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On the Zeta-Function of Zeros of Some Class of Entire Functions

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Using the residue theory, we give an integral representation for the zeta-function that enables us to construct its analytic continuation.

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Introduction

The question of the integral representation for the zeta-function associated with some entire function was studied by V.B. Lidskii and V.A. Sadovnichii in [1], where an entire function $f(z)$ of a certain type was considered.

In [2] A.M. Kytmanov and S.G. Myslivets introduced the concept of the zeta-function associated with a system of meromorphic functions $f = (f_1, \ldots, f_n)$ in $\mathbb{C}^n$. Using the residue theory, these authors gave an integral representation for the zeta-function, but the system of functions $f_1, \ldots, f_n$ was subject to rigid constraints.

Let $f(z)$ be an entire function of order ρ with zeros $z_1, z_2, \ldots$, such that $f(0) \neq 0$. Then, according to Hadamard's theorem on factorization (see, for example, [3, Chapter VIII, S. 8.2.4]), the function $f(z)$ is represented in the form

$$f(z) = e^{Q(z)} P(z),$$

where $P(z)$ is the canonical product constructed by the zero set of the function $f(z)$, and $Q(z)$ is a polynomial with the degree not higher than ρ. In this case, the canonical product $P(z)$ has the form

$$P(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \left(\frac{z}{z_n}\right)^2/2 + \ldots + \left(\frac{z}{z_n}\right)^p/p}$$

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and $p \leqslant \rho$.

Thus, under $f(z) \neq 0$ we have locally (in accordance with [3, Chapter VIII, S. 8.2.4])

$$\begin{aligned}
&\ln f(z) = \ln\left(e^{Q(z)} P(z)\right) = \ln e^{Q(z)} + \ln P(z) = \\
&= Q(z) + \ln\left[\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right) e^{\frac{z}{z_n}+\left(\frac{z}{z_n}\right)^2/2+\ldots+\left(\frac{z}{z_n}\right)^p/p}\right] = \\
&= Q(z) + \sum_{n=1}^{\infty}\ln\left[\left(1-\frac{z}{z_n}\right) e^{\frac{z}{z_n}+\left(\frac{z}{z_n}\right)^2/2+\ldots+\left(\frac{z}{z_n}\right)^p/p}\right] = \\
&= Q(z) + \sum_{n=1}^{\infty}\left[\ln\left(1-\frac{z}{z_n}\right) + \ln e^{\frac{z}{z_n}+\left(\frac{z}{z_n}\right)^2/2+\ldots+\left(\frac{z}{z_n}\right)^p/p}\right] = \\
&= Q(z) + \sum_{n=1}^{\infty}\left[\ln\left(1-\frac{z}{z_n}\right) + \frac{z}{z_n} + \left(\frac{z}{z_n}\right)^2/2 + \ldots + \left(\frac{z}{z_n}\right)^p/p\right].
\end{aligned}$$

Differentiate this relation once. We obtain

$$\begin{aligned}
\frac{f'(z)}{f(z)} &= Q'(z) + \sum_{n=1}^{\infty}\left[\frac{1}{1-\frac{z}{z_n}}\cdot\left(-\frac{1}{z_n}\right) + \frac{1}{z_n} + \frac{z}{z_n}\cdot\frac{1}{z_n} + \ldots + \left(\frac{z}{z_n}\right)^{p-1}\cdot\frac{1}{z_n}\right] = \\
&= Q'(z) + \sum_{n=1}^{\infty}\left[\frac{-1}{z_n - z} + \frac{1}{z_n} + \frac{z}{z_n^2} + \ldots + \frac{z^{p-1}}{z_n^p}\right].
\end{aligned}$$

We rewrite the expression under the summation sign to get

$$\frac{1}{z_n} + \frac{z}{z_n^2} + \ldots + \frac{z^{p-1}}{z_n^p} = \frac{1}{z_n}\cdot\frac{1-\left(\frac{z}{z_n}\right)^p}{1-\frac{z}{z_n}} = \frac{1}{z_n}\cdot\frac{1-\left(\frac{z}{z_n}\right)^p}{z_n - z}\cdot z_n = \frac{1-\left(\frac{z}{z_n}\right)^p}{z_n - z},$$

$$\frac{-1}{z_n - z} + \frac{1}{z_n} + \frac{z}{z_n^2} + \ldots + \frac{z^{p-1}}{z_n^p} = \frac{-1}{z_n - z} + \frac{1-\left(\frac{z}{z_n}\right)^p}{z_n - z} = \frac{-\left(\frac{z}{z_n}\right)^p}{z_n - z} = -\frac{z^p}{(z_n - z)\, z_n^p}.$$

Thus

$$\frac{f'(z)}{f(z)} = Q'(z) - \sum_{n=1}^{\infty}\frac{z^p}{(z_n - z)\, z_n^p}.$$

The resulting series converges absolutely and locally uniformly for $z \neq z_n$ since $p \leqslant \rho_1 \leqslant \rho$ ([3, Chapter VIII, S. 8.2.3]). Here ρ_1 is the index of convergence of zeros.

Recall ([3, Chapter VIII, S. 8.2.2]) that the lower boundary of positive numbers α for which the series $\sum |z_n|^{-\alpha}$ converges is called *the index of convergence of zeros*. Denote it by ρ_1.

Henceforth we assume $|z_1| \leqslant |z_2| \leqslant \ldots \leqslant |z_n| \leqslant \ldots$. Let some number n_0 of zeros of the function $f(z)$ lies inside the circle $|z| = R$, and the rest lies out. Then

$$\sum_{n=1}^{\infty}\frac{z^p}{(z_n - z)\, z_n^p} = \sum_{n=1}^{n_0}\frac{z^p}{(z_n - z)\, z_n^p} + \sum_{n=n_0+1}^{\infty}\frac{z^p}{(z_n - z)\, z_n^p}.$$

Suppose that for the zeros of the function $f(z)$ the following estimates hold:

$$|z - z_n| > \delta\,|z|, \qquad \text{when} \quad |z_n| < |z|, \tag{1}$$

$$|z - z_n| > \delta\,|z_n|, \qquad \text{when} \quad |z_n| > |z|, \tag{2}$$

where for convenience only one constant $\delta > 0$ is introduced. We discuss the conditions under which the estimates (1) and (2) would hold true below.

Estimate the first sum using (1) and the fact that $\dfrac{1}{|z|} < \dfrac{1}{|z_n|}$. We have

$$\left|\sum_{n=1}^{n_0} \frac{z^p}{(z_n - z)\, z_n^p}\right| \leqslant \sum_{n=1}^{n_0} \left|\frac{z^p}{(z_n - z)\, z_n^p}\right| \leqslant |z|^p \sum_{n=1}^{n_0} \frac{1}{|z_n - z|\, |z_n|^p} < |z|^p \sum_{n=1}^{n_0} \frac{1}{\delta\, |z|\, |z_n|^p} <$$
$$< \frac{|z|^p}{\delta} \sum_{n=1}^{n_0} \frac{1}{|z_n|^{p+1}} < |z|^p \cdot \varepsilon_1.$$

Estimate the second sum using (2). We get

$$\left|\sum_{n=n_0+1}^{\infty} \frac{z^p}{(z_n - z)\, z_n^p}\right| \leqslant \sum_{n=n_0+1}^{\infty} \left|\frac{z^p}{(z_n - z)\, z_n^p}\right| \leqslant |z|^p \sum_{n=n_0+1}^{\infty} \frac{1}{|z_n - z|\, |z_n|^p} < |z|^p \sum_{n=n_0+1}^{\infty} \frac{1}{\delta\, |z_n|\, |z_n|^p} =$$
$$= \frac{|z|^p}{\delta} \sum_{n=n_0+1}^{\infty} \frac{1}{|z_n|^{p+1}} < |z|^p \cdot \varepsilon_2.$$

Let us now discuss the conditions (1) and (2). Let

$$|z - z_n| \geqslant |z| - |z_n| > \delta\, |z|\,,$$

i. e.

$$|z| - \delta\, |z| > |z_n|\,, \quad |z| > \frac{|z_n|}{1-\delta}.$$

Setting $z = z_{n+1}$, we obtain the following system of inequalities:

$$|z_{n+1}| > \frac{|z_n|}{1-\delta} > \frac{1}{1-\delta} \cdot \frac{|z_{n-1}|}{1-\delta} = \frac{|z_{n-1}|}{(1-\delta)^2} > \ldots > \frac{|z_1|}{(1-\delta)^n},$$
$$\frac{1}{|z_{n+1}|} < \frac{(1-\delta)^n}{|z_1|}.$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{|z_{n+1}|} < \frac{1}{|z_1|} \sum_{n=1}^{\infty} (1-\delta)^n = \frac{1}{|z_1|} \cdot S,$$
$$\sum_{n=1}^{\infty} \frac{1}{|z_{n+1}|^{\alpha}} < \frac{1}{|z_1|^{\alpha}} \sum_{n=1}^{\infty} \left((1-\delta)^{\alpha}\right)^n = \frac{1}{|z_1|^{\alpha}} \cdot S_{\alpha}, \qquad \alpha > 0,$$

where S and S_α denote the corresponding sums of the series. Thus, we have shown that the conditions (1) and (2) would be true for an entire function with $\rho_1 = 0$.

In what follows we consider entire functions $f(z)$ of the zero order with the index of convergence of zeros $\rho_1 = 0$. The conditions (1) and (2) hold for the zeros of such functions.

Given the above calculations and reasoning, as well as the fact that $p \leqslant \rho_1 \leqslant \rho$ (see [3, Chapter VIII, S. 8.2.3]), we obtain

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{z - z_n} \tag{3}$$

if $z \neq z_n$.

1. The first integral representation

Let $f(z)$ be an entire function of the zero order in $\mathbb{C}$. Consider the equation

$$f(z) = 0. \tag{4}$$

Denote by $N_f = f^{-1}(0)$ the set of all roots of (4) with the multiplicity counted. The number of zeros is at most countable.

Further assume that the following asymptotic representation holds true on a positive part of the real axis (ν_0 is a non-negative integer)

$$\frac{f'(z)}{f(z)} \sim \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu}{z^\nu}, \quad z \to +\infty, \quad \text{i. e.} \quad \frac{f'(z)}{f(z)} - \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu}{z^\nu} = O\left(\frac{1}{|z|^{\nu_0+1}}\right). \tag{5}$$

If $\nu_0 = 0$ then

$$\frac{f'(z)}{f(z)} - \omega_0 = O\left(\frac{1}{|z|}\right). \tag{6}$$

Our goal is to obtain an integral representation for the zeta-function $\zeta_f(s)$ of (4) that was defined in [2] as

$$\zeta_f(s) = \sum_{a \in N_f} (-a)^{-s},$$

where $s \in \mathbb{C}$. We choose the minus sign in the definition of the zeta-function only for convenience in writing the integral formulas; below we explain what value is taken for the multivalued function $(-z)^{-s}$.

Let $z = x + iy$. Suppose that the function f is not equal to zero at any point of $\mathbb{R}_+ := \{z \in \mathbb{C} : x \geqslant 0, y = 0\}$. This means that $N_f \cap \mathbb{R}_+ = \varnothing$.

Consider a domain $D \subset \mathbb{C}$ of the form

$$D = \{z \in \mathbb{C} : r < |z| < R\} \setminus \{z \in \mathbb{C} : r < \text{Re } z < R, \text{ Im } z = 0\}$$

and $0 < r < R$.

Observe that D is a simply connected domain. Its boundary $\gamma = \partial D$ consists of the intervals $[r, R]$ on the real axis, the circle S_R of radius R centered at the origin with positive (counterclockwise) orientation, the interval $[R, r]$ on $\mathbb{R}$ obtained from $[r, R]$ by the change of orientation, and the circle $-S_r$ obtained from S_r by the change of orientation.

Choose the radii r and R so that $\gamma \cap N_f = \varnothing$.

Consider the integral

$$I(s) = \frac{1}{2\pi i} \int_\gamma (-z)^{-s} \frac{f'(z)}{f(z)} \, dz. \tag{7}$$

The functions $(-z)^{-s} = e^{-s \ln(-z)}$ are holomorphic in D, where $\ln \zeta$ denotes the principal branch of the logarithm, i. e. the holomorphic branch $\ln \zeta$ on $\mathbb{C} \setminus \{\zeta \in \mathbb{C} : \text{ Re } \zeta \leqslant 0, \text{ Im } \zeta = 0\}$ equal to zero for $\zeta = 1$. It is obvious that $I(s)$ is an entire function in $s \in \mathbb{C}$.

Note that $I(s)$ can be written in the form (in accordance with the logarithmic residue theorem)

$$I(s) = \frac{1}{2\pi i} \int_\gamma (-z)^{-s} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \int_\gamma (-z)^{-s} \frac{df}{f} = \sum_{a \in N_f \cap D} (-a)^{-s}.$$

Thus

$$I(s) = \sum_{a \in N_f \cap D} (-a)^{-s}.$$

Choose a sequence R_k so that S_{R_k} does not contain the zeros of f.

Lemma 1. *The integral* $\displaystyle\int_{S_{R_k}} (-z)^{-s} \frac{f'(z)}{f(z)}\, dz \to 0$ *as* $R_k \to +\infty$ *and Re* $s > 1$.

Proof. We have

$$\left|(-z)^{-s}\right| = \left|e^{-s(\ln R_k + i(\varphi - \pi))}\right| = e^{-\mathrm{Re}\, s \ln R_k + \mathrm{Im}\, s(\varphi-\pi)} = O\left(R_k^{-\mathrm{Re}\, s}\right),$$

where $\varphi = \arg z$.

Then the estimates hold for the module of the integral

$$\left|\int\limits_{S_{R_k}} (-z)^{-s} \frac{f'(z)}{f(z)}\, dz\right| \leqslant \int\limits_{S_{R_k}} \left|(-z)^{-s}\right| \left|\frac{f'(z)}{f(z)}\right| |dz| \leqslant R_k^{-\mathrm{Re}\, s} \cdot C \cdot R_k = C \cdot R_k^{1-\mathrm{Re}\, s} = \frac{C}{R_k^{\mathrm{Re}\, s - 1}} \to 0,$$

as $R_k \to +\infty$ and Re $s > 1$. □

Denote by $\Gamma_0' = (\infty, r] \cup [r, \infty)$, $\Gamma_0 = S_r \cup \Gamma_0'$. Now, taking $R = R_k$ and letting k tend to infinity in (7), we obtain

$$I(s) = \frac{1}{2\pi i} \int_{\Gamma_0} (-z)^{-s} \frac{f'(z)}{f(z)}\, dz.$$

Lemma 2. *The integral* $I(s)$ *can be continued analytically into the half plane Re* $s > -\nu_0$ *when the condition* (5) *holds.*

Proof. We argue as in [1]. For the proof of this lemma we write the integral $I(s)$ as the sum of four parts:

$$I(s) = \frac{1}{2\pi i} \int_{S_r} (-z)^{-s} \frac{f'(z)}{f(z)}\, dz + \frac{1}{2\pi i} \int_{\Gamma_0'} (-z)^{-s} \left(\frac{f'(z)}{f(z)} - \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu}{z^\nu}\right) dz +$$
$$+ \frac{1}{2\pi i} \int_{\Gamma_0} (-z)^{-s} \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu}{z^\nu}\, dz - \frac{1}{2\pi i} \int_{S_r} (-z)^{-s} \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu}{z^\nu}\, dz = I_1(s) + I_2(s) + I_3(s) + I_4(s).$$

We easily see that $I_1(s)$ and $I_4(s)$ are entire functions in s, and $I_2(s)$, in view of the asymptotic representation (5), can be continued analytically into Re $s > -\nu_0$. Finally, $I_3(s)$ is zero for Re $s > 1$ and so its analytic continuation is equal to zero in the whole plane. □

Suppose further that $\nu_0 = 0$. Then, by Lemma 2, $I(s) = I_1(s) + I_2(s) + I_4(s)$ and for Re $s < 1$ (as in Lemma 1) the integrals $I_1(s)$ and $I_4(s)$ tend to zero when $r \to 0$. Thus, we have

Corollary 1. *For* $0 <$ *Re* $s < 1$

$$\zeta_f(s) = \frac{1}{2\pi i} \int_{\Gamma_0''} \left(\frac{f'(z)}{f(z)} - \omega_0\right) (-z)^{-s}\, dz,$$

where $\Gamma_0'' = (\infty, 0] \cup [0, \infty)$.

Consider the integrals over the intervals with opposite orientation. Since

$$(-z)^{-s} = \left(-xe^{2\pi i}\right)^{-s} = e^{-2\pi i s} (-x)^{-s},$$

we obtain

$$\int_{\infty}^{0}\left(\frac{f'(z)}{f(z)}-\omega_0\right)(-z)^{-s}\,dz=\int_{\infty}^{0}\left(\frac{f'(x)}{f(x)}-\omega_0\right)e^{-2\pi is}(-x)^{-s}\,dx=$$
$$=-e^{-2\pi is}\int_{0}^{\infty}\left(\frac{f'(x)}{f(x)}-\omega_0\right)(-x)^{-s}\,dx.$$

Summing all integrals over the intervals, we obtain

$$\int_{0}^{\infty}\left(\frac{f'(x)}{f(x)}-\omega_0\right)(-x)^{-s}\,dx-e^{-2\pi is}\int_{0}^{\infty}\left(\frac{f'(x)}{f(x)}-\omega_0\right)(-x)^{-s}\,dx=$$
$$=\left(1-e^{-2\pi is}\right)\int_{0}^{\infty}\left(\frac{f'(x)}{f(x)}-\omega_0\right)(-x)^{-s}\,dx.$$

By obvious calculations we have

$$1-e^{-2\pi is}=e^{-\pi is}\left(e^{\pi is}-e^{-\pi is}\right)=e^{-\pi is}2i\frac{e^{i\pi s}-e^{-i\pi s}}{2i}=e^{-\pi is}2i\sin\pi s=(-1)^{-s}\,2i\sin\pi s.$$

Summing all integrals over the intervals, we arrive at the integral

$$2i\sin\pi s\int_{0}^{\infty}\left(\frac{f'(x)}{f(x)}-\omega_0\right)x^{-s}\,dx.$$

Summarizing the above, we obtain the integral representation for the zeta-function $\zeta_f(s)$ in the strip $0<\mathrm{Re}\ s<1$.

Theorem 1.1 *Let $f(z)$ be an entire function of the zero order in $\mathbb{C}$ and satisfy the condition* (6)*. Suppose that $0<\mathrm{Re}\ s<1$. Then*

$$\zeta_f(s)=\frac{\sin\pi s}{\pi}\int_{0}^{\infty}\left(\frac{f'(x)}{f(x)}-\omega_0\right)x^{-s}\,dx,$$

where ω_0 is the limit value of $\dfrac{f'(x)}{f(x)}$ at infinity.

The method of proof shows that if the asymptotic condition (5) holds, we have the following result.

Corollary 2. *Suppose that the asymptotic condition* (5) *holds. Then for $-\nu_0<\mathrm{Re}\ s<1$ the following holds*

$$\zeta_f(s)=\frac{\sin\pi s}{\pi}\int_{0}^{\infty}\left(\frac{f'(x)}{f(x)}-\sum_{\nu=0}^{\nu_0}\frac{\omega_\nu}{x^\nu}\right)x^{-s}\,dx.$$

To conclude this section we compare the obtained integral representation with the integral representation for the classical Riemann zeta-function $\zeta(s)$ (see, for example, [4, Chapter 2, S. 9]) in the strip $0<\mathrm{Re}\ s<1$. Namely,

$$\zeta(s)=-\frac{\sin\pi s}{\pi}\int_{0}^{\infty}\left\{\frac{\Gamma'(1+x)}{\Gamma(1+x)}-\ln x\right\}x^{-s}\,dx,\quad 0<\mathrm{Re}\ s<1.$$

2. The second integral representation

Consider an entire function $f(z)$ of order ρ. In this section we obtain another integral representation for the zeta-function $\zeta_f(s)$ of the zeros z_n of f that have the form

$$z_n = -q_n + is_n, \quad q_n > 0. \tag{8}$$

For this purpose we consider the integral $\int_0^\infty x^{s-1} e^{z_n x}\, dx$, in which we make the change of variables

$$z_n \cdot x = -y, \quad x = \frac{y}{-z_n}, \quad dx = \frac{1}{-z_n}\, dy.$$

Thus, by (8)

$$\int_0^\infty x^{s-1} e^{z_n x}\, dx = \int_l \frac{y^{s-1}}{(-z_n)^{s-1}} e^{-y} \frac{1}{(-z_n)}\, dy = \frac{1}{(-z_n)^s} \int_0^\infty y^{s-1} e^{-y}\, dy = \frac{\Gamma(s)}{(-z_n)^s},$$

where l is a ray (corresponding to the change of variables $z_n \cdot x = -y$) from the origin, and $\Gamma(s)$ is the Euler gamma-function defined by the formula

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

Further, we consider the product

$$\Gamma(s) \cdot \zeta_f(s) = \Gamma(s) \sum_{n=1}^\infty (-z_n)^{-s} = \Gamma(s) \sum_{n=1}^\infty \frac{1}{(-z_n)^s} = \sum_{n=1}^\infty \frac{\Gamma(s)}{(-z_n)^s} = \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{z_n x}\, dx =$$
$$= \int_0^\infty \sum_{n=1}^\infty x^{s-1} e^{z_n x}\, dx = \int_0^\infty x^{s-1} \sum_{n=1}^\infty e^{z_n x}\, dx.$$

Denoting

$$F(f, x) = \sum_{n=1}^\infty e^{z_n x}, \tag{9}$$

we obtain

$$\Gamma(s) \cdot \zeta_f(s) = \int_0^\infty x^{s-1} F(f, x)\, dx,$$

or

$$\zeta_f(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} F(f, x)\, dx.$$

It is necessary to justify the change of the order of summation and integration and explain why the series (9) converges.

To prove that the series (9) is convergent we use the Cauchy criterion. Consider

$$|e^{z_n x}| = \left|e^{(-q_n + is_n)x}\right| = \left|e^{-q_n x} \cdot e^{is_n x}\right| = \left|e^{-q_n x}\right| = e^{-q_n x}.$$

Then for the convergence of the series (9) it is necessary and sufficient that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{e^{-q_n x}} = \overline{\lim_{n\to\infty}} e^{-\frac{q_n x}{n}} = \overline{\lim_{n\to\infty}} \frac{1}{e^{\frac{q_n x}{n}}} < 1,$$

i.e.

$$\underline{\lim}_{n\to\infty} \frac{q_n}{n} > 0. \tag{10}$$

To justify the change of the order of summation and integration it is necessary to prove the uniform convergence of the series $\sum\limits_{n=1}^{\infty} x^{s-1}e^{z_n x}$ on the set $[0;+\infty)$. We enumerate the zeros $z_1,\dots,z_n,\dots$ in the order of increasing absolute values of the real parts, i.e., $q_1 \leqslant q_2 \leqslant \dots \leqslant q_n \leqslant \dots$, and let Re $s=\sigma>1$. Consider the series of modules

$$\sum_{n=1}^{\infty}\left|x^{s-1}e^{z_n x}\right| = \sum_{n=1}^{\infty} x^{\sigma-1}e^{-q_n x} = \sum_{n=1}^{\infty} x^{\sigma-1}e^{-q_1 x}e^{(q_1-q_n)x} = e^{-q_1 x}\sum_{n=1}^{\infty} x^{\sigma-1}e^{(q_1-q_n)x}.$$

Consider the function $g_n(x) = x^{\sigma-1}e^{(q_1-q_n)x}$ and find its extremums. Consider the equation

$$g_n'(x) = 0.$$

For root x_0 of this equation we have the relation

$$\sigma-1+x(q_1-q_n)=0,$$
$$\sigma-1=x(q_n-q_1),$$
$$x_0=\frac{\sigma-1}{q_n-q_1}.$$

It is easy to see that for $x>x_0$ the function $g_n(x)$ is decreasing, and for $0<x<x_0$ the function $g_n(x)$ is increasing. Thus, the point x_0 is a local maximum of the function $g_n(x)$.

Then

$$\sum_{n=1}^{\infty}\left|x^{s-1}e^{z_n x}\right| = e^{-q_1 x}\sum_{n=1}^{\infty} x^{\sigma-1}e^{(q_1-q_n)x} \leqslant e^{-q_1 x}\sum_{n=1}^{\infty}\left(\frac{\sigma-1}{q_n-q_1}\right)^{\sigma-1} e^{(q_1-q_n)\frac{\sigma-1}{q_n-q_1}} =$$
$$= e^{-q_1 x}\sum_{n=1}^{\infty}\left(\frac{\sigma-1}{q_n-q_1}\right)^{\sigma-1} e^{-(\sigma-1)} = e^{-q_1 x}\sum_{n=1}^{\infty}\left(\frac{\sigma-1}{e}\right)^{\sigma-1}\left(\frac{1}{q_n-q_1}\right)^{\sigma-1} =$$
$$= e^{-q_1 x}\left(\frac{\sigma-1}{e}\right)^{\sigma-1}\sum_{n=1}^{\infty}\left(\frac{1}{q_n-q_1}\right)^{\sigma-1}.$$

For the uniform convergence of the series under study it is necessary that the series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{q_n-q_1}\right)^{\sigma-1}$ converges. The convergence of this series is equivalent to the convergence of the series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{q_n}\right)^{\sigma-1}$. Thus, to change the order of summation and integration it is necessary that

$$\text{the series} \quad \sum_{n=1}^{\infty}\left(\frac{1}{q_n}\right)^{\sigma-1} \quad \text{is convergent.} \tag{11}$$

We have proved the following result.

Theorem 2.1 *Suppose that the conditions* (10) *and* (11) *are satisfied and Re* $s>1$. *Then*

$$\zeta_f(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}F(f,x)\,dx, \tag{12}$$

where $F(f,x)$ *is defined by formula* (9).

Corollary 3. *Suppose that the conditions of Theorem 2.1 are satisfied. Then for* $0 < Re \ s < 1$ *the following formula holds*

$$\zeta_f(s)\,\Gamma(s) = \int_0^\infty \left(F(f,x) - \frac{1}{x}\right) x^{s-1}\,dx,$$

where $F(f,x)$ *is defined by formula* (9).

Proof. We write the expression (12) in the form

$$\zeta_f(s)\,\Gamma(s) = \int_0^\infty F(f,x)\,x^{s-1}\,dx = \int_0^1 \left(F(f,x) - \frac{1}{x}\right) x^{s-1}dx + \int_0^1 \frac{1}{x}x^{s-1}dx + \int_1^\infty F(f,x)\,x^{s-1}dx.$$

In the last equation we calculate the second integral. We have

$$\int_0^1 \frac{1}{x}x^{s-1}\,dx = \int_0^1 x^{s-2}\,dx = \left.\frac{x^{s-1}}{s-1}\right|_0^1 = \frac{1}{s-1},$$

since Re $s > 1$.

Thus, for Re $s > 1$ the following equalities hold

$$\zeta_f(s)\,\Gamma(s) = \int_0^1 \left(F(f,x) - \frac{1}{x}\right) x^{s-1}\,dx + \frac{1}{s-1} + \int_1^\infty F(f,x)\,x^{s-1}\,dx.$$

According to the principle of analytic continuation, this formula holds for Re $s > 0$. Moreover, for $0 <$ Re $s < 1$ we have

$$-\int_1^\infty \frac{x^{s-1}}{x}\,dx = -\int_1^\infty x^{s-2}\,dx = -\left.\frac{x^{s-1}}{s-1}\right|_1^\infty = -\left.\frac{1}{s-1}\cdot\frac{1}{x^{1-s}}\right|_1^\infty = -\frac{1}{s-1}(0-1) = \frac{1}{s-1}.$$

Hence we obtain

$$\zeta_f(s)\,\Gamma(s) = \int_0^1 \left(F(f,x) - \frac{1}{x}\right) x^{s-1}\,dx - \int_1^\infty \frac{x^{s-1}}{x}\,dx + \int_1^\infty F(f,x)\,x^{s-1}\,dx.$$

Simplifying the expression, we obtain the statement of the corollary. □

In the conclusion of this section we give (see, for example, [4, Chapter 2, S. 4]) one more integral representation for the classical Riemann zeta-function $\zeta(s)$. Namely,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx, \quad \text{Re } s > 1. \tag{13}$$

If $0 <$ Re $s < 1$ the integral representation (13) can be written (see, for example, [4, Chapter 2, S. 7]) in the form

$$\zeta(s)\,\Gamma(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1}\,dx, \quad 0 < \text{Re } s < 1.$$

The difference between the classical integral representation (13) and the obtained integral representation (12) is that in the classical case it is possible to calculate explicitly the series (9), since the Riemann zeta-function is defined by the zeros of $\dfrac{1}{\Gamma(1+x)}$, i.e., $z_n = -1, -2, -3, \ldots$. Therefore, this classical formula follows from the formula of Corollary 3.

3. Examples

In this section we consider the examples of entire functions $f(z)$ of zero order, constructed by the zero set z_n, for which the following relation holds

$$\lim_{x\to+\infty}\frac{f'(x)}{f(x)}=\omega_0,$$

where the ratio $\dfrac{f'(z)}{f(z)}$ is defined by formula (3).

It is well-known that the limit of the sum of a series is equal to the sum of the series consisting of the limits of its terms, when there is the uniform convergence, i. e.

$$\lim_{x\to a}\sum_{n=1}^{\infty}u_n(x)=\sum_{n=1}^{\infty}\left\{\lim_{x\to a}u_n(x)\right\}. \tag{14}$$

Example 1. Let $z_n=-2^n$. Then, in accordance with formula (3)

$$\frac{f'(x)}{f(x)}=\sum_{n=1}^{\infty}\frac{1}{x+2^n}.$$

Since $\dfrac{1}{x+2^n}\leqslant\dfrac{1}{2^n}$, where $x\geqslant 0$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{x+2^n}$ converges uniformly on the set $[0,+\infty)$ in accordance with the Weierstrass criterion of a uniform convergence of functional series.

Given formula (14), we have

$$\lim_{x\to+\infty}\frac{f'(x)}{f(x)}=\lim_{x\to+\infty}\sum_{n=1}^{\infty}\frac{1}{x+2^n}=\sum_{n=1}^{\infty}\lim_{x\to+\infty}\frac{1}{x+2^n}=0,$$

i. e. $\omega_0=0$ and $\dfrac{f'(x)}{f(x)}\sim 0$ as $x\to+\infty$.

Example 2. Let $z_n=-q_n+is_n$, $q_n>0$. Then, in accordance with formula (3)

$$\frac{f'(x)}{f(x)}=\sum_{n=1}^{\infty}\frac{1}{x-z_n}.$$

Estimate the terms:

$$\frac{1}{|x-z_n|}=\frac{1}{|x+q_n-is_n|}=\frac{1}{\sqrt{(x+q_n)^2+s_n^2}}\leqslant\frac{1}{\sqrt{q_n^2+s_n^2}}=\frac{1}{|z_n|},\qquad x\geqslant 0.$$

Since the series $\displaystyle\sum\frac{1}{|z_n|^{\alpha}}$ converges ([3, Chapter VIII, S. 8.2.2]), when $\alpha>\rho$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{|z_n|}$ converges. Then the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{x-z_n}$ converges uniformly on the set $[0,+\infty)$ in accordance with the Weierstrass criterion of a uniform convergence of functional series.

Given formula (14), we have

$$\lim_{x\to+\infty}\frac{f'(x)}{f(x)}=\lim_{x\to+\infty}\sum_{n=1}^{\infty}\frac{1}{x+q_n-is_n}=\sum_{n=1}^{\infty}\lim_{x\to+\infty}\frac{1}{x+q_n-is_n}=0,$$

i. e. $\omega_0 = 0$ and $\dfrac{f'(x)}{f(x)} \sim 0$ as $x \to +\infty$.

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О дзета-функции корней одного класса целых функций

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С использованием теории вычетов дается интегральное представление для дзета-функции, которое позволяет построить аналитическое продолжение дзета-функции.

Ключевые слова: дзета-функция, интегральное представление, целая функция.