Journal of Siberian Federal University. Mathematics & Physics 2014, 7(4), 466–471

УДК 517.55

# Boundary Version of the Morera Theorem for a Matrix Ball of the Second Type

**Gulmirza Kh. Khudayberganov***
**Zokirbek K. Matyakubov**†
National University of Uzbekistan
Vuzgorodok, Tashkent, 100174
Uzbekistan

Received 24.05.2014, received in revised form 26.08.2014, accepted 15.09.2014

*In this article we prove a boundary Morera theorem for a matrix ball of the second type.*

*Keywords: matrix ball, automorphism, Poisson kernel, Morera theorem.*

In this article we consider a boundary version of the Morera theorem for a matrix ball of the second type. Our starting point is Nagel and Rudin's result (see [1]), which says that if $f$ is a continuous function on the boundary of a ball in $\mathbb{C}^n$ and the integral

$$\int_0^{2\pi} f\left(\psi(e^{i\varphi},\ 0,\ \dots,\ 0)\right) e^{i\varphi} d\varphi = 0,$$

for all (holomorphic) automorphisms $\psi$ of a ball, then the function $f$ is holomorphically extends into the ball. For classical domains an analog of a boundary Morera theorem was obtained in [7].

Let $\mathbb{C}\,[m \times m]$ be the space of $[m \times m]$-matrices with complex elements. We denote by $\mathbb{C}^n\,[m \times m]$ the Cartesian product of $n$ copies of $\mathbb{C}\,[m \times m]$ :

$$\mathbb{C}^n\,[m \times m] = \mathbb{C}\,[m \times m] \times \ldots\ldots \times \mathbb{C}\,[m \times m]\,.$$

Set (see, for example [2])

$$B_I = \left\{Z \in \mathbb{C}^n\,[m \times m] : \ \ I - \langle Z, Z\rangle > 0\right\},$$

where $\langle Z, Z\rangle = Z_1 Z_1^* + Z_2 Z_2^* + \ldots + Z_n Z_n^*$ is a 'scalar' product, $I$ is the identity matrix $[m \times m]$, $Z_\nu^* = \overline{Z'}_v$ is the adjoint and transposed matrix to $Z_\nu$, $\nu = 1, 2, \ldots, n$. $B_I$ is called a *matrix ball* (of the first type). Here $I - \langle Z, Z\rangle > 0$ means that the Hermite matrix $I - \langle Z, Z\rangle$ is positively defined, i.e. all eigen values are positive.

The skeleton of $B_I$ is the set

$$X_I = \left\{Z \in \mathbb{C}^n\,[m \times m] : \ \ \langle Z, Z\rangle = I\right\}.$$

The domain $B_{II}$ in spaces $\mathbb{C}^n\,[m \times m]$:

$$B_{II} = \left\{Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z\rangle > 0, \quad Z'_v = Z_\nu,\ \nu = 1, 2, \ldots, n\right\}, \tag{1}$$

where $I$ is, as usual, the identity matrix of order $m$, is called a *matrix ball of the second type* (see [3]).

The skeleton of this domain is the following manifold:

$$X_{II} = \left\{Z \in \mathbb{C}^n\,[m \times m] : \langle Z, Z\rangle = I, \quad Z'_v = Z_\nu,\ \nu = 1, 2, \ldots, n\right\}.$$

*gkhudaiberg@mail.ru
†zokirbek.1986@mail.ru

**Lemma 1.** *The domain $B_{II}$ has the following properties:*
*1) $B_{II}$ is bounded;*
*2) $B_{II}$ is a complete circular domain;*
*3) $B_{II}$ and its skeleton $X_{II}$ are invariant under unitary transformations.*

*Proof.* **1.** The definition of the domain implies that each diagonal element of the matrix $\langle Z, Z\rangle$ is positive and less than 1, and the sum of the squares of the modules of all elements in $Z_\nu$, $\nu = 1, ..., n$, does not exceed $m$. This implies that the matrix ball of the second type is bounded.

**2.** If $Z \in B_{II}$ and $\alpha \in \mathbb{C}, |\alpha| \leq 1$, then

$$I - \langle \alpha Z, \alpha Z\rangle = I - |\alpha|^2 \langle Z, Z\rangle = I(1 - |\alpha|^2) + |\alpha|^2 (I - \langle Z, Z\rangle) > 0.$$

**3.** Invariance under unitary transformations means that if $U$ is a unitary matrix of order $m$, then for $Z \in B_{II}$ we have $UZ \in B_{II}$ and $ZU \in B_{II}$. Indeed,

$$I - \langle UZ, UZ\rangle = I - UZ_1\overline{Z_1}U^* - UZ_2\overline{Z_2}U^* - ..... - UZ_n\overline{Z_n}U^* =$$
$$= I - U\left(Z_1\overline{Z_1} + Z_2\overline{Z_2} + .... + Z_n\overline{Z_n}\right)U^* = I - U\langle Z, Z\rangle U^* = U(I - \langle Z, Z\rangle)U^* > 0,$$

and

$$\langle ZU, ZU\rangle = \langle Z, Z\rangle .$$

The invariance of the skeleton is proved similarly. □

We consider normalized Lebesgue measures $\mu$ in $B_{II}$ and $\sigma$ on the skeleton $X_{II}$, i.e.

$$\int_{B_{II}} d\mu(Z) = 1 \quad \text{and} \quad \int_{X_{II}} d\sigma(Z) = 1.$$

We define the space $H^1(B_{II})$ as follows: a function $f$ belongs to $H^1(B_{II})$ if it is holomorphic in $B_{II}$ and

$$\sup_{0<r<1} \int_{X_{II}} |f(rZ)|\, d\sigma(Z) < \infty.$$

We fix a point $\Lambda^0 \in X_{II}$ ($\Lambda^0 = (\Lambda_1^0, ..., \Lambda_n^0)$) and consider the following embedding of a unit disk $\Delta$ in the domain $B_{II}$

$$\left\{W \in \mathbb{C}^n [m \times m] : \ W_\nu = \xi\Lambda_\nu^0, \ |\xi| < 1, \ \nu = 1, ..., n, \right\}. \tag{2}$$

By this embedding the boundary $T$ of the disk $\Delta$ transforms into the disk on $X_{II}$. If $\psi$ is an automorphism of the domain $B_{II}$, then the set (2) under the action of this automorphism becomes some analytic disk with the boundary on $X_{II}$.

**Theorem 1.** *Let $f$ be a continuous function on $X_{II}$. If $f$ satisfies*

$$\int_T f(\psi(\xi\Lambda^0))d\xi = 0 \tag{3}$$

*for all automorphisms $\psi$ of the domain $B_{II}$, then the function $f$ has a holomorphic extension $F$ in $B_{II}$ of the class $C(\overline{B}_{II})$.*

*Proof.* On $X_{II}$ the subgroup of the automorphisms leaving 0 fixed acts transitively (see [3]). Since $X_{II}$ is invariant with respect to unitary transformations, the condition (3) is satisfied for any point $\Lambda \in X_{II}$.

First of all, we parametrize manifold $X_{II}$ as follows: for $Z \in X_{II}$ we put $Z = e^{i\theta}U$, where $0 \leqslant \theta \leqslant 2\pi$, and in the matrix $U_1$ the element $u_{11}^{(1)}$ in the left top corner is positive. We denote the manifold of such matrices by $X^+$. This way we parametrize not the whole set $X_{II}$, but some smaller set, which differs from $X_{II}$ by a set of zero measure.

The normalized Lebesgue measure $d\sigma$ can be written as (Lemma 8.4 in [2])

$$d\sigma = \frac{d\theta}{2\pi} d\sigma_1(U) = \frac{1}{2\pi i}\frac{d\xi}{\xi} d\sigma_1(U),$$

where $\xi = e^{i\theta}$, and the measure $\sigma_1$ is positive on $X^+$.

Multiplying equality (3) by $d\sigma_1$ and integrating over $X^+$, from (3) we obtain

$$\int_{X_{II}} f(\psi(Z)) z^{\nu}_{pq} d\sigma(Z) = 0, \tag{4}$$

where $z^{\nu}_{pq}$ are components of vector $Z = (Z_1, Z_2, \ . \ . \ . \ , Z_n)$, $p, q = 1, ...., m$ , $\nu = 1, ...., n$.

We consider the automorphism $\psi_A$ translating the point $A = (A_1, ..., A_n)$ from $B_{II}$ into 0 (see [3]). It is defined up to a generalized unitary transformation.

Then we substitute the automorphism $\psi_A^{-1}$ in (4) instead of $\psi$ and change variables $W = \psi_A^{-1}(Z)$. We get

$$\int_{X_{II}} f(W)\psi^{A,\nu}_{pq}(W) d\sigma(\psi_A(W)) = 0, \tag{5}$$

where $\psi^{A,\nu}_{pq}$ are components of the automorphism $\psi_A$.

By Corollary 7.7 from [2] we have

$$d\sigma(\psi_A(W)) = P(A, W) d\sigma(W),$$

where $P\,(A, W)$ is an invariant Poisson kernel for the matrix ball $B_{II}$ of the second type.

Then, from the condition (5) we have that

$$\int_{X_{II}} f(W)\psi^{A,\nu}_{pq}(W) P(A, W) d\sigma(W) = 0 \tag{6}$$

for all points $A = (A_1, ..., A_n)$ from $B_{II}$ and all $p, q = 1, ...., m$ , $\nu = 1, ...., n$.

Thus, taking into account the properties of the Poisson integral of continuous functions, Theorem 1 follows from the next assertion.

**Theorem 2.** *If for $f \in L^1(X_{II})$ the equality (6) holds for all automorphisms $\psi_A$ of domain $B_{II}$, then $f$ is a radial boundary value of some function $F \in H^1(B_{II})$.*

*Proof.* The invariant Poisson kernel for a matrix ball of the second type has the form

$$P(A, W) = \frac{\left(\det(I - A_1\bar{A}_1 - ..... - A_n\bar{A}_n)\right)^{\frac{(m+1)n}{2}}}{\left|\det(I - A_1\bar{W}_1 - ..... - A_n\bar{W}_n)\right|^{(m+1)n}} =$$

$$= \frac{\left(\det(I - A_1\bar{A}_1 - ..... - A_n\bar{A}_n)\right)^{\frac{(m+1)n}{2}}}{\left(\det(I - A_1\bar{W}_1 - ..... - A_n\bar{W}_n)\right)^{\frac{(m+1)n}{2}}\left(\det(I - W_1\bar{A}_1 - ..... - W_n\bar{A}_n)\right)^{\frac{(m+1)n}{2}}}.$$

We write the elements of matrices $A$ and $W$ in the vector form:

$$A = (A_1, ..., A_n) = \left(a^1_{11}, ..., a^1_{1m}; ....; a^1_{m1}, ..., a^1_{mm}; ....;\ a^n_{11}, ..., a^n_{1m}; ...\right.$$

$$\left. ...; a^n_{m1}, ..., a^n_{mm}\right) = \left(\left\|a^1_{pq}\right\|, ...., \left\|a^n_{pq}\right\|\right),$$

$$W = (W_1, ..., W_n) = \left(w^1_{11}, ..., w^1_{1m}; ....; w^1_{m1}, ..., w^1_{mm}; ....;\ w^n_{11}, ..., w^n_{1m}; ...\right.$$

$$...; w^n_{m1}, ..., w^n_{mm}) = \left(\left\|w^1_{pq}\right\|, ...., \left\|w^n_{pq}\right\|\right),$$

where $\left\|a^\nu_{pq}\right\| = \left\|a^\nu_{qp}\right\|$, $\left\|w^\nu_{pq}\right\| = \left\|w^\nu_{qp}\right\|$, $p, q = 1, ..., m$, $\nu = 1, ..., n$.

We shall compute

$$\sum_{p,q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial P(A, W)}{\partial \bar{a}^\nu_{pq}}. \tag{7}$$

Denote

$$I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n = \|\alpha_{sj}\| \qquad (s, j = 1, ..., m),$$

where

$$\alpha_{sj} = \delta_{sj} - \sum_{k=1}^{m} \sum_{\nu=1}^{n} w^\nu_{sk} \bar{a}^\nu_{jk}, \quad a^\nu_{jk} = a^\nu_{kj}, \quad w^\nu_{sk} = w^\nu_{ks}, \quad s, j = 1, ..., m,$$

and $\delta_{sj}$ is the Kronecker symbol.
Calculations show that

$$\sum_{q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial \det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)}{\partial \bar{a}^\nu_{pq}} =$$

$$= \det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n) - \det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)[p, p],$$

where $\det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)[p, p]$ denotes the cofactor of the element $\alpha_{pp}$ in the matrix $I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n$.

Then

$$\sum_{p,q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial \det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)}{\partial \bar{a}^\nu_{pq}} =$$

$$= m \det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n) - \sum_{p=1}^{m} \det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)\,[p, p].$$

Similarly

$$\sum_{p,q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial \det(I - A_1 \bar{A}_1 - ... - A_n \bar{A}_n)}{\partial \bar{a}^\nu_{pq}} =$$

$$= m \det(I - A_1 \bar{A}_1 - ... - A_n \bar{A}_n) - \sum_{p=1}^{m} \det(I - A_1 \bar{A}_1 - ... - A_n \bar{A}_n)\,[p, p].$$

Therefore, the expression (7) is equal to

$$\frac{m\,(m+1)}{2} n P(A, W) \left[ \frac{\sum\limits_{p=1}^{m} \det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)[p, p]}{\det(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)} - \right.$$

$$\left. - \frac{\sum\limits_{p=1}^{m} \det(I - A_1 \bar{A}_1 - ... - A_n \bar{A}_n)[p, p]}{\det(I^{(m)} - A_1 \bar{A}_1 - ... - A_n \bar{A}_n)} \right] =$$

$$= \frac{m\,(m+1)}{2} n P(A, W) [\mathrm{Sp}\,(I - W_1 \bar{A}_1 - ... - W_n \bar{A}_n)^{-1} - \mathrm{Sp}\,(I - A_1 \bar{A}_1 - ... - A_n \bar{A}_n)^{-1}]. \tag{8}$$

Here Sp, as usual, is the matrix trace.

An automorphism of the domain $B_{II}$ has the form (see [3])

$$\psi_A(W) = \bar{R}^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\sum_{\nu=1}^{n}\left(W_\nu - A_\nu\right)R_{\nu k}, \quad k = 1, ..., n,$$

where $R$ is a block matrix satisfying the condition

$$R'\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)\bar{R} = I.$$

If the condition (6) is satisfied for the components of the map $\psi_A(W)$, the same condition is satisfied for the components of the map

$$\varphi_A(W) = \left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\sum_{\nu=1}^{n}\left(W_\nu - A_\nu\right),$$

since matrices $R$, $\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)$ are nonsingular and depend only on $A$. Then from (6) we get

$$\int_{X_{II}} f(W)\varphi_{pq}^{A,\nu}(W)P(A,W)d\sigma(W) = 0, \tag{9}$$

where $\varphi_{pq}^{A,\nu}(W)$ are the components of the map $\varphi_A(W)$, $(p, q = 1, ..., m, \ \ \nu = 1, ..., n)$.

Now we compute the sum

$$\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\varphi_{p,q}^{A,\nu}.$$

It is obvious that this expression is equal to Sp $\langle\varphi_A(W), A\rangle$, since

$$\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\varphi_{p,q}^{A,\nu} = \text{Sp}\left[\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\times\right.$$
$$\left.\times\left(W_1\bar{A}_1 + ... + W_n\bar{A}_n - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)\right] =$$
$$= \text{Sp}\left[\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\times\right.$$
$$\left.\times\left(\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right) - \left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)\right)\right] =$$
$$= \text{Sp}\left[\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1} - \left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\right], \tag{10}$$

Using this, we get from (9)

$$\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\frac{\partial F(A)}{\partial\bar{a}_{pq}^{\nu}} = 0, \tag{11}$$

where

$$F(A) = \int_{X_{II}} f(W)P(A,W)d\sigma(W) \tag{12}$$

is the Poisson integral of the function $f$.

The function $F(A)$ is real analytic in the domain $B_{II}$. We expand $F(A)$ in a Taylor series in a neighborhood of 0,

$$F(A) = \sum_{|\alpha|,|\beta|\geq 0} C_{\alpha,\beta}a^{\alpha}\bar{a}^{\beta},$$

where $\alpha = \left(\|\alpha_{pq1}\|, ..., \|\alpha_{pqn}\|\right)$ and $\beta = \left(\|\beta_{pq1}\|, ..., \|\beta_{pqn}\|\right)$, $(p, q = 1, ..., m)$ are matrices with nonnegative integer elements and

$$|\alpha| = \sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\alpha_{pq\nu}, \qquad a^{\alpha} = \prod_{p,q=1}^{m}\prod_{\nu=1}^{n}a_{pq\nu}^{\alpha_{pq\nu}}.$$

Then (11) implies

$$\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\frac{\partial F(A)}{\partial \bar{a}_{pq}^{\nu}} = \sum_{|\alpha|,|\beta|} |\beta|\, C_{\alpha,\beta} a^{\alpha}\bar{a}^{\beta} = 0.$$

It follows that for $|\beta| > 0$ all coefficients $C_{\alpha,\beta}$ are equal to zero. So, the function $F(A)$ is holomorphic in $B_{II}$ and belongs to the class $H^1(B_{II})$.

If $f$ is continuous on $X_{II}$, then the function $F$ belongs to $C(\bar{B}_{II})$ and its boundary values on $X_{II}$ concide with $f$. □

The proof of this theorem shows that it remains true if the conditions (3) and (6) are satisfied only for those automorphisms $\psi_A$, for which the point $A = (A_1, ..., A_n)$ lies in some open set $V \subset B_{II}$. Therefore the following statement is true.

**Theorem 3.** *If a function $f \in L^1(X_{II})$ satisfies the condition* (6) *for all points lying in some open set $V \subset B_{II}$ and for all components of the automorphism $\psi_A$, then $f$ is a radial boundary value for some function $F \in H^1(B_{II})$ on $X_{II}$.*

# References


[1] A.Nagel, W.Rudin, Moebius-invariant functions spaces on balls and spheres, *Duke Math. J.*, **43**(1976), no. 4, 841–865.

[2] G.Khudayberganov, A.M.Kytmanov, B.Shaimkulov, Complex analysis in matrix domains, Krasnoyarsk, Siberian Federal University, 2011 (in Russian).

[3] G.Khudayberganov, B.B.Hidirov, U.S.Rakhmonov, Automorphisms of matrix balls, *Doklady NUUz*, (2010), no. 3, 205–210 (in Russian).

[4] S.Kosbergenov, On multidimensional boundary Morera's theorem for matrix ball, *Izvestiya VUZov. Matematika*, (2001), no. 4, 28–32 (in Russian).

[5] P.Lankaster, The theory of matrices, Academic Press, New York–London, 1969.

[6] F.R.Gantmakher, The theory of matrices, Chelsea Publition Company, 1977.

[7] S.Kosbergenov, A.M.Kytmanov, S.G.Myslivets, On a boundary Morera theorem for the classical domains, *Sib. Math. J.*, **40**(1999), no. 3, 506–514.


# Граничный вариант теоремы Морера для матричного шара второго типа


**Гулмирза Х. Худайберганов**
**Зокирбек М. Матайкубов**



*В этой статье доказывается граничная теорема Морера для матричного шара второго типа.*



*Ключевые слова: матричный шар, автоморфизм, ядро Пуассона, теорема Морера.*