $tQ_i$, where $Q_i$ are defined by (12) and do not lie on coordinate subspaces. Since $w_j$ do not lie on coordinate subspaces, then $z_{jm} = \frac{1}{w_{jm}}$, $m = 1, \ldots, n$ and therefore we have finite number of zeros. Consequently $p \leq s$.

**Theorem 2** ([12]). The following equality holds:

$$\sum_{j=1}^P \frac{1}{z_{j1}(t)^{\gamma_1+1} \cdot z_{j2}(t)^{\gamma_2+1} \cdot \ldots \cdot z_{jn}(t)^{\gamma_n+1}} = \sum_{K \in \mathbb{R}} (-t)^{||K||+n} \sum_J (-1)^s(J) \frac{1}{\beta(K, J)!} \left. \frac{\partial ||\beta||}{\partial w_j} \left[ \tilde{\Delta}(t) \cdot w_1^{\gamma_1+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \tilde{Q}^K q^{K+I(J)} \right] \right|_{w=a_j}. $$

Thus, the power sum of the zeros of (15) is a polynomial on $t$, and therefore, the equality in Theorem 2 also holds for $t = 1$.

Denote $\sigma_{\gamma+1} = \sum_{j=1}^P \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdot \ldots \cdot z_{jn}^{\gamma_n+1}}$, where $z^{(j)} = (z_{j1}, \ldots, z_{jn}) = (z_{j1}(1), \ldots, z_{jn}(1))$, $j = 1, \ldots, n$.

**Theorem 3** ([12]). For the system (2) with functions $f_j$ defined by (4) and $Q_i$ defined by (12) the following formulas are valid:

$$\sigma_{\gamma+1} = \sum_{j=1}^P \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdot \ldots \cdot z_{jn}^{\gamma_n+1}} = \frac{1}{(2\pi \sqrt{-1})^n} \sum_{||K|| \geq 0} (-1)^{||K||+n} \sum_J (-1)^s(J) \int_{\Gamma_h, a_j} \tilde{\Delta} \cdot w_1^{\gamma_1+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \tilde{Q}^K q^{K+I(J)} dz,$$

where $z^{(j)} = z^{(j)}(1)$.
3. Examples

Example 1.1. Consider the system of equations in two complex variables
\[
\begin{align*}
    f_1(z_1, z_2) &= (1 + a_1 z_1 - a_2 z_2) e^{(c_1 z_1 + c_2 z_2)} = 0, \\
    f_2(z_1, z_2) &= (1 - b_1 z_1 + b_2 z_2) e^{(d_1 z_1 + d_2 z_2)} = 0.
\end{align*}
\] (19)

Jacobian \( \Delta = a_1 b_2 - a_2 b_1 \) different from zero.

The root of system (19) is \( z_1 = -\frac{a_2 + b_2}{\Delta}, \quad z_2 = -\frac{a_1 + b_1}{\Delta}. \) Here we suppose, that the root not lie on the coordinate planes. Therefore \( a_1 + b_1 \neq 0, \ a_2 + b_2 \neq 0, \) then
\[
\sigma_{\gamma+1} = \frac{(-1)^{\gamma_1+\gamma_2} \cdot \Delta^{\gamma_1+\gamma_2+2}}{(a_1 + b_1)^{\gamma_2+1}(a_2 + b_2)^{\gamma_1+1}}.
\]

In particular,
\[
\sigma_{(2,2)} = \frac{\Delta^4}{(a_1 + b_1)^2(a_2 + b_2)^2}.
\]

We make the change of variables \( z_1 = \frac{1}{w_1} \) and \( z_2 = \frac{1}{w_2}. \) System will go into
\[
\begin{align*}
    \tilde{f}_1 &= w_1 w_2 + a_1 w_1 - a_2 w_2 = (w_1 + a_1)(w_2 - a_2) + a_1 a_2 = 0, \\
    \tilde{f}_2 &= w_1 w_2 - b_1 w_1 + b_2 w_2 = (w_1 - b_1)(w_2 + b_2) - b_1 b_2 = 0,
\end{align*}
\] (20)

its Jacobian in \( \tilde{\Delta} = (\tilde{w}_2 - \tilde{a}_2)(\tilde{w}_1 - \tilde{b}_1) - (\tilde{w}_1 + \tilde{a}_1)(\tilde{w}_2 + \tilde{b}_2). \)

Now Theorem 3 implies
\[
J_{\gamma} = \sigma_{\gamma+1} = \sum_{\kappa \in \mathbb{R}} \frac{1}{(2\pi i)^2} \int_{\tilde{\Gamma}_{\kappa\sigma,j}} \frac{w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot (a_1 a_2)^{k_1} (b_1 b_2)^{k_2} \cdot \tilde{\Delta}}{(w_1 + a_1)^{k_1+1} \cdot (w_2 - a_2)^{k_1+1} \cdot (w_1 - b_1)^{k_2+1} \cdot (w_2 + b_2)^{k_2+1}} \, dw_1 \wedge dw_2,
\] (21)

where \( R = \{ \gamma | \exists \iota : \gamma_i + 2 > k_1 + k_2, \quad i = 1, 2 \}, \) a \( \tilde{\Gamma}_{\kappa\sigma,j} \) are cycles of the form \( \{ |w_1 + a_1| = r_{11}, |w_2 + b_2| = r_{22} \}, \) taken with positive orientation and \( \{ |w_2 - a_2| = r_{12}, |w_1 - b_1| = r_{21} \}, \) taken with negative orientation.

Calculate these integrals, in particular, we have
\[
J_{(1,1)} = a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_1 a_2 b_1^2 \frac{a_1 + b_1}{a_1 + b_2} - 2a_2 a_1 b_2^2 \frac{a_1 + b_2}{a_2 + b_1} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2}.
\]

Therefore
\[
\frac{\Delta^4}{(a_1 + b_1)^2(a_2 + b_2)^2} = a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_1 a_2 b_1^2 \frac{a_1 + b_1}{a_1 + b_2} - 2a_2 a_1 b_2^2 \frac{a_1 + b_2}{a_2 + b_1} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2} + \frac{2a_1 a_2 b_1 b_2}{(a_1 + b_1)^2 + (a_2 + b_2)^2}.
\] (22)

Example 1.2. Recall the expansion of \( \Gamma \)–function an infinite product:
\[
\frac{1}{\Gamma(1 - z)} = e^{-\gamma z} \prod_{k=1}^{\infty} \left( 1 - \frac{z}{k} \right) e^{\frac{z}{k}},
\]

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where $\gamma$ is Euler constant.

Consider the system of equations

\[
\begin{cases}
f_1(z_1, z_2) = \frac{e^{\gamma(-a_1 z_1 + a_2 z_2)}}{\Gamma(1 - (-a_1 z_1 + a_2 z_2))} = \prod_{k=1}^{\infty} \left( 1 - \frac{-a_1 z_1 + a_2 z_2}{k} \right) e^{-\frac{-a_1 z_1 + a_2 z_2}{k}} = 0, \\
f_2(z_1, z_2) = \frac{e^{\gamma(b_1 z_1 - b_2 z_2)}}{\Gamma(1 - (b_1 z_1 - b_2 z_2))} = \prod_{s=1}^{\infty} \left( 1 - \frac{b_1 z_1 - b_2 z_2}{s} \right) e^{\frac{b_1 z_1 - b_2 z_2}{s}} = 0.
\end{cases}
\tag{23}
\]

Each function is expanded into an infinite product of functions from the system of type (20). The roots of the system (23) are the points

\[
\left( \frac{a_2 s + b_2 k}{a_1 b_2 - a_2 b_1}, \frac{a_1 s + b_1 k}{a_1 b_2 - a_2 b_1} \right).
\]

In our case $a_1 b_2 \neq a_2 b_1$.

Therefore

\[
\sigma(2, 2) = J_{(1, 1)} = \sum_{k,s=1}^{\infty} \frac{(a_1 b_2 - a_2 b_1)^4}{(a_1 s + b_1 k)^2(a_2 s + b_2 k)^2}.
\]

This series converges when $\frac{k^2}{s^2} \neq \frac{a_1}{b_1}$ and $\frac{k^2}{s^2} \neq \frac{a_2}{b_2}$.

Thus

\[
\sigma(2, 2) = J_{(1, 1)} = \sum_{k,s=1}^{\infty} \frac{a_1^2 b_1^2 + a_2^2 b_2^2}{k^2 s^2} - \sum_{k,s=1}^{\infty} \frac{2a_1^2 a_2 b_1^2}{k^2 s(a_1 s + b_1 k)} - \sum_{k,s=1}^{\infty} \frac{2a_1 a_2^2 b_2^2}{k^2 s(a_1 s + b_1 k)} - \sum_{k,s=1}^{\infty} \frac{2a_1^2 a_2 b_2^2}{k^2 s(a_1 s + b_1 k)}
\]

\[
- \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} + \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} + \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} + \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} + \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} + \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2}.
\]

Therefore from (22) we have, that

\[
\sum_{k,s=1}^{\infty} \frac{a_1^2 b_2^2 + a_2^2 b_1^2}{k^2 s^2} = \sum_{k,s=1}^{\infty} \frac{(a_1 b_2 - a_2 b_1)^4}{(a_1 s + b_1 k)^2(a_2 s + b_2 k)^2} - \sum_{k,s=1}^{\infty} \frac{2a_1^2 a_2^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} - \sum_{k,s=1}^{\infty} \frac{2a_1 a_2^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} - \sum_{k,s=1}^{\infty} \frac{2a_1^2 a_2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} - \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2} - \sum_{k,s=1}^{\infty} \frac{2a_1^2 b_1 b_2^2}{k^2 s(a_1 s + b_1 k)^2}.
\]

Use the identity [14, Ch. 5, Item 5.1. no. 2,12]

\[
\sum_{k=0}^{\infty} \frac{1}{(k + a)^n} = \frac{(-1)^n}{(n - 1)!} \psi^{(n-1)}(a),
\]

- 523 -
\[ \sum_{k=1}^{\infty} \frac{1}{k(kn+m)} = \frac{1}{m} \left[ \psi \left( \frac{m}{n} + 1 \right) + C \right], \]

where \( \psi(t) = \frac{\Gamma'(t)}{\Gamma(t)} \).

We obtain
\[ \sum_{k,s=1}^{\infty} \frac{1}{k^2(s+ak)^2} = - \sum_{k=1}^{\infty} \frac{1}{ak^4} + \sum_{k=1}^{\infty} \frac{1}{k^2} \psi'(ak), \]

\[ \sum_{k,s=1}^{\infty} \frac{1}{s^2k(ak+bs)} = \sum_{s=1}^{\infty} \frac{1}{bs^3} \left[ \psi \left( \frac{bs}{a} + 1 \right) + C \right] = \sum_{s=1}^{\infty} C = \sum_{s=1}^{\infty} \frac{1}{bs^3} \left[ \psi \left( \frac{bs}{a} \right) + \frac{a}{bs} \right], \]

\[ \sum_{k,s=1}^{\infty} \frac{1}{k^2(k+as)^2} = \sum_{k,s=1}^{\infty} \frac{1}{a^2s^3} \left[ \psi(\frac{k}{as} + 1) + C \right] + \sum_{s=1}^{\infty} \frac{1}{a^2s^3} \psi'(as). \]

Transform the expression
\[ \sum_{k,s=1}^{\infty} \frac{(a_1b_2 - a_2b_1)^4}{\pi^4(a_1s + b_1k)^2(a_2s + b_2k)^2} = (a_1^2b_2^3 + a_2^2b_1^3) \sum_{k,s=1}^{\infty} \frac{1}{k^2s^2} + \]
\[ + \left( \frac{4a_2b_1b_2}{a_1} + \frac{4a_1b_1b_2}{a_2} - 8a_1^2a_2 - 8b_1^2b_2 \right) \sum_{k=1}^{\infty} \frac{1}{k^4} - (2a_1^2a_2b_2 + 2a_1a_2^2b_1) \sum_{k=1}^{\infty} \frac{C}{k^4} - \]
\[ -2a_1^2a_2b_2 \sum_{k=1}^{\infty} \frac{\psi \left( \frac{b_1k}{a_2} \right)}{k^3} - 2a_1a_2^2b_1 \sum_{k=1}^{\infty} \frac{\psi \left( \frac{b_2k}{a_1} \right)}{k^3} + \]
\[ + (2a_1b_1b_2 - 2a_2b_1^2b_2) \sum_{s=1}^{\infty} \frac{\psi \left( \frac{a_2s}{b_2} \right)}{s^3} + (2a_2b_1^2b_2 - 2a_1b_1b_2^2) \sum_{s=1}^{\infty} \frac{\psi \left( \frac{a_1s}{b_1} \right)}{s^3} + \]
\[ + 2a_1^2b_2^2 \sum_{k=1}^{\infty} \frac{\psi \left( \frac{b_2k}{a_1} \right)}{k^2} + 2a_2^2b_1^2 \sum_{k=1}^{\infty} \frac{\psi \left( \frac{b_1k}{a_2} \right)}{k^2} + \]
\[ + (2a_2b_1^2 - 2a_1a_2b_1b_2) \sum_{s=1}^{\infty} \frac{\psi \left( \frac{a_2s}{b_2} \right)}{s^2} + (2a_1b_2^2 - 2a_1a_2b_1b_2) \sum_{s=1}^{\infty} \frac{\psi \left( \frac{a_1s}{b_1} \right)}{s^2} \]

Consider the expression
\[ \sum_{k=1}^{\infty} \frac{\psi(tk)}{k^3}. \]

Differentiate its by \( t \). We have
\[ \left( \sum_{k=1}^{\infty} \frac{\psi(tk)}{k^3} \right)' = \sum_{k=1}^{\infty} \frac{\psi'(tk)}{k^2}. \]

Therefore, our double series expressed in terms of one-dimensional series of the same type.
Example 2.1. Consider the system of equations in three complex variables

\[
\begin{align*}
f_1(z_1, z_2, z_3) &= 1 - a_1 z_1 - a_2 z_2 - a_3 z_3 + a_1 a_2 z_1 z_2 + a_1 a_3 z_1 z_3 + a_2 a_3 z_2 z_3 = \\
&= (1 - a_1 z_1)(1 - a_2 z_2)(1 - a_3 z_3) + a_1 a_2 a_3 z_1 z_2 z_3 = 0, \\
f_2(z_1, z_2, z_3) &= 1 - b_1 z_1 - b_2 z_2 - b_3 z_3 + b_1 b_2 z_1 z_2 + b_1 b_3 z_1 z_3 + b_2 b_3 z_2 z_3 = \\
&= (1 - b_1 z_1)(1 - b_2 z_2)(1 - b_3 z_3) + b_1 b_2 b_3 z_1 z_2 z_3 = 0, \\
f_3(z_1, z_2, z_3) &= 1 - c_1 z_1 - c_2 z_2 - c_3 z_3 + c_1 c_2 z_1 z_2 + c_1 c_3 z_1 z_3 + c_2 c_3 z_2 z_3 = \\
&= (1 - c_1 z_1)(1 - c_2 z_2)(1 - c_3 z_3) + c_1 c_2 c_3 z_1 z_2 z_3 = 0.
\end{align*}
\]

(24)

The roots of system (24) are \((z_{j1}, z_{j2}, z_{j3}), j = 1, 2, 3\).

We make the change of variables \(z_1 = \frac{1}{w_1}, z_2 = \frac{1}{w_2}\) and \(z_3 = \frac{1}{w_3}\). Our system transforms into

\[
\begin{align*}
\tilde{f}_1 &= w_1 w_2 w_3 - a_1 w_2 w_3 - a_2 w_1 w_3 - a_3 w_1 w_2 + a_1 a_2 w_3 + a_1 a_3 w_2 + a_2 a_3 w_1 = \\
&= (w_1 - a_1)(w_2 - a_2)(w_3 - a_3) + a_1 a_2 a_3 = 0, \\
\tilde{f}_2 &= w_1 w_2 w_3 - b_1 w_2 w_3 - b_2 w_1 w_3 - b_3 w_1 w_2 + b_1 b_2 w_3 + b_1 b_3 w_2 + b_2 b_3 w_1 = \\
&= (w_1 - b_1)(w_2 - b_2)(w_3 - b_3) + b_1 b_2 b_3 = 0, \\
\tilde{f}_3 &= w_1 w_2 w_3 - c_1 w_2 w_3 - c_2 w_1 w_3 - c_3 w_1 w_2 + c_1 c_2 w_3 + c_1 c_3 w_2 + c_2 c_3 w_1 = \\
&= (w_1 - c_1)(w_2 - c_2)(w_3 - c_3) + c_1 c_2 c_3 = 0,
\end{align*}
\]

(25)

where \(\tilde{\Delta}\) is Jacobian of system (25)

\[
\tilde{\Delta} = (w_2 - a_2)(w_3 - a_3)(w_1 - b_1)(w_3 - b_3)(w_1 - c_1)(w_2 - c_2) - (w_1 - b_1)(w_2 - b_2)(w_1 - c_1)(w_3 - c_3) -
\]

\[(w_1 - a_1)(w_3 - a_3)(w_2 - b_2)(w_3 - b_3)(w_1 - c_1)(w_2 - c_2) - (w_1 - b_1)(w_2 - b_2)(w_2 - c_2)(w_3 - c_3) +
\]

\[(w_1 - a_1)(w_2 - a_2)(w_3 - b_3)(w_1 - c_1)(w_3 - c_3) - (w_1 - b_1)(w_3 - b_3)(w_2 - c_2)(w_3 - c_3).
\]

Now Theorem 3 implies

\[
J_{(0,0,0)} = \sigma_{(1,1,1)} = \sum_{k_1+k_2+k_3<2} \frac{1}{(2\pi i)^2} \int_{\tilde{r}_{q,a,j}} \frac{w_1 w_2 w_3 \cdot (a_1 a_2 a_3)^k_1 (b_1 b_2 b_3)^k_2 (c_1 c_2 c_3)^k_3 \cdot \tilde{\Delta}}{(w_1 - a_1)^{k_1+1}(w_2 - a_2)^{k_2+1}(w_3 - a_3)^{k_3+1}} \times
\]

\[
\frac{dw_1 \wedge dw_2 \wedge dw_3}{(w_1 - b_1)^{k_2+1}(w_2 - b_2)^{k_2+1}(w_3 - b_3)^{k_3+1} \cdot (w_1 - c_1)^{k_1+1}(w_2 - c_2)^{k_2+1}(w_3 - c_3)^{k_3+1}},
\]

where \(\tilde{r}_{q,a,j}\) are cycles of the form \{\(|w_1 - a_1| = r_{11}, |w_2 - b_2| = r_{22}, |w_3 - c_3| = r_{33}\); \{|w_3 - a_3| = r_{13}, |w_3 - b_3| = r_{23}, |w_2 - c_2| = r_{32}\}; \{|w_2 - a_2| = r_{12}, |w_2 - b_2| = r_{22}, |w_1 - c_1| = r_{31}\}\) taken with positive orientation and \{\(|w_1 - a_1| = r_{11}, |w_3 - b_3| = r_{33}, |w_2 - c_2| = r_{23}\); \{|w_3 - a_3| = r_{13}, |w_1 - c_1| = r_{31}, |w_2 - b_2| = r_{22}\}; \{|w_2 - a_2| = r_{12}, |w_1 - b_1| = r_{21}, |w_3 - c_3| = r_{33}\}\) taken with negative orientation.
Calculate these integrals. We obtain

\[
J_{(0,0)} = c_{(1,1,1)} = a_1 b_2 c_3 + a_1 b_3 c_2 + a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 + a_3 b_2 c_1 +
\]
\[
+ a_3 c_1 c_2 c_3 \cdot \left[ \frac{b_1}{b_1 - c_1} + \frac{b_2}{b_2 - c_1} \right] + a_3 b_1 b_2 b_3 \cdot \left[ \frac{c_3}{c_3 - b_3} + \frac{c_2}{c_2 - b_2} \right]
\]
\[
+ a_2 b_1 b_2 b_3 \cdot \left[ \frac{c_3}{c_3 - b_3} + \frac{c_1}{c_1 - b_1} \right] + \frac{a_1 c_1}{a_1 - c_1} \cdot \left[ \frac{b_2 c_2 c_3}{b_2 - c_2} + \frac{b_3 c_2 c_3}{b_3 - c_2} + \frac{a_2 a_3 b_2}{a_2 - b_2} + \frac{a_2 a_3 b_3}{a_3 - b_3} \right]
\]
\[
+ a_2 c_2 \cdot \left[ \frac{b_1 c_1 c_3}{b_1 - c_1} + \frac{b_3 c_1 c_3}{b_3 - c_1} + \frac{a_1 a_3 b_1}{a_1 - b_1} \right].
\]

(26)

Example 2.2. Consider the system of equations

\[
\begin{align*}
J_1(z_1, z_2, z_3) &= \frac{\sin \sqrt{a_1 z_1 + a_2 z_2 + a_3 z_3 - a_1 a_2 z_1 z_2 - a_1 a_3 z_1 z_3 - a_3 a_2 z_2 z_3}}{k^2 \pi^2} = 0, \\
J_2(z_1, z_2, z_3) &= \frac{\sin \sqrt{\frac{1}{b_1 z_1 + b_2 z_2 + b_3 z_3 - b_1 b_2 z_1 z_2 - b_1 b_3 z_1 z_3 - b_2 b_3 z_2 z_3}}}{b_1^2 \pi^2} = 0, \\
J_3(z_1, z_2, z_3) &= \frac{\sin \sqrt{c_1 z_1 + c_2 z_2 + c_3 z_3 - c_1 c_2 z_1 z_2 - c_1 c_3 z_1 z_3 - c_2 c_3 z_2 z_3}}{c_1^2 \pi^2} = 0.
\end{align*}
\]

(27)

Each function is expanded into an infinite product of functions from the systems of the type (25). Transform Formula (26). We obtain

\[
J_{(0,0)} = \sum_{k,s,m=1}^{\infty} \frac{a_1 b_2 c_3 + a_1 b_3 c_2 + a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 + a_3 b_2 c_1 + a_3 c_1 c_2 c_3 \cdot \left[ \frac{b_1}{b_1 m^2 - c_1 s^2} + \frac{b_2}{b_2 m^2 - c_2 s^2} \right] + a_1 b_1 b_2 b_3 \cdot \left[ \frac{c_3}{c_3 m^2 - b_3 s^2} + \frac{c_2}{c_2 s^2 - b_2 m^2} \right] + a_2 b_1 b_2 b_3 \cdot \left[ \frac{c_3}{c_3 s^2 - b_3 m^2} + \frac{c_1}{c_1 m^2 - b_1 s^2} \right] + a_3 b_1 b_2 b_3 \cdot \left[ \frac{c_2}{c_2 s^2 - b_2 m^2} + \frac{c_1}{c_1 m^2 - b_1 s^2} \right] + a_3 c_1 \cdot \left[ \frac{b_2 c_2 c_3}{m^2 (b_2 m^2 - c_2 s^2)} + \frac{b_3 c_2 c_3}{m^2 (b_3 m^2 - c_3 s^2)} \right] + a_2 a_3 b_1 \cdot \left[ \frac{a_2 a_3 b_2}{k^2 (a_2 s^2 - b_2 k^2)} + \frac{a_2 a_3 b_3}{k^2 (a_3 s^2 - b_3 k^2)} \right]}{\pi^6 k^2 s^2 m^2}.
\]
Integrating by parts, easy to show, that
\[ \int \sum_{s=1}^{\infty} \frac{a_{2}c_{2}}{\pi^{2}(a_{2}m^{2} - c_{2}k^{2})} \cdot \left[ \frac{b_{1}c_{1}c_{3}}{m^{2}(b_{1}m^{2} - c_{1}s^{2})} + \frac{b_{3}c_{1}c_{3}}{m^{2}(b_{3}m^{2} - c_{3}s^{2})} \right] + \]
\[ \sum_{k,s,m=1}^{\infty} \frac{a_{2}c_{2}}{\pi^{2}(a_{2}m^{2} - c_{2}k^{2})} \cdot \left[ \frac{a_{1}a_{2}b_{3}}{k^{2}(a_{2}s^{2} - b_{3}k^{2})} + \frac{a_{1}a_{3}b_{1}}{k^{2}(a_{1}s^{2} - b_{1}k^{2})} \right]. \]

Second member of identity has the form
\[ \sum_{k,s,m=1}^{\infty} \frac{1}{\pi^{2}s^{2}(as^{2} - bk^{2})(cs^{2} - dm^{2})}. \]

Use identity [14, Ch. 5, Item 5.125, no. 4] (if \( a > 0 \))
\[ \sum_{k=0}^{\infty} \frac{1}{(k^{2} + a^{2})} = \frac{1}{2a^{2}} + \frac{\pi \csc(\pi a)}{2a}. \]
Then we have
\[ \sum_{k,s,m=1}^{\infty} \frac{1}{\pi^{2}s^{2}(as^{2} - bk^{2})(cs^{2} - dm^{2})} = \sum_{s=1}^{\infty} \frac{1}{\pi^{2}bsd^{2}} \cdot \left[ \frac{-1}{2(-a/b)s^{2}} + \frac{\pi \csc(\pi a/b)}{2\sqrt{-a/b}} \right] \times \]
\[ \times \left[ \frac{-1}{2(-c/d)s^{2}} + \frac{\pi \csc(\pi a/b)}{2\sqrt{-c/d}} \right] = \]
\[ \sum_{s=1}^{\infty} \frac{1}{4\pi^{2}acs^{6}} + \sum_{s=1}^{\infty} \frac{\csc(\pi \sqrt{-c/d})}{4\pi^{2}-cds^{5}} + \sum_{s=1}^{\infty} \frac{\csc(\pi \sqrt{-a/b})}{4\pi^{2}-abcds^{5}} + \sum_{s=1}^{\infty} \frac{\csc(\pi \sqrt{-a/b}) \cdot \csc(\pi \sqrt{-c/d})}{4\pi^{4}-abcds^{4}}. \]

Let \( \Phi_{1}(e^{2t}, e^{2t}; e^{4t}, x) \) be a basic hypergeometric series (see, for example, [14, p. 793]).
Consider known formula [14, Ch. 5, Item 5.2.18, no. 13]
\[ \sum_{s=1}^{x+1} \frac{x^{s}}{e^{2ts} - 1} = \frac{x}{x} \sum_{s=1}^{x} \frac{x^{s}}{e^{2ts} - 1} = \frac{x}{e^{4t} - 1} \cdot \Phi_{1}(e^{2t}, e^{2t}; e^{4t}, x) = \frac{1}{e^{4t} - 1} \cdot \Phi_{1}(e^{2t}, e^{2t}; e^{4t}, x). \]

Therefore
\[ \sum_{s=1}^{\infty} \frac{\csc(ts)}{s^{5}} = \sum_{s=1}^{\infty} \frac{1}{s^{5}} + 2 \sum_{s=1}^{\infty} \frac{1}{s^{5}(e^{2ts} - 1)} = \]
\[ = \sum_{s=1}^{\infty} \frac{1}{s^{5}} + \frac{2}{e^{4t} - 1} \int_{0}^{1} dy \int_{0}^{y} \frac{1}{x} dx \int_{0}^{x} \frac{1}{w} dw \int_{0}^{w} \frac{1}{v} dv \int_{0}^{w} 2\Phi_{1}(e^{2t}, e^{2t}; e^{4t}, u) du. \]

Integrating by parts, easy to show, that
\[ \int_{0}^{1} dy \int_{0}^{y} \frac{1}{x} dx \int_{0}^{x} \frac{1}{w} dw \int_{0}^{w} \frac{1}{v} dv \int_{0}^{w} 2\Phi_{1}(e^{2t}, e^{2t}; e^{4t}, u) du = \frac{1}{4} \int_{0}^{1} \ln^{3} y \cdot 2\Phi_{1}(e^{2t}, e^{2t}; e^{4t}, y) dy. \]

Therefore
\[ \sum_{s=1}^{\infty} \frac{\csc(ts)}{s^{5}} = \zeta(5) + \frac{1}{2(e^{4t} - 1)} \int_{0}^{1} \ln^{3} y \cdot 2\Phi_{1}(e^{2t}, e^{2t}; e^{4t}, y) dy. \]

Simplify the expression
\[ \sum_{s=1}^{\infty} \frac{\csc(as) \cdot \csc(bs)}{s^{4}} = \sum_{s=1}^{\infty} \frac{1 + e^{-2as}}{s^{4}} + \frac{1 + e^{-2bs}}{s^{4}} = \]
\[ = \sum_{s=1}^{\infty} \frac{1}{s^{4}} \cdot \left[ 1 + \frac{2}{e^{2as} - 1} + \frac{2}{e^{2bs} - 1} + \frac{4}{(e^{2as} - 1)(e^{2bs} - 1)} \right]. \]
Now consider the expression \( \frac{4}{(e^{2as} - 1)(e^{2bs} - 1)} \). Substituting \( \frac{a}{b} = 2 \), we have

\[
\frac{4}{(e^{2as} - 1)(e^{2bs} - 1)} = \frac{4}{(e^{2bs} - 1)^2(e^{2bs} + 1)} = \frac{1}{2(e^{2bs} - 1)^2} - \frac{1}{4(e^{2bs} - 1)} + \frac{1}{4(e^{2bs} + 1)}.
\]

We note that

\[
\frac{\partial}{\partial b} \left[ \frac{1}{e^{2bs} - 1} \right] = -\frac{2se^{2bs}}{(e^{2bs} - 1)^2} = -\frac{2s}{(e^{2bs} - 1)} - \frac{2s}{(e^{2bs} - 1)^2}.
\]

Thus

\[
\frac{1}{(e^{2bs} - 1)^2} = -\frac{1}{e^{2bs} - 1} - \frac{1}{2s} \cdot \frac{\partial}{\partial b} \left[ \frac{1}{e^{2bs} - 1} \right].
\]

Therefore,

\[
\frac{\partial}{\partial b} \left[ \sum_{s=1}^{\infty} \frac{1}{2s^2} \frac{1}{e^{2bs} - 1} \right] = \frac{1}{8(4e^{2bs} - 1)} + \frac{1}{8(4e^{2bs} - 1)^2}.
\]

Of the formulas obtained above the sum (28) is equal to \( \frac{1}{8} \int_0^1 \ln^4 y \cdot \frac{\partial}{\partial b} \left[ 2\Psi(1; e^{2b}; e^{4b}; y) \right] dy \).

Therefore

\[
\sum_{s,t,m=1}^{\infty} \frac{1}{\pi^6 s^2 (a, s^2 - bk^2)(c, s^2 - dm^2)} = \frac{1}{3780abc} + \frac{1}{4\pi^3 \sqrt{-cda}} \left( \zeta(5) + \frac{1}{2(e^{4\pi \sqrt{-c/d}} - 1)} \int_0^1 \ln^4 y \cdot 2\Psi(1; e^{2\pi \sqrt{-c/d}}, e^{2\pi \sqrt{-c/d}}, e^{4\pi \sqrt{-c/d}}; y) dy \right) + \frac{1}{4\pi^3 \sqrt{-abc}} \left( \zeta(5) + \frac{1}{2(e^{4\pi \sqrt{-a/b}} - 1)} \int_0^1 \ln^4 y \cdot 2\Psi(1; e^{2\pi \sqrt{-a/b}}, e^{2\pi \sqrt{-a/b}}, e^{4\pi \sqrt{-a/b}}; y) dy \right),
\]

\[
- \frac{1}{4\pi^4 \sqrt{abcd}} \cdot \frac{1}{(e^{4\pi \sqrt{-a/b}} - 1)} \int_0^1 \ln^4 y \cdot 2\Psi(1; e^{2\pi \sqrt{-a/b}}, e^{2\pi \sqrt{-a/b}}, e^{4\pi \sqrt{-a/b}}; y) dy,
\]

\[
- \frac{1}{4\pi^4 \sqrt{abcd}} \cdot \frac{1}{(e^{4\pi \sqrt{-c/d}} - 1)} \int_0^1 \ln^4 y \cdot 2\Psi(1; e^{2\pi \sqrt{-c/d}}, e^{2\pi \sqrt{-c/d}}, e^{4\pi \sqrt{-c/d}}; y) dy + \frac{3}{32\pi^4 \sqrt{abcd}} \cdot \frac{1}{(e^{4\pi \sqrt{-c/d}} - 1)} \int_0^1 \ln^4 y \cdot 2\Psi(1; e^{2\pi \sqrt{-c/d}}, e^{2\pi \sqrt{-c/d}}, e^{4\pi \sqrt{-c/d}}; y) dy - \frac{1}{8\pi^4 \sqrt{abcd}} \cdot \int_0^1 \ln^4 y \cdot \frac{\partial}{\partial y} \left[ \frac{1}{e^{4\pi \sqrt{-c/d}} - 1} \cdot 2\Psi(1; e^{2\pi \sqrt{-c/d}}, e^{2\pi \sqrt{-c/d}}, e^{4\pi \sqrt{-c/d}}; y) \right] dy - \frac{1}{32\pi^4 \sqrt{abcd}} \cdot \frac{1}{64\pi^4 \sqrt{abcd}} \cdot \int_0^1 \ln^4 y \cdot 2\Psi(1; e^{2\pi \sqrt{-c/d}}, -1; e^{2\pi \sqrt{-c/d}}; y) dy,
\]

substituting given \( \sqrt{\frac{ad}{bc}} = 2 \).

Thus we have, that \( \sigma_{(1,1,1)} \) calculated in terms of well-known expression.

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References


Некоторые примеры нахождения сумм кратных рядов

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Рассмотрен метод нахождения вычетных интегралов для определенных систем неалгебраических уравнений. Такие интегралы связаны со степенными суммами корней систем уравнений. Показано, как эти результаты можно применить к нахождению сумм кратных рядов.

Ключевые слова: вычетный интеграл, степенная сумма, кратные ряды.