EDN: UVZXKX
УДК 517.10

On Weakly Contractions Via w-distances

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Received 10.08.2024, received in revised form 27.09.2024, accepted 29.11.2024

Abstract. In this article, we will check whether the known results remain valid if the metric d is replaced by the w-distance p. we show that in some contractive conditions where w-distance p participates instead of metric d, symmetry of w-distance p can be assumed and the proofs can be shorter. We are talking about results such as Banach's contraction principle, Kannan's theorem, Boyd–Wong, Meir–Keeler, Chatterje's, Reich's, Hardy–Rogers', Karapinars' and Wardowskis' theorems and many others.

By doing so, we would obtain generalizations of the above results.

Keywords: fixed point, w-distance, p-interpolative Kannan type contraction, p-Hardy–Rogers contraction, $(\digamma, p)$-contraction.

Citation: H. Lakzian, S. Barootkoob, N. Fabiano, S. Radenović, On Weakly Contractions Via w-distances, J. Sib. Fed. Univ. Math. Phys., 2025, 18(1), 109–118. EDN: UVZXKX.

1. Introduction and preliminaries

One of the generalizations of the well-known Banach theorem from 1922 is the introduction after 75 years of the so-called w-distance p in the given metric space (X, d). Thus we obtained an ordered triple (X, d, p) where (X, d) is the given metric space and p is a function from $X \times X$ in $[0, +\infty)$ that satisfies the following three axioms:

p1) $p(x, z) \leqslant p(x, y) + p(y, z)$ *for all* $x, y, z \in X$;

p2) *For any* $x \in X$, *the function* $p(x, \cdot) : X \to [0; +\infty)$ *is* $d-$*lower semi-continuous;*

p3) *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $p(z, x) \leqslant \delta$ *and* $p(z, y) \leqslant \delta$ *imply* $d(x, y) \leqslant \varepsilon$.

Then, p *is called a* w*-distance on* X.

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The last two axioms are new compared to those known for metric space. The second represents the lower semi-continuity in the second variable and the third connects the metric d and the w-distance p.

A typical example of a w-distance is the metric d itself defined on a nonempty set X. Actually, **p1**) is fulfilled as a triangle relation. Since the metric d is a continuous function with 2 variables, it is also semi-continuous from below in the second variable. Indeed, if y_n is a sequence in X that converges to y by metric d then $p(x,y) = d(x,y) = \lim_{n\to+\infty} d(x,y_n) = \liminf_{n\to+\infty} d(x,y_n)$ and **p2**) is fulfilled. Assuming that $\delta = \dfrac{\varepsilon}{2}$, we get that **p3**) is fulfilled, because $p(x,y) = d(x,y) \leqslant d(x,z) + d(y,z) \leqslant \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$.

Now we list several typical examples of w-distances, some of which were also mentioned in the first paper on w-distances.

1. Let (X,d) be a metric space. Then a function $p : X \times X \to [0,+\infty)$ defined by $p(x,y) = c$ for every $x, y \in X$ is a w-distance on X, where c is a positive real number.

2. Let X be a normed linear space with norm $\|\cdot\|$. Then a function $p : X \times X \to [0,+\infty)$ defined by $p(x,y) = \|x\| + \|y\|$ for every $x, y \in X$ is a w-distance on X.

3. The similar as example 3. only the function $p : X \times X \to [0,+\infty)$ is defined by $p(x,y) = \|x\|$ for all $x, y \in X$.

For several examples of w-distances see [2], pages 382, 383, 384.

An important note about **p1)** and **p3)**. Since Example 1.3. from [3] (see also Example 4 from [3]) shows that the w-distance in the general case is not symmetric, i.e., it is not $p(x,y) = p(y,x)$ for every $x, y \in X$, then the triangle relation as well as the axiom **p3)** should be understood as the introduced order x, z; x, y; y, z and, z, x; z, y and finally x, y.

The following Lemma is one of the most important that is used in the study of w-distance metric spaces. It relates metric convergence to w-distance convergence. It is also important because it gives us the information (a sufficient condition) when the sequence x_n is Cauchy in the metric space (X,d). In the sequel, we denote by $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{N}$, the sets of positive real numbers, real numbers and natural numbers, respectively.

Lemma 1.1 ([2], Lemma 1.)**.** *Let X be a metric space with metric d and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,+\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(i) *If $p(x_n,y) \leqslant \alpha_n$ and $p(x_n,z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;*

(ii) *if $p(x_n,y_n) \leqslant \alpha_n$ and $p(x_n,z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*

(iii) *if $p(x_n,x_m) \leqslant \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(iv) *if $p(y,x_n) \leqslant \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

The similar as in the context of metric spaces ([1,6]) we recall the following two lemmas that we will use in the proofs of our results. These both lemmas are important and are used to prove the Cauchyness of the sequence $x_n = fx_{n-1}, n \in \mathbb{N}$.

Lemma 1.2. *Let $\{u_n\}$ be a Picard sequence in metric space (X,d) with the w-distance p such that*

$$p(u_{n+1},u_n) < p(u_n,u_{n-1}) \tag{1}$$

or

$$p(u_n,u_{n+1}) < p(u_{n-1},u_n) \tag{2}$$

in both cases for all $n \in \mathbb{N}$. Then $u_n \neq u_m$ whenever $n \neq m$.

Proof. Consider the case (1). Suppose on the contrary that $u_n = u_m$ for some $n < m$. Then, $u_{n+1} = fu_n = fu_m = u_{m+1}$, hence

$$p(u_{n+1},u_n) = p(u_{m+1},u_m) < p(u_m,u_{m-1}) < \cdots < p(u_{n+1},u_n),$$

we obtain a contradiction. For the case (2) the proof is the same. □

Lemma 1.3. *Let (X, d) be a metric space with w-distance p and let $\{u_n\}$ be a sequence in X such that both $p(u_{n+1}, u_n)$ and $p(u_n, u_{n+1})$ tend to 0 as $n \to +\infty$. If $\{u_n\}$ is not a Cauchy sequence in metric space (X, d), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε^+ when $k \to +\infty$:*

$$\left\{p\left(u_{m(k)}, u_{n(k)}\right)\right\}, \left\{p\left(u_{m(k)}, u_{n(k)-1}\right)\right\}, \left\{p\left(u_{m(k)+1}, u_{n(k)}\right)\right\},$$
$$\left\{p\left(u_{m(k)-1}, u_{n(k)+1}\right)\right\}, \left\{p\left(u_{m(k)+1}, u_{n(k)+1}\right)\right\}, \ldots \tag{3}$$

or

$$\left\{p\left(u_{n(k)}, u_{m(k)}\right)\right\}, \left\{p\left(u_{n(k)-1,} u_{m(k)}\right)\right\}, \left\{p\left(u_{n(k)}, u_{m(k)+1}\right)\right\},$$
$$\left\{p\left(u_{n(k)+1}, u_{m(k)-1}\right)\right\}, \left\{p\left(u_{n(k)+1}, u_{m(k)+1}\right)\right\}, \ldots \tag{4}$$

Proof. Since $\{u_n\}$ is not a d-Cauchy sequence, from Lemma 1 (iii) of [2], it follows that $p(u_n, u_m)$ does not tend to 0 as $n, m \to +\infty$. This means that there exist $\varepsilon > 0$ and subsequences $\{n(k)\}, \{m(k)\}$ such that $m(k) > n(k) > k$ and

$$p\left(u_{n(k)}, u_{m(k)}\right) \geqslant \varepsilon \ \text{ and } \ p\left(u_{n(k)-1}, u_{m(k)}\right) < \varepsilon.$$

Then, using the axiom **(p1)** and the fact that both $p(u_{n+1}, u_n)$ and $p(u_n, u_{n+1})$ tend to 0 as $n \to +\infty$ it follows, in the same way as in metric spaces (see for instance [6]) that the given sequences tend to ε^+. □

The w-distance p is symmetric if $p(x, y) = p(y, x)$ for all $x, y \in X$. For such a w-distances, if $a \neq b$ then $p(a, b) > 0$, i.e., from $p(a, b) = 0$ follows $a = b$. In many contractive conditions, symmetry of the w-distance can be assumed. This is achieved by introducing a new function q from $X \times X$ to $[0, +\infty)$ defined by $q(x, y) = \max\{p(x, y), p(y, x)\}$. It is easy to show that q is a w-distance. Often when proving fixed point results in metric spaces with w-distance one finds that the condition $x \neq y$ implies that $p(x, y) > 0$. But many examples show that if $p(x, y)$ is not a symmetric w-distance that this need not be true. Such an example is: $X = [0, +\infty)$, $p(x, y) = y$. Indeed, $1 \neq 0$ while $p(1, 0) = 0$. So, for many contractive conditions, symmetry of the w-distance can be assumed. For this purpose, we introduce the following function: $q(x, y) = \max\{p(x, y), p(y, x)\}$. It is easily shown that it is a symmetric w-distance. The proof uses the fact that $p(x, y)$ is semi-continuous from below in the second variable if and only if $p(y, x)$ it is semi-continuous from below in the first variable. And then for the function q we have:

$$\begin{aligned} q(x, y_n) &= \max\{p(x, y_n), p(y_n, x)\} \geqslant \\ &\geqslant \max\left\{\lim \inf_{n \to +\infty} p(x, y_n), \lim \inf_{n \to +\infty} p(y_n, x)\right\} \geqslant \\ &\geqslant \max\{p(x, y), p(y, x)\} = \\ &= q(x, y), \end{aligned}$$

that is., $q(x, y) \leqslant \liminf_{n \to +\infty} q(x, y_n)$, i.e., $q(x, y)$ is semi-continuous from below in second variable.

In some contractive conditions where w-distance p participates instead of metric d, symmetry of w-distance p can be assumed. Namely, using the property that from $0 \leqslant a \leqslant b$ and $0 \leqslant c \leqslant d$ follows $\max\{a, c\} \leqslant \max\{b, d\}$, from which yields that the w-distance p, one moves to the above symmetric w-distance q.

The following results are natural in the framework of complete metric spaces with w-distance, and for mapping $T: X \to X$, the next will be assumed either T is continuous or the infimum of

the number set
$\{p(x,y)+p(x,Tx) : x\in X\}$ where y is not a fixed point of T, is positive.

$p-$Banach contraction theorem: $p(Tx,Ty)\leqslant k\cdot p(x,y), k\in(0,1)$

$p-$Kannan contraction: $p(Tx,Ty)\leqslant l\cdot[p(x,Tx)+p(y,Ty)], l\in\left(0,\frac{1}{2}\right)$

$p-$Chatterjea contraction: $p(Tx,Ty)\leqslant m\cdot[p(x,Ty)+p(y,Tx)], m\in\left(0,\frac{1}{2}\right)$

$p-$Reich contraction: $p(Tx,Ty)\leqslant A\cdot p(x,y)+B\cdot p(x,Tx)+C\cdot p(y,Ty)$, $A,B,C\geqslant 0$ and $A+B+C<1$

$p-$Hardy–Rogers contraction: $p(Tx,Ty)\leqslant A\cdot p(x,y)+B\cdot p(x,Tx)+C\cdot p(y,Ty)+D\cdot p(x,Ty)+E\cdot p(y,Tx), A,B,C,D,E\geqslant 0$ and $A+B+C+D+E<1$

$p-$ *Ćirić* (I) $p(Tx,Ty)\leqslant\lambda_1\cdot\max\left\{p(x,y),\frac{p(x,Tx)+p(y,Ty)}{2},\frac{p(x,Ty)+p(y,Tx)}{2}\right\}$, $\lambda_1\in(0,1)$

$p-$*Ćirić* (II) $p(Tx,Ty)\leqslant\lambda_2\cdot\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{p(x,Ty)+p(y,Tx)}{2}\right\}$, $\lambda_2\in(0,1)$

$p-$*Ćirić* (III) $p(Tx,Ty)\leqslant\lambda_3\cdot\max\left\{p(x,y),\frac{p(x,Tx)+p(y,Ty)}{2},p(x,Ty),p(y,Tx)\right\}$, $\lambda_3\in(0,1)$

$p-$ *Ćirić* (IV) $p(Tx,Ty)\leqslant\lambda_4\cdot\max\{p(x,y),p(x,Tx),p(y,Ty),p(x,Ty),p(y,Tx)\}$, $\lambda_4\in(0,1)$

$p-$Bryant contraction: $p(T^n x,T^n y)\leqslant r\cdot p(x,y), r\in(0,1), n\in\mathbb{N}$

If (X,d) is compact metric space and if

$p-$Nemytzki contraction: $p(Tx,Ty)<p(x,y)$ whenever $x\neq y$.

$p-$Browder contraction: $p(Tx,Ty)\leqslant\phi(p(x,y)),\phi$ nondecreasing and continuous from the right function from $(0,+\infty)$ into$(0,+\infty)$ such that $\phi(t)<t$

$p-$Boyd–Wong contraction: $p(Tx,Ty)\leqslant\phi(p(x,y)),\phi$ is a real function, upper semi-continuous from the right, satisfying $\phi(t)<t$ for $t>0$.

$p-$Meir–Keeler contraction: For all $\varepsilon>0$ there exists $\delta>0$ such that
$\varepsilon\leqslant p(x,y)<\varepsilon+\delta$ implies $p(Tx,Ty)<\varepsilon$

Further there are Jungck, Fisher and many other contractions.

2. p-Hardy-Rogers contraction

In section 5 of [2] the authors proved one theorem and three corollaries. In all four results, they assume that the contraction coefficient k belongs to the set $[0,1)$ or $[0,1/2)$. It is easy to see that this is imprecise and that the assumption must be that k belongs to $(0,1)$ or $(0,1/2)$. This is the difference obtained when, under the known contractive conditions of metric spaces, the metric d is replaced by the w-distance p. For example, in Theorem 4 from [2], if $k=0$ is set, $p(Tx,T^2x)=0$ is obtained. Whence it does not have to follow as with metric spaces that then $Tx=T^2x$ because the equality $p(a,b)=0$ does not necessarily follow $a=b$.

In this article, we will check, among other things, whether the known theorems (results) remain valid if the metric d is replaced by the w-distance p. We are talking about results such as Banach's contraction principle, Kanan's theorem, Chatterje's, Reich's and Hardy–Rogers' theorems. Then the Boyd–Wong and Meir–Keeler theorems and many others.

By doing so, we would obtain generalizations of the above results because each metric is a w-distance. One of the following sufficient conditions may be used for the existence of a fixed point:

(i) The mapping of T from X to X is continuous;

(ii) The number set $\{p(x,y)+p(x,Tx) : x\in X,\ y\in X$ but such that y is different from $Ty\}$ has a positive infimum.

First, we will formulate and prove the Hardy–Rogers theorem within metric spaces with w-distance p.

Theorem 2.1. *Let (X,d,p) be a complete metric space with w-distance p, T a mapping from X to X and let there exist non-negative constants A,B,C,D and E such that $A+B+C+D+E<1$ and* *that for every x, y from X it holds* $p(Tx, Ty) \leqslant Ap(x, y) + Bp(x, Tx) + Cp(y, Ty) + Dp(x, Ty) + Ep(y, Tx)$.

Then T has a unique fixed point say $z \in X$ such that $p(z, z) = 0$ if at least one of the above conditions (i) or (ii) holds.

Proof. Let us first assume that z is a fixed point of the mapping T and show that then $p(z, z) = 0$. Then by putting in the contractive condition $x = y = Tx = Ty$ we get: $p(x, x) \leqslant (A + B + C + D + E)p(x, x) < p(x, x)$ which is not possible if $p(x, x) > 0$. In order to show the uniqueness of a possible fixed point of the mapping T, let us assume that there are two different fixed points of it u and v. Using the fact that according to the already shown $p(u, u) = p(v, v) = 0$, then based on that and putting $x = u$, $y = v$ in the contractive condition we get: $p(u, v) \leqslant Ap(u, v) + Bp(u, u) + Cp(v, v) + Dp(u, v) + Ep(v, u) = Ap(u, v) + Dp(u, v) + Ep(v, u)$ i.e., $(1 - A - D)p(u, v) \leqslant Ep(v, u)$. Similarly, we get that $(1 - A - D)p(v, u) \leqslant Ep(u, v)$. By taking the maximum of the left and right sides, we get $(1 - A - D)q(u, v) \leqslant Eq(u, v)$ where $q(a, b) = \max\{p(a, b), p(b, a)]$. If it is assumed that $q(u, v) > 0$, we get a contradiction with $A + B + C + D + E < 1$. Otherwise, from $q(u, v) = 0$ and since q is a symmetric w-distance, we conclude $u = v$.

The rest of the proof is very simialar to the one for the metric spaces. □

3. p-interpolative Kannan type contraction

In 1969, Kannan [4] proved the following fixed point theorem.

Theorem 3.1. *Let $f : X \to X$ be a Kannan contraction mapping, i.e., $d(fx; fy) \leqslant k(d(x; fx) + d(y; fy))$ for all $x; y \in X$ and some $0 \leqslant k < \frac{1}{2}$, of a f-orbitally complete metric space. Then f has a unique fixed point.*

Afterwards, T. Suzuki published a nice paper [7] in which generalized Kannan's result in two new ones. He introduced the concept of weakly Kannan contraction mappings and non-weakly Kannan contraction mappings. For more details, see Section 4 and 5 in that paper.

Recently, the concept of interpolative Kannan type contraction mappings was introduced by E. Karapinar in [5]; and he proved the following theorem:

Theorem 3.2. *Let (X, d) be a complete metric space. Suppose that $f : X \to X$ is a interpolative Kannan type contraction self-map; i.e. if there exist a constant $\alpha \in (0; 1)$ and $k \in [0; 1)$ such that either of the followings hold:*

$$d(fx; fy) \leqslant k[d(x; fx)]^{\alpha}[d(y; fy)]^{1-\alpha}; \tag{5}$$

for all $x, y \in X$ with $x \neq fx$. Then f has a unique fixed point in X.

In this section, we introduced the concept of weakly interpolative Kannan type contraction mappings and we will prove and generalize Karapinar's theorem in the setting of w-distances.

We know that a w-distance p is not symmetric; i.e. $p(x; y)$ is not equal to $p(y; x)$ in general. So, we can define weakly interpolative Kannan type contractions as follows.

Definition 3.1. *Let $(X; d)$ be a metric space. The mapping $f : X \to X$ is said to be a weakly interpolative Kannan type contraction or p-interpolative Kannan type contraction, if there exist a constant $\alpha \in (0; 1)$ and $k \in [0; 1)$ such that either of the followings hold for all $x; y \in X$:*

$$p(fx, fy) \leqslant k[p(fx, x)]^{\alpha}[p(fy, y)]^{1-\alpha}, \tag{6}$$

or

$$p(fx, fy) \leqslant k[p(fx, x)]^{\alpha}[p(y, fy)]^{1-\alpha}, \tag{7}$$

or

$$p\,(fx, fy) \leqslant k[p\,(x, fx)]^{\,\alpha}[p\,(y, fy)]^{1-\,\alpha}, \tag{8}$$

or

$$p\,(fx, fy) \leqslant k[p\,(x, fx)]^{\,\alpha}[p\,(fy, y)]^{1-\,\alpha}. \tag{9}$$

If $p = d$ then f is called interpolative Kannan type contraction [5].

The following example shows that the class of weakly interpolative Kannan type contraction is more than interpolative Kannan type contraction.

Example 3.1. *Let $X = \{x, y, z\}$. Consider the metric d and the w-distance p on X, as follows.*

$$d(x, x) = d(y, y) = d(z, z) = 0, \quad d(x, y) = d(y, x) = 3,$$

$$d(x, z) = d(z, x) = 1, \quad d(y, z) = d(z, y) = 2.$$

$$p(x, x) = p(z, z) = 1, \quad p(y, y) = 0, \quad p(z, y) = \frac{3}{2}, \quad p(y, z) = 3;$$

$$p(y, x) = p(x, y) = 2, \quad p(x, z) = p(z, x) = 4.$$

Also, define $f : X \to X$ by $f(x) = f(y) = y$ and $f(z) = x$. Then for $\alpha = \frac{1}{2}$ and for each $k \in \left(\frac{1}{\sqrt{2}}, 1\right)$ the contractions (6)–(9) are true. while these contractions are not true for d, for each $k \in (0, 1)$ and each $\alpha \in (0, 1)$. For example,

$$d(fx, fz) = d(y, x) = 3 > 3^{\alpha} > k3^{\alpha} = kd(y, x)^{\alpha}d(x, z)^{1-\alpha} = kd(fx, x)^{\alpha}d(fz, z)^{1-\alpha}.$$

Obviousely y is the fixed point of f. Now define $q(x, y) = \max\{p(x, y), p(y, x)\}$ which is a symmetric w-distance. Then

$$q(x, x) = q(y, y) = 1, \quad q(y, y) = 0,$$

$$q(x, y) = 2, \quad q(x, z) = 4, \quad q(y, z) = 3.$$

and for $\alpha = \frac{1}{2}$ and for each $k \in \left(\frac{1}{\sqrt{2}}, 1\right)$, the contractions (6) and (7) are true for q.

In the latter theorem we conclude that for proving the existence of fixed point of a self map it sufficies to consider the symmetric w-distances. We apply the following remark for proving this main theorem.

Remark 1. *Note that if p is a symmetric w-distance, then for each $x \neq y$ we have $p(x, y) > 0$. Since if $p(x, y) = 0$, then*

$$p(x, x) \leqslant p(x, y) + p(y, x) = 2p(x, y) = 0.$$

Therefore $p(x, x) = 0 = p(x, y)$. So Lemma 1.1 implies $x = y$, a contradiction.

Therefore we have the following theorem.

Theorem 3.3. *Let p be a w-distance on a complete metric space (X, d). Suppose that $f : X \to X$ is a weakly interpolative Kannan type contraction self-map. Then f has a fixed point x in X such that $p(x, x) = 0$. In addition, if one of the equations (6)–(9) holds for all $x, y \in X$, then x is unique.*

Proof. Let $x_0 \in X$. Define by induction $x_n = f(x_{n-1})$. If for some n, $x_{n+1} = x_n$, then x_n is a fixed point of f. Otherwise if $x_{n+1} \neq x_n$, for each n, without loss of generality, we may assume p is symmetric. Since otherwise we can define $q(x, y) = \max\{p(x, y), p(y, x)\}$ which is a symmetric w-distance. Then if each of the equations (6)–(9) holds, then

$$p(fx, fy) \leqslant k\,[q(fx, x)]^{\alpha}\,[q(fy, y)]^{1-\alpha}\,,$$

and similarly

$$p(fy, fx) \leqslant k\,[q(fy, y)]^{\alpha}\,[q(fx, x)]^{1-\alpha}\,.$$

Therefore for each x, y with $fx \neq x$ and $fy \neq y$ we have

$$q(fx, fy) \leqslant k\,[q(fx, x)]^{\alpha}\,[q(fy, y)]^{1-\alpha}\,,$$

or

$$q(fx, fy) \leqslant k\,[q(fy, y)]^{\alpha}\,[q(fx, x)]^{1-\alpha}\,,$$

Note that since q is symmetric and $x_n \neq x_{n+1}$, we have $q(x_{n+1}, x_n) \neq 0$. So for each n, we have

$$q(x_{n+1}, x_n) \leqslant k\,[q(x_{n+1}, x_n)]^{\alpha}\,[q(x_n, x_{n-1})]^{1-\alpha}\,;$$

or

$$q(x_{n+1}, x_n) \leqslant k\,[q(x_{n+1}, x_n)]^{\beta}\,[q(x_n, x_{n-1})]^{1-\beta}\,;$$

where $0 < \beta = 1 - \alpha < 1$ and so,

$$[q(x_{n+1}, x_n)]^{1-\alpha} \leqslant k\,[q(x_n, x_{n-1})]^{1-\alpha}\,;$$

or

$$[q(x_{n+1}, x_n)]^{1-\beta} \leqslant k\,[q(x_n, x_{n-1})]^{1-\beta}\,.$$

Now we will have

$$[q(x_{n+1}, x_n)]^{1-\alpha} \leqslant k\,[q(x_n, x_{n-1})]^{1-\alpha} \leqslant k^2\,[q(x_{n-1}, x_{n-2})]^{1-\alpha}\,;$$

or

$$\begin{aligned}
[q(x_{n+1}, x_n)]^{1-\alpha} &\leqslant k\,[q(x_n, x_{n-1})]^{1-\alpha} = \\
&= k\left([q(x_n, x_{n-1})]^{1-\beta}\right)^{\frac{1-\alpha}{1-\beta}} \leqslant \\
&\leqslant k^2\left([q(x_{n-1}, x_{n-2})]^{1-\beta}\right)^{\frac{1-\alpha}{1-\beta}} = \\
&= k^2\,[q(x_{n-1}, x_{n-2})]^{1-\alpha}\,.
\end{aligned}$$

Therefore by an inductive method we conclude that $[q(x_{n+1}, x_n)]^{1-\alpha} \leqslant \ldots \leqslant k^n\,[q(x_1, x_0)]^{1-\alpha}$. This implies that $\lim_n q(x_{n+1}, x_n) = 0$ (since $0 < k < 1$). (Note that if p is symmetric, then we can replace q with p and also, all of the statements after "or" can be omitted and the proof is shorter.)

In the sequel, since applying "or" and working with β is very similar to working with α, we omitted them and we assume p is symmetric.

Now for each m, n we have

$$p(x_n, x_m) \leqslant k\,[p(x_n, x_{n-1})]^{\alpha}\,[p(x_m, x_{m-1})]^{1-\alpha} \to 0.$$

Therefore $\lim_{n,m} p(x_n, x_m) = 0$ and so by Lemma 1.1, $\{x_n\}$ is a Cauchy sequence. Now since (X, d) is complete, $\{x_n\}$ is convergent. Hence, there is $x \in X$ such that $\lim_n x_n = x$. In the

sequel we show that x is the fixed point of f.
By contrary if x is not the fixed point of f, then by Remark 1 $p(fx, x) \neq 0$ and

$$p(fx, x) \leqslant \liminf_n p(fx, x_{n+1}) = \liminf_n p(fx, fx_n) \leqslant k\,[p(fx, x)]^{\alpha}\,[p(x_{n+1}, x_n)]^{1-\alpha}.$$

and so,

$$[p(fx, x)]^{1-\alpha} \leqslant k\,[p(x_{n+1}, x_n)]^{1-\alpha} \to 0.$$

Therefore for each $\epsilon > 0$, $p(fx, x) < \epsilon$, a contradiction. Therefore $fx = x$ and $p(x, x) = p(fx, x) = 0$. For uniqueness, let x, y are fixed points of f. Then

$$p(x, y) = p(fx, fy) \leqslant k\,[p(fx, x)]^{\alpha}\,[p(fy, y)]^{1-\alpha} = 0.$$

That is, $p(x, y) = 0$ and so by Remark 1 $x = y$.

□

4. $(\mathcal{F}, p)$-contraction

Suppose that $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following properties.
$(\mathcal{F}_1)$ The mapping $\mathcal{F}$ is strictly increasing;
$(\mathcal{F}_2)$ For every sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to+\infty} t_n = 0$ if and only if $\lim_{n\to+\infty} \mathcal{F}(t_n) = -\infty$.
$(\mathcal{F}_3)$ $\lim_{t\to 0^+} t^k \mathcal{F}(t) = 0$ for some $k \in (0, 1)$.

Definition 4.1. *Let (X, d) be a metric space with a w-distance p. A mapping $f : X \to X$ is said to be a weakly $\mathcal{F}$-contraction or $(\mathcal{F}, p)$-contraction, if there exist a constant $\alpha > 0$ such that*

$$p(fx, fy) > 0 \ \textit{implies}\ \ \alpha + \mathcal{F}\big(p(fx, fy)\big) \leqslant \mathcal{F}\big(p(x, y)\big), \tag{1}$$

for all $x, y \in X$.
If $p = d$ then f is called $\mathcal{F}$-contraction; [8].

The following example shows that the category of $(\mathcal{F}, p)$-contractions are bigger than its for $\mathcal{F}$-contractions:

Example 4.1. *Consider $X = \{x, y, z\}$ with the metric d which is defined by*

$$d(x, x) = d(y, y) = d(z, z) = 0;$$

$$d(x, y) = d(y, x) = d(x, z) = d(z, x) = d(y, z) = d(z, y) = 2.$$

Define $f : x \to X$ by $fx = fy = x$ and $fz = y$ and assume $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfies in at least $(\mathcal{F}_1)$, Then f is not an $\mathcal{F}$-contraction. Indeed for each $\alpha > 0$ we have

$$\alpha + \mathcal{F}(d(fx, fz)) = \alpha + \mathcal{F}(2) > \mathcal{F}(2) = \mathcal{F}(d(x, y)).$$

Now define the w-distance p with

$$p(x, x) = p(y, y) = 0; \quad p(z, z) = 1;$$

$$p(x, y) = 1; \quad p(y, x) = \frac{1}{2}; \quad p(y, z) = p(z, y) = p(x, z) = p(z, x) = 2.$$

then for $\mathcal{F}(r) = Ln(r)$ and for $\alpha = Ln2$ and each of the positive cases $p(fx, fz)$, $p(fy, fz)$, $p(fz, fx)$, $p(fz, fy)$ the contraction (1) hold. Therefore f is $(\mathcal{F}, p)$-contraction.

Wardowski in [8] proved the following theorem.

Theorem 4.1. *Each $\mathcal{F}$-contraction f on a complete metric space (X, d) has a unique fixed point. Moreover, for each $x_0 \in X$, the corresponding Picard sequence $\{f^n x_0\}$ converges to that fixed point.*

Then in [1] it is shown that the proof of the above theorem needs only the condition $(\mathcal{F}_1)$. Indeed $(\mathcal{F}_1)$ implies that $\mathcal{F}$ is almost every where continous and moreover the left and right limits exist in each $a \in (0, +\infty)$ and $\lim_{r \to a^+} \mathcal{F}(r) = \mathcal{F}(a^+)$. Then two Lemmas similar as Lemmas 1.2 and 1.3 for the metric d applied for the proof.

In the sequel we prove this theorem for $(\mathcal{F}, p)$-contraction.

Theorem 4.2. *Let $f : X \to X$ be a $(\mathcal{F}, p)$-contraction mapping on a complete metric space (X, d) with a w-distance p. If f is continuous, or for every $w \in X$ with $w \neq fw$, we have $\inf\{p(x, w) + p(x, fx) : x \in X\} > 0$, then f has a unique fixed point $u \in X$; and every sequence $\{f^n x_0\}_{n \in \mathbb{N}}$ is convergent to u, for every $x_0 \in X$.*

Proof. As we see in Theorem 4.2, we may consider p symmetric. Then the proof is similar to [1, Theorem 2.3] in the case where f is continuos. If f is not continous then similar to [1, Theorem 2.3] we can show that $f^n x_0 \to x$ and if x is not the fixed point of f, then

$$0 < \inf\{p(y, x) + p(y, fy) : y \in X\} \leqslant \inf\{p(x_n, x) + p(x_n, x_{n+1}) : n \in \mathbb{N}\} = 0.$$

Which is a contradiction. So x must be a fixed point. For uniqueness let x, y be the distinct fixed points of f, then since we consider p as a symmetric w-distance and $x \neq y$, we have $p(fx, fy) = p(x, y) > 0$ and so

$$\alpha + \mathcal{F}\big(p(x, y)\big) = \alpha + \mathcal{F}\big(p(fx, fy)\big) \leqslant \mathcal{F}\big(p(x, y).$$

Which means that $\alpha \leqslant 0$, a contradiction. So x is the unique fixed point of f. □

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О слабых сокращениях через w-расстояния

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Аннотация. В этой статье мы проверим, остаются ли известные результаты верными, если метрику d заменить на w-расстояние p. Мы показываем, что в некоторых условиях сжатия, где вместо метрики d участвует w-расстояние p, можно предположить симметрию w-расстояния p, и доказательства могут быть короче. Мы говорим о таких результатах, как принцип сжатия Банаха, теорема Каннана, теоремы Бойда–Вонга, Мейра–Килера, Чаттерье, Райха, Харди–Роджерса, Карапинарса и Вардовский и многих других.

Сделав это, мы получим обобщения приведенных выше результатов.

Ключевые слова: фиксированная точка, w-расстояние, p-интерполяционное сокращение типа Каннана, p-сокращение Харди–Роджерса, $(\mathcal{F}, p)$-сокращение.