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# On Long-Term Behavior of Continuous-Time Markov Branching Processes Allowing Immigration

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*We observe the continuous-time Markov Branching Process allowing Immigration. Limit properties of transition functions and their convergence to invariant measures are investigated. A speed of this convergence is defined.*

*Keywords: Markov Branching Process, immigration, transition functions, invariant measures, rate of convergence.*

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## 1. Introduction and preliminaries

We are interested in the model of evolution of particles so-called a branching process allowing immigration. The mentioned process can have a simple physical interpretation: a population size changes not only as a result of reproduction and disappearance of existing particles, but also at the random stream of inbound "extraneous" particles of the same type from outside. Similar processes, apparently, have been considered first by Bartlett in [3]. Sevastyanov [11] has defined the processes allowing immigration as a special case of two-type branching process. In a case of birth and death process the similar model was considered by Karlin and McGregor [7]. We adhere on the model of population growth entered by Sevastyanov, called the Markov Branching Process allowing Immigration (MBPI) in which states form a homogeneous Markov chain on the set of  $\mathbb{N}_0 = 0 \cup \mathbb{N}$ .

Let  $X(t)$ ,  $t \in \mathcal{T} = [0; +\infty)$ , be the population size in MBPI, in which evolution of individuals occurs by the following scheme. Each individual existing at epoch  $t$  independently of his history and of each other for a small time interval  $(t; t + \varepsilon)$  transforms into  $j \in \mathbb{N}_0 \setminus \{1\}$  individuals with probability  $a_j \varepsilon + o(\varepsilon)$  and, with probability  $1 + a_1 \varepsilon + o(\varepsilon)$  stays to live or makes evenly one descendant (as  $\varepsilon \rightarrow 0$ ). Here  $\{a_j\}$  represent intensities of individuals' transformation that  $a_j \geq 0$  for  $j \in \mathbb{N}_0 \setminus \{1\}$  and  $0 < a_0 < -a_1 = \sum_{j \in \mathbb{N}_0 \setminus \{1\}} a_j < \infty$ . Independently of these for this time interval  $j \in \mathbb{N}$  new individuals enter the population with probability  $b_j \varepsilon + o(\varepsilon)$  and immigration is absent with probability  $1 + b_0 \varepsilon + o(\varepsilon)$ . Immigration intensities  $b_j \geq 0$  for  $j \in \mathbb{N}$  and  $0 < -b_0 = \sum_{j \in \mathbb{N}} b_j < \infty$ . Appeared individuals undergo transformations under the reproduction law generated by intensities  $\{a_j\}$ . So MBPI  $X(t)$  is completely defined by infinitesimal generating

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functions (GFs) (see [11])

$$f(s) = \sum_{j \in \mathbb{N}_0} a_j s^j \quad \text{and} \quad g(s) = \sum_{j \in \mathbb{N}_0} b_j s^j.$$

We know that  $X(t)$  is homogenous continuous-time Markov chain. Owing to the Markovian nature of this process transition functions

$$p_{ij}(t) := \mathbf{P}_i \{X(t) = j\} = \mathbf{P} \{X(t + \tau) = j | X(\tau) = i\}$$

satisfy to Kolmogorov-Chapman equation

$$p_{ij}(t) = \sum_{k \in \mathbb{N}} p_{ik}(\tau) \cdot p_{kj}(t - \tau), \quad \tau \leq t, \quad (1.1)$$

for all  $i, j \in \mathbb{N}$  and  $\tau, t \in \mathcal{T}$ . A corresponding probability GF

$$\mathcal{P}_i(t; s) := \mathbf{E}_i s^{X(t)} = \mathbf{E} \left[ s^{X(t)} | X(0) = i \right] = \sum_{j \in \mathbb{N}_0} p_{ij}(t) s^j$$

has a following form (see [11]):

$$\mathcal{P}_i(t; s) = F_i(t; s) \exp \left\{ \int_0^t g(F(\tau; s)) d\tau \right\}, \quad (1.2)$$

where the GF  $F_i(t; s) = \mathbf{E}_i s^{Z(t)}$  and  $Z(t)$  represents Markov Branching Process (MBP) without immigrations generated by GF  $f(s)$ . From the fundamental extinction theorem it follows that  $F_i(t; s) = [F(t; s)]^i \rightarrow q^i$  uniformly for  $0 \leq s < 1$ , where  $q$  is the extinction probability of the MBP  $Z(t)$ ; see [12, p.53]. Therefore, in view of the formula (1.2)

$$\frac{\mathcal{P}_i(t; s)}{\mathcal{P}(t; s)} \rightarrow q^i < \infty, \quad (1.3)$$

where  $\mathcal{P}(t; s) := \mathcal{P}_0(t; s)$ .

Moments of  $X(t)$  for any  $t \in \mathcal{T}$  are expressed by corresponding factorial moments of GF  $f(s)$  and  $g(s)$ . Designating

$$a = f'(1) \quad \text{and} \quad \alpha = g'(1),$$

we see that  $a\varepsilon + o(\varepsilon)$  denote the mean per capita number of single individual during  $(t; t + \varepsilon)$  as  $\varepsilon \rightarrow 0$ , and  $\alpha\varepsilon + o(\varepsilon)$  is mean of immigrants for this time interval. The case  $\alpha = 0$  corresponds to the MBP without immigration since then  $g(s) \equiv 0$ . In this sense the process  $X(t)$  is generalisation of MBP  $Z(t)$ .

Classification of states is the fundamental problem of the theory of MBPI. Differentiating (1.2) in a point of  $s = 1$  entails

$$\mathbf{E}_i X(t) = \sum_{j \in \mathbb{N}_0} j p_{ij}(t) = \begin{cases} \left( \frac{\alpha}{a} + i \right) e^{at} - \frac{\alpha}{a}, & a \neq 0, \\ \alpha t + i, & a = 0. \end{cases} \quad (1.4)$$

From (1.4) follows that in case of  $a < 0$  a limit

$$\lim_{t \rightarrow \infty} \mathbf{E}_i X(t) = \frac{\alpha}{|a|} < \infty,$$

and in the supercritical case  $\mathbf{E}_i X(t)$  has an asymptotic exponential growth:

$$\mathbf{E}_i X(t) \sim \left( \frac{\alpha}{a} + i \right) e^{at}, \quad t \rightarrow \infty.$$

Last statements denote various behaviors of trajectories of the process  $X(t)$  depending on value of parameter  $a = f'(1)$ . According to the general classification the MBPI is designated as subcritical, critical and supercritical, if  $a < 0$ ,  $a = 0$  and  $a > 0$ , respectively.

In this paper we observe limit properties of transition function  $p_{ij}(t)$ , and also problems concerning an ergodic property of states and existence of an invariant (stationary) measure of process  $X(t)$ .

Ergodic properties of arbitrary continuous-time Markov chain are in detail investigated in the monograph of Anderson [1, Chapter 6]. First results concerning existence of invariant measures for MBPI have been received by Sevastyanov in his fundamental researches [11]. Conner [4] investigated invariant properties of MBPI in the critical case. Seneta [10] has established a unique correspondence between properties of invariant measures of a branching process with immigration and those of the process without immigration in a discrete-time case. Yang [13] considers a subcritical case. Pakes [9] studied all cases. Li, Chen and Pakes [8] have generalized results of paper [9].

Sevastyanov [12] has proved that if the first moment of immigration intensity  $g'(1)$  is finite, then for subcritical case there are finite limits  $\lim_{t \rightarrow \infty} p_{0j}(t)$  and corresponding GF  $\mathcal{P}(t; s)$  converges to the limit one:

$$\mathcal{P}(t; s) \rightarrow \exp \left\{ \int_s^1 \frac{g(x)}{f(x)} dx \right\}, \quad t \rightarrow \infty. \quad (1.5)$$

Yang [13] has improved Sevastyanov's result, having established that GF defined in (1.5) at minimal moment condition of  $\sum_{j \in \mathbb{N}} b_j \ln j < \infty$  generates an invariant distribution.

It is easy to see that GF  $\widehat{\mathcal{P}}(t; s) = \mathcal{P}(t; qs)$  generates subcritical MBPI in which offspring law obeys the GF  $\widehat{f}(s) = f(qs)/q$  and the immigration size law has the GF  $\widehat{g}(s) = g(qs)$ . According to the convergence (1.5), if  $\sum_{j \in \mathbb{N}} j \ln \widehat{b}_j < \infty$ , where  $\widehat{b}_j$  are positive coefficients in the power series expansion of  $\widehat{g}(s)$ , then

$$\widehat{\mathcal{P}}(t; s) \rightarrow \exp \int_s^1 \frac{\widehat{g}(x)}{\widehat{f}(x)} dx, \quad t \rightarrow \infty.$$

We re-join our designation and receive that if  $\sum_{j \in \mathbb{N}} b_j q^j \ln j < \infty$  then in case of  $a \neq 0$

$$\mathcal{P}(t; s) \rightarrow \exp \left\{ \int_s^q \frac{g(x)}{f(x)} dx \right\}, \quad t \rightarrow \infty, \quad (1.6)$$

for all  $0 \leq s < q$ .

In critical case Sevastyanov [11] proved that if the offspring law has a finite variance and the immigration size law has a finite mean then the normalized process  $2X(t)/f''(1)t$  has a limiting Gamma distribution function  $\Gamma_{1, \lambda}(x)$ ,  $x \geq 0$ , where  $\lambda = 2g'(1)/f''(1)$ . In this case Pakes [9] has proved a convergence of  $t^\lambda \mathcal{P}_i(t; s)$  to a limit GF  $\sum_{j \in \mathbb{N}_0} \pi_j s^j$ , where non-negative numbers  $\{\pi_j\}$  represent an invariant measure for  $X(t)$ .

In Section 2 we observe limit properties of transition functions  $p_{ij}(t)$  and their convergence to invariant measures. In supercritical case results of paper [8] are recurred and discussed. In

critical case the new proof of mentioned theorem from [9] about convergence to invariant measure at minimal moment conditions is shown.

Section 3 is devoted to estimate of speed of convergence of  $\pi_j$  to invariant measures. In particular, in the critical case we prove that a rate of speed of convergence of  $t^\lambda p_{ij}(t)$  to the  $\pi_j$  is  $O(\ln t/t)$ .

## 2. Ergodic properties of transition functions

Observing limit properties of transition functions  $p_{ij}(t)$ , in this section we are interested in ergodicity property of the chain  $X(t)$  and observe a problem of existence of invariant measure. For our purpose we need to the statement about a limit behavior of ratio  $p_{ij}(t)/p_{00}(t)$ . In particular, putting  $s = 0$  in (1.3) gives  $p_{i0}(t)/p_{00}(t) \rightarrow q^i$ . The following more general statement, the monotone ratio lemma is proved in [8].

**Lemma 1** ([8]). *For all  $j \in \mathbb{N}$*

$$\frac{p_{ij}(t)}{p_{00}(t)} \uparrow q^i v_j < \infty, \quad t \rightarrow \infty, \quad (2.1)$$

where positive numbers  $v_j = \lim_{t \rightarrow \infty} p_{0j}(t)/p_{00}(t)$  are in the power series expansion of

$$\mathcal{U}(s) = \exp \left\{ \int_0^s \frac{g(q) - g(u)}{f(u)} du \right\}, \quad (2.2)$$

that converges on set of  $0 \leq s < 1$ .

From Kolmogorov-Chapman equation (1.1) it follows

$$\frac{p_{0j}(t + \tau)}{p_{00}(t + \tau)} \cdot \frac{p_{00}(t + \tau)}{p_{00}(t)} = \sum_{k \in \mathbb{N}_0} \frac{p_{0k}(t)}{p_{00}(t)} \cdot p_{kj}(\tau).$$

In other hand it is easily to see

$$\frac{p_{00}(t + \tau)}{p_{00}(t)} \uparrow e^{g(q)\tau}, \quad t \rightarrow \infty,$$

for any  $\tau \in \mathcal{T}$ . Then taking limit as  $t \rightarrow \infty$  from last but one relation and considering (2.1) directly appears the invariant equation

$$e^{g(q)t} \cdot v_j = \sum_{i \in \mathbb{N}_0} v_i p_{ij}(t), \quad j \in \mathbb{N}_0. \quad (2.3)$$

Let's consider the case  $a \neq 0$ . Statements (2.1), (2.3) suggest to consider the normalized GF  $\mathcal{P}(t; s)/e^{g(q)t}$ . So due to (1.3) and (1.6) come out that if  $\sum_{j \in \mathbb{N}} b_j q^j \ln j < \infty$  then

$$e^{|g(q)|t} \cdot \mathcal{P}_i(t; s) \rightarrow q^i \cdot \mathcal{C}(s), \quad t \rightarrow \infty, \quad (2.4)$$

for all  $0 \leq s < q$ , where limiting GF  $\mathcal{C}(s) = \sum_{j \in \mathbb{N}_0} \sigma_j s^j$  has a form of

$$\mathcal{C}(s) = \exp \left\{ \int_s^q \frac{g(x) - g(q)}{f(x)} dx \right\} \quad (2.5)$$

In [9] the assertion (2.4) stated in virtue of corresponding discrete time result. Putting  $s = 0$  gives the following local limit property:

$$e^{|g(q)|t} \cdot p_{00}(t) \rightarrow \mathcal{C}(0) < \infty, \quad t \rightarrow \infty \quad (2.6)$$

since integrand in (2.5) is finite as  $s \uparrow q$ . Considering together (2.2) and (2.4)–(2.6) ensues a following formula about interrelation of functions  $\mathcal{U}(s)$  and  $\mathcal{C}(s)$ :

$$\mathcal{U}(s) = \frac{\mathcal{C}(s)}{\mathcal{C}(0)}. \quad (2.7)$$

The last form in the context of transition functions could be written as

$$\frac{p_{ij}(t)}{p_{00}(t)} \uparrow q^i \frac{\sigma_j}{\mathcal{C}(0)}, \quad t \rightarrow \infty.$$

From last reasons and Lemma 1 it directly follows

$$e^{g(q)t} \cdot \mathcal{C}(s) = \mathcal{P}(t; s) \cdot \mathcal{C}(F(t; s)) \quad (2.8)$$

for  $0 \leq s < q$ . The relation (2.8) shows that for case  $a > 0$  transition functions  $p_{ij}(t)$  are exponentially decrease to zero. The limit

$$\lambda_X = - \lim_{t \rightarrow \infty} \frac{\ln p_{ii}(t)}{t}$$

denotes a decay parameter of the state space of MBPI. The process  $X(t)$  is called as  $\lambda_X$ -recurrent if  $\int_0^{+\infty} e^{\lambda_X t} p_{ii}(t) dt = \infty$  and  $\lambda_X$ -transient otherwise. Mote over the chain is subdivided as  $\lambda_X$ -positive if  $\lim_{t \rightarrow \infty} e^{\lambda_X t} p_{ii}(t) > 0$  and  $\lambda_X$ -null if this limit is zero. According to results of [8] if  $\sum_{j \in \mathbb{N}} b_j q^j \ln j < \infty$ , then  $\lambda_X = |g(q)|$  and  $X(t)$  is  $\lambda_X$ -positive. The set of non-negative numbers  $\{\sigma_j\}$  generated by GF  $\mathcal{C}(s)$  is the unique (up to multiplicative constant)  $\lambda_X$ -invariant measure.

In subcritical case the set  $\{\sigma_j\}$  is an invariant distribution having a finite mean

$$\mathcal{C}'(1) = \sum_{j \in \mathbb{N}} j \sigma_j = g'(1)/|a|.$$

Now consider the case  $a = 0$ . After minor reasoning from (1.2) it is possible to be convinced that  $p_{00}(t) = O(t^{-\lambda})$  as  $t \rightarrow \infty$ . Ipso facto Pakes [9] observed the limit  $\pi(s) := \lim_{t \rightarrow \infty} t^\lambda \mathcal{P}_i(t; s)$ . He proves this limit exists if

$$\sum_{j \in \mathbb{N}} a_j j^2 \ln j < \infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} b_j j \ln j < \infty, \quad (2.9)$$

and it has a form

$$\pi(s) = \frac{1}{[b(1-s)]^\lambda} \exp \left\{ \int_s^1 \left[ \frac{g(u)}{f(u)} + \frac{\lambda}{1-u} \right] du \right\} \quad (2.10)$$

Herewith embedding techniques for corresponding discrete time result are used. We show below that abovementioned result holds if moments  $\sum_{j \in \mathbb{N}} j^2 a_j$  and  $\sum_{j \in \mathbb{N}} j b_j$  are finite instead of conditions (2.9).

**Theorem 1.** *If in critical MBPI  $2b := f''(1) < \infty$ ,  $\alpha := g'(1) < \infty$  and  $\lambda = \alpha/b$ , then*

$$t^\lambda \mathcal{P}_i(t; s) \rightarrow \pi(s), \quad t \rightarrow \infty, \quad (2.11)$$

where GF  $\pi(s) = \sum_{j \in \mathbf{N}_0} \pi_j s^j$  has the form of (2.10) and set of non-negative numbers  $\{\pi_j\}$  is invariant measure for  $X(t)$ .

*Proof.* According to relation (1.3), it suffices to consider the case  $i = 0$ . Designating  $R(t; s) = 1 - F(t; s)$  and setting  $u = F(\tau; s)$  it follows from (1.2) that

$$\begin{aligned} t^\lambda \mathcal{P}(t; s) &= \exp \left\{ \lambda \ln t + \int_0^t g(F(\tau; s)) d\tau \right\} = \\ &= \exp \left\{ \lambda \ln [tR(t; s)] - \lambda \ln R(t; s) + \int_s^{F(t; s)} \frac{g(u)}{f(u)} du \right\} = \\ &= \exp \left\{ \lambda \ln [tR(t; s)] + \lambda \int_0^{F(t; s)} \frac{1}{1-u} du + \int_s^{F(t; s)} \frac{g(u)}{f(u)} du \right\} = \\ &= \exp \left\{ \lambda \ln [tR(t; s)] + \int_s^{F(t; s)} \left[ \frac{g(u)}{f(u)} + \frac{\lambda}{1-u} \right] du + \ln(1-s)^{-\lambda} \right\}. \end{aligned}$$

In turn, it is known that if the second moment  $2b := f''(1)$  is finite then  $tR(t; s) \rightarrow b$  as  $t \rightarrow \infty$  for all  $0 \leq s < 1$ ; see [12, p.73]. Hence taking limit as  $t \rightarrow \infty$  we receive (2.11).

Now from the formula (1.2) we will write out the following chain of equalities:

$$\begin{aligned} \mathcal{P}(t + \tau; s) &= \exp \left\{ \int_0^{t+\tau} g(F(u; s)) du \right\} = \\ &= \mathcal{P}(\tau; s) \cdot \exp \left\{ \int_\tau^{t+\tau} g(F(u; s)) du \right\} = \\ &= \mathcal{P}(\tau; s) \cdot \exp \left\{ \int_0^t g(F(u; F(\tau; s))) du \right\}. \end{aligned}$$

In junction of last equality we replaced  $v = u - \tau$  and used the well-known functional equation  $F(t + \tau; s) = F(t; F(\tau; s))$ ; see [12, p. 24]. Thus

$$\mathcal{P}(t + \tau; s) = \mathcal{P}(\tau; s) \cdot \mathcal{P}(t; F(\tau; s)).$$

Considering (2.11) it follows from this the invariant functional equation

$$\pi(s) = \mathcal{P}(t; s) \cdot \pi(F(t; s)),$$

that has a transition functions version as

$$\pi_j = \sum_{i \in \mathbf{N}_0} \pi_i p_{ij}(t), \quad j \in \mathbf{N}_0.$$

The theorem is proved. □

The following theorem describes main properties of GF  $\pi(s)$ .

**Theorem 2.** *In conditions of Theorem 1 GF  $\pi(s) = \sum_{j \in \mathbf{N}_0} \pi_j s^j$  is positive and strongly increasing for  $0 \leq s < 1$ . If in addition suppose  $\sum_{j \in \mathbf{N}} b_j j \ln j < \infty$ , then*

$$\frac{1}{n^\lambda} [\pi_0 + \pi_1 + \dots + \pi_n] \rightarrow \frac{1}{\Gamma(\lambda + 1) b^\lambda}, \quad n \rightarrow \infty, \quad (2.12)$$

where  $\Gamma(\ast)$  is the Euler's Gamma function.

*Proof.* Positiveness of  $\pi(s)$  is obvious. Direct differentiating implies

$$\pi'(s) = -\frac{g(s)}{f(s)}\pi(s).$$

In the considering case GF  $f(s)$  monotonously decreases from  $f(0) > 0$  to  $f(1) = 0$ , and GF  $g(s)$  monotonously increases from  $g(0) < 0$  to  $g(1) = 0$ . Therefore  $\pi'(s) > 0$  for all  $0 \leq s < 1$ .

It is easy to see that in additional condition the function

$$\mathcal{B}(s) := \exp \left\{ - \int_s^1 \left[ \frac{g(u)}{f(u)} + \frac{\lambda}{1-u} \right] du \right\} \quad (2.13)$$

is bounded for  $0 \leq s < 1$ . So that

$$\pi(s) \sim \frac{1}{b^\lambda (1-s)^\lambda}, \quad s \uparrow 1.$$

According to Hardy-Littlewood Tauberian theorem, from last formula it follows (2.12).  $\square$

**Corollary 1.** *If conditions of Theorem 1 occur and in addition  $\sum_{j \in \mathbb{N}} b_j j \ln j < \infty$ , then*

$$t^\lambda p_{00}(t) \rightarrow \frac{1}{b^\lambda \mathcal{B}(0)}, \quad t \rightarrow \infty,$$

where function  $\mathcal{B}(s)$  is defined in (2.13).

### 3. A speed rate of convergence to invariant measures

Recall the GF  $F(t; s) = \mathbf{E}s^{Z(t)}$ , where  $Z(t)$  is MBP without immigration. This GF is the solution of backward Kolmogorov equation (see [12, p.27])

$$\frac{\partial F(t; s)}{\partial t} = f(F(t; s)), \quad (3.1)$$

with initial condition  $F(0; s) = s$ , here  $f(s)$  is infinitesimal GF defined in Section 1.

Let  $a \neq 0$ . Multiplying to  $f'(q) \cdot (F(t; s) - q)$  the equation (3.1) we transform as

$$\frac{dF(t; s)}{F(t; s) - q} \cdot \left[ 1 - \frac{f(F(t; s)) - f'(q) \cdot (F(t; s) - q)}{f(F(t; s))} \right] = f'(q) dt.$$

Integrating this equation on  $[0; t] \subset \mathcal{T}$  it receives

$$\frac{R(t; s)}{R(0; s)} = \beta^t \exp \left\{ \int_s^{F(t; s)} \left[ \frac{1}{u - q} - \frac{f'(q)}{f(u)} \right] du \right\}, \quad (3.2)$$

where  $R(t; s) = q - F(t; s)$  and hereinafter  $\beta := \exp \{f'(q)\}$ . Since  $R(0; s) = q - s$  and  $\sup_{0 \leq s < 1} F(t; s) \rightarrow q$ , taking limit in (3.2) as  $t \rightarrow \infty$  entails the following assertion.

**Lemma 2.** *If  $a \neq 0$ , then*

$$R(t; s) = \mathcal{A}(s) \cdot \beta^t (1 + o(1)), \quad t \rightarrow \infty, \quad (3.3)$$

for  $0 \leq s < 1$ , where

$$\mathcal{A}(s) = (q - s) \exp \left\{ \int_s^q \left[ \frac{1}{u - q} - \frac{f'(q)}{f(u)} \right] du \right\}. \quad (3.4)$$

Note that the Lemma 2 in [6] was proved for the case of  $a > 0$  only.

In considering case our discussion will depend on the function  $\mathcal{A}(s)$ . Thereby we have to observe properties of this function in detail.

**Lemma 3.** *The function  $\mathcal{A}(s)$  is continuously, monotone decreasing and concave for  $0 \leq s < 1$ . Moreover if  $a > 0$  or  $a < 0$  and*

$$\sum_{j \in \mathbb{N}} a_j j \ln j < \infty, \quad (3.5)$$

then  $0 < \mathcal{A}(0) < \infty$ ,  $\mathcal{A}(q) = 0$ ,  $\mathcal{A}'(q) = -1$ . This function is a solution of the Schroeder equation

$$\mathcal{A}(F(t; s)) = \beta^t \cdot \mathcal{A}(s) \quad (3.6)$$

and this solution is unique for  $0 \leq s < q$ .

*Proof.* In fact the function  $\mathcal{A}(s)$  is defined on the set of  $0 \leq s < 1$ , since that is result of (3.2) as  $t \rightarrow \infty$ . Its continuity is obvious. From (3.4) we have

$$\mathcal{A}'(s) = \frac{f'(q)}{f(s)} \mathcal{A}(s). \quad (3.7)$$

It is known that GF  $f(s)$  is convex everywhere. For  $0 \leq s < q$  it is strictly positive and monotone decreasing. As  $\mathcal{A}(s) > 0$  and  $f'(q) < 0$  it follows  $\mathcal{A}'(s) < 0$ . Hence the function  $\mathcal{A}(s)$  is monotone decreasing. By the same reasoning we will be convinced that  $\mathcal{A}(s)$  to be monotone decreasing for  $q \leq s < 1$ .

We know that in point of  $s = q$  the GF  $f(s)$  changes its sign from plus to minus and its derivative  $f'(s)$  monotonously increase. Therefore considering  $\mathcal{A}'(s) < 0$  we find out that

$$\mathcal{A}''(s) = \frac{f'(q) - f'(s)}{f(s)} \cdot \mathcal{A}'(s) < 0.$$

This implies the concavity of  $\mathcal{A}(s)$ .

In case  $a < 0$  the condition (3.5) is equivalent to that

$$\int_0^1 \frac{f(u) - a(u-1)}{(u-1)f(u)} du = \ln \mathcal{A}(0) < \infty; \quad (3.8)$$

see [12, p.57]. We see that that  $\mathcal{A}(0) > 0$  and this is finite. In the case  $a > 0$  we can easily be convinced that  $0 < \mathcal{A}(0) < \infty$  from (3.4). The assertion  $\mathcal{A}(q) = 0$  directly follows from (3.8) in the case  $a < 0$ . If  $a > 0$ , then the integrand in (3.4) stays bounded as  $s \rightarrow q$  and hence  $\mathcal{A}(q) = 0$ . Considering  $f(s) \sim f'(q)(s-q)$  as  $s \rightarrow q$ , it follows from (3.4) and (3.7) that

$$\mathcal{A}'(q) = \lim_{s \rightarrow q} \mathcal{A}'(s) = \lim_{s \rightarrow q} \frac{f'(q)}{f(s)} \mathcal{A}(s) = \lim_{s \rightarrow q} \frac{\mathcal{A}(s)}{s-q} = -1.$$

Now designating  $K(u)$  the integrand in (3.4) we see that function  $\mathcal{A}(s)$  actually satisfies the equation (3.6):

$$\begin{aligned} \beta^t \mathcal{A}(s) &= (q-s)\beta^t \exp \left\{ \int_s^{F(t;s)} K(u) du \right\} \exp \left\{ \int_{F(t;s)}^q K(u) du \right\} = \\ &= (q-F(t;s)) \exp \left\{ \int_{F(t;s)}^q K(u) du \right\} = \mathcal{A}(F(t;s)). \end{aligned}$$



In the last equality we used (3.2).

To observe the uniqueness of the solution of equation (3.6) we follow the method from [2, p.14]. Suppose  $\tilde{\mathcal{A}}(s)$  to be an arbitrary solution of (3.6). Then it as well as  $\mathcal{A}(s)$  satisfies to equation

$$\mathcal{A}'(F(t; s)) \cdot F'(t; s) = \beta^t \cdot \mathcal{A}'(s). \quad (3.9)$$

Hereinafter, if not otherwise stated, the derivative symbol for the function  $F(t; s)$  should be understood by  $s$ . It follows from (3.9)

$$\frac{\mathcal{A}'(s)}{\tilde{\mathcal{A}}'(s)} = \frac{\mathcal{A}'(F(t; s))}{\tilde{\mathcal{A}}'(F(t; s))}. \quad (3.10)$$

We have already proved that the solution of (3.6) is concave, hence both  $\mathcal{A}'(s)$  and  $\tilde{\mathcal{A}}'(s)$  are monotone decrease. Since  $F(t; 0) \uparrow q$  for all  $0 \leq s < q$ , there always exists some  $\tau \in \mathcal{T}$  and some arbitrary small  $\varepsilon \in \mathcal{T}$  such that  $F(\tau; 0) \leq s \leq F(\tau + \varepsilon; 0)$ . Then by combining the equalities (3.9) and (3.10) we can write following relations:

$$\begin{aligned} \frac{\mathcal{A}'(s)}{\tilde{\mathcal{A}}'(s)} &\leq \frac{\mathcal{A}'(F(t; F(\tau; 0)))}{\tilde{\mathcal{A}}'(F(t; F(\tau + \varepsilon; 0)))} \leq \\ &\leq \frac{\mathcal{A}'(F(t + \tau; 0))}{\tilde{\mathcal{A}}'(F(t + \tau; 0))} \cdot \frac{\tilde{\mathcal{A}}'(F(t + \tau; 0))}{\tilde{\mathcal{A}}'(F(t + \tau + \varepsilon; 0))} \leq \\ &\leq \frac{\mathcal{A}'(0)}{\tilde{\mathcal{A}}'(0)} \cdot \frac{F'(t + \tau + \varepsilon; 0)}{F'(t + \tau; 0) \cdot \beta^\varepsilon} = \frac{\mathcal{A}'(0)}{\tilde{\mathcal{A}}'(0)} \cdot \frac{F'(\varepsilon; F(t + \tau; 0))}{\beta^\varepsilon}. \end{aligned} \quad (3.11)$$

Since  $F(t; 0) \uparrow q$ , we see  $F'(\varepsilon; F(t; 0)) \uparrow \beta^\varepsilon$  as  $t \rightarrow \infty$ . Undoubtedly that  $F'(t; q) = \beta^t$ . So taking limit as  $t \rightarrow \infty$  of right side of (3.11) gives

$$\frac{\mathcal{A}'(s)}{\tilde{\mathcal{A}}'(s)} \leq \frac{\mathcal{A}'(0)}{\tilde{\mathcal{A}}'(0)}.$$

A similarly reasoning implies a converse inequality. Thus we have

$$\frac{\mathcal{A}'(s)}{\tilde{\mathcal{A}}'(s)} = \frac{\mathcal{A}'(0)}{\tilde{\mathcal{A}}'(0)} = \text{const},$$

As  $\mathcal{A}(0) = \tilde{\mathcal{A}}(0)$ , then  $\mathcal{A}(s) = \tilde{\mathcal{A}}(s)$ . The Lemma 3 is proved completely.  $\square$

Further, according to Lemma 1

$$\frac{\mathcal{P}(t; s)}{\mathcal{P}(t; 0)} = \exp \left\{ \int_0^t [g(F(\tau; s)) - g(F(\tau; 0))] d\tau \right\} \rightarrow \mathcal{U}(s).$$

Using this relation gives

$$\begin{aligned} e^{|g(q)|t} \mathcal{P}(t; s) &= \exp \left\{ \int_0^t [g(F(\tau; s)) - g(q)] d\tau \right\} \sim \\ &\sim \mathcal{U}(s) \cdot \exp \left\{ \int_0^t [g(F(\tau; 0)) - g(q)] d\tau \right\} \sim \\ &\sim \mathcal{U}(s) \cdot \exp \left\{ \int_0^{F(t; 0)} \frac{g(u) - g(q)}{f(u)} d\tau \right\} \sim \\ &\sim \mathcal{U}(s) \cdot \mathcal{C}(0) \cdot \exp \left\{ \int_q^{F(t; 0)} \frac{g(u) - g(q)}{f(u)} d\tau \right\}, \end{aligned}$$

as  $t \rightarrow \infty$ . From here, having designation

$$\mathcal{H}(s) := \exp \left\{ \int_s^q \frac{g(q) - g(u)}{f(u)} du \right\},$$

for  $0 \leq s < q$  and taking into account (2.7), obtain

$$e^{|g(q)|t} \mathcal{P}(t; s) \sim \mathcal{C}(s) \cdot \mathcal{H}(F(t; 0)), \quad t \rightarrow \infty. \quad (3.12)$$

Using the Taylor expansion for  $\mathcal{H}(s)$  it follows

$$\mathcal{H}(s) \sim 1 + \frac{g'(q)}{f'(q)}(s - q), \quad s \uparrow q. \quad (3.13)$$

Combining now relations (3.12) and (3.13), taking into account convergence  $F(t; 0) \rightarrow q$  we draw a conclusion that

$$e^{|g(q)|t} \mathcal{P}(t; s) \sim \mathcal{C}(s) \cdot \left( 1 + \frac{g'(q)}{|f'(q)|} R(t) \right), \quad t \rightarrow \infty.$$

We use the received asymptote together with the formula (3.3) in equality (1.2). Then considering that  $F_i(t; s) \sim q^i - iq^{i-1}R(t; s)$ , we write the following theorem which gives an estimation of speed of convergence in (2.4).

**Theorem 3.** *Let  $a \neq 0$ . If  $\sum_{j \in \mathbb{N}} b_j q^j \ln j < \infty$ , then*

$$e^{|g(q)|t} \mathcal{P}_i(t; s) = q^i \mathcal{C}(s) \cdot \left( 1 + \left( \frac{g'(q)}{|f'(q)|} - \frac{i}{q} \right) \mathcal{A}(s) \beta^t (1 + o(1)) \right),$$

as  $t \rightarrow \infty$ , where limit GF  $\mathcal{C}(s)$  has the form of (2.5) and function  $\mathcal{A}(s)$  defined in (3.4) and  $\beta = \exp \{f'(q)\}$  as before.

Using the continuity theorem of GF attracts from the Theorem 3 the following statement.

**Corollary 2.** *In conditions of Theorem 3 a following representation holds:*

$$e^{|g(q)|t} p_{ij}(t) = q^i \sigma_j \cdot \left( 1 + \left( \frac{g'(q)}{|f'(q)|} - \frac{i}{q} \right) \mathcal{A}(0) \beta^t (1 + o(1)) \right),$$

as  $t \rightarrow \infty$ .

In critical case we have to use the following lemma.

**Lemma 4** ([5]). *Let  $a = 0$  and  $2b := f''(1)$ . If  $c = f'''(1) < \infty$ , then*

$$R(t; s) = \frac{1}{bt} + \frac{c}{6b^3} \frac{\ln bt(1-s)}{t^2} + \varepsilon(t; s), \quad (3.14)$$

as  $t \rightarrow \infty$ , where

$$\sup_{0 \leq s < 1} |\varepsilon(t; s)| = o\left(\frac{\ln t}{t^2}\right).$$

The following theorem holds.

**Theorem 4.** *Let in critical MBPI  $2b := f''(1)$ ,  $\alpha := g'(1)$  and  $\lambda = \alpha/b$ . If  $c := f'''(1) < \infty$  and  $g''(1) < \infty$ , then*

$$t^\lambda \mathcal{P}(t; s) = \pi(s) \cdot \left( 1 + \frac{\alpha c}{6b^3} \cdot \frac{\ln bt(1-s)}{t} (1 + o(1)) \right), \quad t \rightarrow \infty. \quad (3.15)$$

*Proof.* Repeating initial reasoning in the proof of Theorem 1, we have

$$\begin{aligned}
t^\lambda \mathcal{P}(t; s) &= \exp \left\{ \lambda \ln \left[ \frac{tR(t; s)}{1-s} \right] + \int_s^{F(t; s)} \left[ \frac{g(u)}{f(u)} + \frac{\lambda}{1-u} \right] du \right\} \\
&= \pi(s) \cdot \exp \left\{ \lambda \ln [btR(t; s)] - \int_{F(t; s)}^1 \left[ \frac{g(u)}{f(u)} + \frac{\lambda}{1-u} \right] du \right\} \\
&= \pi(s) \cdot [btR(t; s)]^\lambda \cdot \mathcal{B}(F(t; s)), \tag{3.16}
\end{aligned}$$

where the function  $\mathcal{B}(s)$  is defined in (2.13). Owing to Taylor expansion and the Lemma 4

$$\mathcal{B}(F(t; s)) \sim 1 - \frac{g''(1)}{2b^2t}, \quad t \rightarrow \infty. \tag{3.17}$$

In other hand according to Lemma 4 again

$$btR(t; s) = 1 + \frac{c}{6b^2} \frac{\ln bt(1-s)}{t} (1 + o(1)), \quad t \rightarrow \infty. \tag{3.18}$$

Now formula (3.15) follows from (3.16)–(3.18).  $\square$

Setting  $s = 0$ , it follows from Theorem 4 the assertion below.

**Corollary 3.** *In conditions of Theorem 4 the following asymptote holds:*

$$t^\lambda p_{00}(t) = \frac{1}{b^\lambda \mathcal{B}(0)} \cdot \left( 1 + \frac{\alpha c}{6b^3} \cdot \frac{\ln t}{t} + o\left(\frac{\ln t}{t}\right) \right), \quad t \rightarrow \infty.$$

The following theorem is generalization of Theorem 5 for all  $i \in \mathbb{N}$ .

**Theorem 5.** *Let conditions of Theorem 4 hold. Then*

$$t^\lambda \mathcal{P}_i(t; s) = \pi(s) \cdot \left( \delta_i(t) + \frac{\alpha c}{6b^3} \cdot \frac{\ln bt(1-s)}{t} (1 + o(1)) \right) \tag{3.19}$$

as  $t \rightarrow \infty$ , where  $\delta_i(t) = 1 - i/bt$ .

*Proof* repeats the reasoning in previous theorems and it follows

$$t^\lambda \mathcal{P}_i(t; s) = \pi(s) \cdot F_i(t; s) \cdot [btR(t; s)]^\lambda \cdot \mathcal{B}(F(t; s)).$$

We get on to statement (3.16) using (3.14), (3.17), (3.18), seeing  $F_i(t; s) \sim 1 - iR(t; s)$ .  $\square$

Finally, from Theorem 5 we have the following

**Corollary 4.** *In conditions of Theorem 4*

$$t^\lambda p_{i0}(t) = \frac{1}{b^\lambda \mathcal{B}(0)} \cdot \left( \delta_i(t) + \frac{\alpha c}{6b^3} \cdot \frac{\ln t}{t} + o\left(\frac{\ln t}{t}\right) \right), \quad t \rightarrow \infty,$$

and for all  $j \in \mathbb{N}$  the following asymptote occurs:

$$t^\lambda p_{ij}(t) = \pi_j \cdot \left( \delta_i(t) + \frac{\alpha c}{6b^3} \cdot \frac{\ln t}{t} + o\left(\frac{\ln t}{t}\right) \right), \quad t \rightarrow \infty,$$

where  $\delta_i(t)$  as in Theorem 5.

*Proof* of first assertion follows from (3.16) setting in it  $s = 0$ . The second one is consequence of use the continuity theorem for GF.  $\square$

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## О предельном поведении марковских ветвящихся процессов с иммиграцией непрерывного времени

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*Мы рассмотрим марковский ветвящийся процесс с иммиграцией. Исследуются предельные свойства переходных вероятностей и их сходимость к инвариантным мерам. Определяется скорость этой сходимости.*

*Ключевые слова: марковский ветвящийся процесс, иммиграция, переходные вероятности, инвариантные меры, скорость сходимости к инвариантным мерам.*