Journal of Siberian Federal University. Mathematics & Physics 2025, 18(1), 25–31

EDN: GMRCJR
УДК 517.9

Characteristic Determinant of a Perturbed Regular Third-order Differential Operator on an Interval

Nurlan S. Imanbaev*
Zhanibekov South Kazakhstan Pedagogical University
Shymkent, Kazakhstan
Institute of Mathematics and Mathematical Modeling
Almaty, Kazakhstan

Received 10.08.2024, received in revised form 13.09.2024, accepted 01.11.2024

Abstract. The spectral problem for a third-order operator with an integral perturbation of one of the boundary value conditions are considered in the paper. Conditions are regular and at the same time strongly regular. A feature of the problem is that the conjugate operator is a loaded third-order differential operator with regular (strongly regular) boundary value conditions. Characteristic determinant of the spectral problem is constructed, and conclusions about eigenvalues of the perturbed operator are drawn.

Keywords: differential operator, integral perturbation, conjugate operator, loaded, eigenvalues, eigenfunctions, characteristic determinant, entire function.

Citation: N.S. Imanbaev, Characteristic Determinant of a Perturbed Regular Third-order Differential Operator on an Interval, J. Sib. Fed. Univ. Math. Phys., 2025, 18(1), 25–31. EDN: GMRCJR.

Introduction and problem statement

In the functional space $W_2^3(0,1)$, let us consider the following differential operator L_1

$$l(u) = -u'''(x) = \lambda u(x), \quad 0 < x < 1 \tag{1}$$

with the "perturbed" boundary value conditions

$$U_1(u) \equiv u(0) = \int_0^1 \overline{P(x)} u(x)\, dx, \quad U_2(u) \equiv u'(0) = 0, \quad U_3(u) \equiv u(1) = 0, \tag{2}$$

where λ is a spectral parameter, $U_1(u), U_2(u), U_3(u)$ are linear homogeneous independent forms that are regular by G. D. Birkhoff [1, 2], $P(x) \in L_2(0,1)$. It was noted by M. A. Naimark ([3], p. 67) that all differential operators of odd order with strongly regular boundary value conditions are also with regular boundary value conditions. A third-order linear differential operator with nonlocal boundary value conditions under integral perturbation was studied [4]. Eigenvalue problems with periodic boundary value conditions, which are regular boundary value conditions, were studied [5–7]. The questions of regularity and strongly regularity of boundary value conditions for the Sturm–Liouville operator are related to the questions of basis property of the system of root vectors. In this case, when boundary value conditions are strongly regular the results of V. P. Mikhailov [8] and G. M. Keselman [9] imply the Riesz basis property of the systems of eigen and associated functions of the Sturm–Liouville operator in $L_2(0,1)$. For regular boundary value conditions, A. A. Shkalikov [10] proved unconditional basis property with

*imanbaevnur@mail.ru https://orcid.org/0000-0002-5220-9899

brackets. A. M. Minkin [11] proved the opposite statement, namely, unconditional basis property of a system of root vectors implies regularity of the operator. The issues of stability and instability of the basis property of systems of eigen and associated functions of the multiple differentiation operator with regular, but not strongly regular, boundary value conditions were studied [12–17].

Calculation of eigenvalues and eigenfunctions of third-order equations of composite type is described in [18]. The spectrum of the boundary value problem with shift for the wave equation was studied in [19].

The characteristic determinant of "perturbed" spectral problem (1)–(2) for operator L_1 will be constructed. Conclusions about eigenvalues of operator L_1 are established on the basis of the obtained formula.

Conjugate problem

Let us define operator L_1^*. Using Lagrange formula $(L_1, u, \nu) - (u, L_1^*, \nu) = \int_0^1 l(u)\,\overline{v(x)}dx - \int_0^1 u(x)\,\overline{l^*(v)}dx$ for all functions $u \in D(L_1)$ and $v \in D(L_1^*)$ and boundary value conditions (2), one can find

$$-\int_0^1 u'''(x)\,\overline{v(x)}dx = \overline{v(0)}u''(0) - \overline{v(1)}u''(1) + u'(1)\,\overline{v'(1)} -$$

$$-u'(0)\,\overline{v'(0)} - u(1)\,\overline{v''(1)} + \int_0^1 u(x)\,dx\left[P(x)\cdot\overline{v''(0)} + \overline{v'''(x)}\right].$$

Due to linear independence of forms $U_1(u)$, $U_2(u)$, $U_3(u)$ and $V_1(v)$, $V_2(v)$, $V_3(v)$ operator L_1^* is given by the loaded differential relation

$$L_1^* v \equiv l^*(v) = v'''(x) + P(x)\,v''(0) = \overline{\lambda}v(x), \quad 0 < x < 1,\ P(x) \in L_2(0,1) \tag{1*}$$

and boundary value conditions

$$V_1(v) \equiv v(0) = 0, \quad V_2(v) = v(1) = 0, \quad V_3(v) = v'(1) = 0, \tag{2*}$$

which is one of the features of considered spectral problem (1)–(2).

Construction of the characteristic determinant of spectral problem (1)-(2)

The general solution of equation (1) has the form

$$u(x) = C_1 e^{2\rho x} + \left(C_2\cdot\cos\sqrt{3}\rho x + C_3\cdot\sin\sqrt{3}\rho x +\right)e^{-\rho x}, \tag{3}$$

where C_1, C_2, C_3 are arbitrary constants and

$$\rho = \frac{\sqrt[3]{-\lambda}}{2}. \tag{4}$$

Putting (3) into boundary value condition (2), one can obtain the linear system with respect to coefficients C_1, C_2, C_3:

$$\begin{cases} C_1\left(1-\int\limits_0^1 \overline{P(x)}e^{2\rho x}dx\right)+C_2\left(1-\int\limits_0^1 \overline{P(x)}e^{-\rho x}\cos\sqrt{3}\rho x dx\right)- \\ -C_3\int\limits_0^1 \overline{P(x)}e^{-\rho x}\sin\sqrt{3}\rho x dx=0, \\ 2C_1-C_2+C_3\sqrt{3}=0, \\ C_1\cdot e^{3\rho}+C_2\cos\sqrt{3}\rho+C_3\sin\sqrt{3}\rho=0. \end{cases}$$

Determinant of this system is the characteristic determinant of spectral problem (1)–(2):

$$\Delta_1(\rho)=\begin{vmatrix} 1-\int\limits_0^1 \overline{P(x)}e^{2\rho x}dx & 1-\int\limits_0^1 \overline{P(x)}e^{-\rho x}\cos\sqrt{3}\rho x dx & -\int\limits_0^1 \overline{P(x)}e^{-\rho x}\sin\sqrt{3}\rho x dx \\ 2 & -1 & \sqrt{3} \\ e^{3\rho} & \cos\sqrt{3}\rho & \sin\sqrt{3}\rho \end{vmatrix}=$$

$$=\left(-\sin\sqrt{3}\rho-\sqrt{3}\cos\sqrt{3}\rho\right)\left(1-\int_0^1 \overline{P(x)}e^{2\rho x}dx\right)-$$

$$-\left(2\sin\sqrt{3}\rho-\sqrt{3}e^{3\rho}\right)\left(1-\int_0^1 \overline{P(x)}e^{-\rho x}\cos\sqrt{3}\rho x dx\right)=$$

$$=\left(-\sin\sqrt{3}\rho-\sqrt{3}\cos\sqrt{3}\rho\right)\left(1-\int_0^1 \overline{P(x)}e^{2\rho x}dx\right)-\left(2\sin\sqrt{3}\rho-\sqrt{3}e^{3\rho}\right)\times$$

$$\times\left(1-\int_0^1 \overline{P(x)}e^{-\rho x}\cos\sqrt{3}\rho x dx\right)-\left(2\cos\sqrt{3}\rho+e^{3\rho}\right)\int_0^1 \overline{P(x)}e^{-\rho x}\sin\sqrt{3}\rho x dx. \quad (5)$$

When $P(x)=0$ it is characteristic determinant of the "unperturbed" spectral problem for operator L_0 given by differential relation (1) with boundary value conditions

$$u(0)=0,\ u'(0)=0,\ u(1)=0. \quad (6)$$

Let us denote it by $\Delta_0(\rho)=-3\sin\sqrt{3}\rho-\sqrt{3}\cos\sqrt{3}\rho+\sqrt{3}e^{3\rho}$.

Following the previously obtained results [5, 6] and setting determinant $\Delta_0(\rho)$ to zero, one can obtain the condition for existence of non-trivial solutions $u(x)$:

$$\sqrt{3}\sin\sqrt{3}\rho+\cos\sqrt{3}\rho=e^{3\rho}. \quad (7)$$

Equation (7) has a solution if $\rho\leqslant 0$. Equation (7) is reduced to the form

$$2\cos\frac{\pi}{6}\sin\sqrt{3}\rho+2\sin\frac{\pi}{6}\cos\sqrt{3}\rho=e^{3\rho},$$

i.e.,

$$\sin\left(\frac{\pi}{6}+\sqrt{3}\rho\right)=\frac{1}{2}e^{3\rho},\ \rho<0 \quad (8)$$

Eigenvalues of "unperturbed" operator L_0 are roots of this equation. These roots are defined as the abscissa of the intersection points of the curves

$$y=\sin\left(\frac{\pi}{6}+\sqrt{3}\rho\right),\quad y=\frac{1}{2}e^{3\rho x},\ x\leqslant 0.$$

When $x=0$ both curves have common point $y=\frac{1}{2}$. Zero points of function $\sin\left(\frac{\pi}{6}+\sqrt{3}\rho\right)$ are $\mu_k=-\frac{\pi}{6\sqrt{3}}-(k-1)\pi$, $k=1,2,\ldots$. Then eigenvalues ρ_k of operator L_0 are

$$\rho_k=\mu_k+(-1)^k\varepsilon_k,\ k=1,\,2,\ldots \quad (9)$$

and $\lim\limits_{k\to\infty}\varepsilon_k = 0$.

Eigenfunctions of operator L_0 are

$$u_k(x) = e^{2\rho_k x} - \left(\cos\sqrt{3}\rho_k x + \sqrt{3}\sin\sqrt{3}\rho_k x\right)e^{-\rho_k x}. \tag{10}$$

Function (10) satisfies all conditions of problem (1)–(6).

Conjugate operator L_0^* has the form

$$L_0^* v = l_0^*(v) = v'''(x) - \bar{\lambda} v(x) = 0, \quad v(0) = 0,\ v(1) = 0,\ v'(1) = 0. \tag{11}$$

Eigenvalues of operator L_0^* coincide with eigenvalues of operator L_0, and corresponding eigenfunctions of operator L_0^* are

$$v_k(x) = e^{-2\rho_k x} - \left(\cos\sqrt{3}\rho_k x + N\sin\sqrt{3}\rho_k x\right)e^{\rho_k x}, \tag{12}$$

where $N = \dfrac{\cos\sqrt{3}\rho_k - e^{3\rho_k}}{\sin\sqrt{3}\rho_k}$.

Function $P(x)$ under the integral in (5) is represented in the form of a bi-orthogonal expansion in Fourier series by the system $\{v_k(x)\}$:

$$P(x) = \sum_{k=1}^{\infty} a_k v_k(x). \tag{13}$$

Calculating integrals in (5) and taking (12), (13) into account, one can obtain the following form of determinant $\Delta_1(\rho)$:

$$\begin{aligned}
\Delta_1(\rho) = &\left(-\sin\sqrt{3}\rho - \sqrt{3}\cos\sqrt{3}\rho\right) + \left(\sin\sqrt{3}\rho - \sqrt{3}\cos\sqrt{3}\rho\right)\times\\
&\times\left(\sum_{k=1}^{\infty}\alpha_k\left[\frac{e^{2(\rho-\rho_k)}}{2(\rho-\rho_k)} - \frac{1}{2(\rho-\rho_k)} + \frac{2\rho+\rho_k}{(2\rho+\rho_k)^2+3\rho_k^2} - \frac{e^{2\rho+\rho_k}}{(2\rho+\rho_k)^2+3\rho_k^2}\times\right.\right.\\
&\times\left(\sqrt{3}\rho_k\sin\sqrt{3}\rho_k + (2\rho+\rho_k)\cos\sqrt{3}\rho_k\right) + N\frac{\sqrt{3}\rho_k}{(2\rho+\rho_k)^2+3\rho_k^2} +\\
&\left.\left.+N\frac{e^{2\rho+\rho_k}}{(2\rho+\rho_k)^2+3\rho_k^2}\left((2\rho+\rho_k)\sin\sqrt{3}\rho_k - \sqrt{3}\cos\sqrt{3}\rho_k\right)\right]\right) -\\
&-\left(2\sin\sqrt{3}\rho - \sqrt{3}e^{3\rho}\right) + \left(2\sin\sqrt{3}\rho - \sqrt{3}e^{3\rho}\right)\left(\sum_{k=1}^{\infty}\alpha_k\left[\frac{2\rho+\rho_k}{(2\rho_k+\rho)^2-\left(\sqrt{3}\rho\right)^2} +\right.\right.\\
&+\frac{e^{-(2\rho+\rho_k)}}{(2\rho_k+\rho)^2-\left(\sqrt{3}\rho\right)^2}\left(\sqrt{3}\rho\sin\sqrt{3}\rho - (2\rho_k+\rho)\cos\sqrt{3}\rho\right) + \frac{1}{8(\rho_k-\rho)} -\\
&-\frac{e^{\rho_k-\rho}}{8(\rho_k-\rho)}\cdot\left(\cos\sqrt{3}(\rho_k-\rho) + \sqrt{3}\sin\sqrt{3}(\rho_k-\rho)\right) - \frac{1}{2}\cdot\frac{\rho_k-\rho}{(\rho_k-\rho)^2+3(\rho_k+\rho)^2} +\\
&+\frac{1}{2}\cdot\frac{e^{\rho_k-\rho}}{(\rho_k-\rho)^2+3(\rho_k+\rho)^2}\cdot\left(\sqrt{3}(\rho_k+\rho)\sin\sqrt{3}(\rho_k+\rho) + (\rho_k-\rho)\cos\sqrt{3}(\rho_k+\rho)\right) +\\
&+\frac{N}{2}\cdot\frac{\sqrt{3}(\rho_k+\rho)}{(\rho_k-\rho)^2+3(\rho_k+\rho)^2} + \frac{N}{2}\cdot\frac{e^{\rho_k-\rho}}{(\rho_k-\rho)^2+3(\rho_k+\rho)^2}\left((\rho_k+\rho)\sin\sqrt{3}(\rho_k+\rho) -\right.
\end{aligned}$$

$$-\sqrt{3}\,(\rho_k+\rho)\cos\sqrt{3}\,(\rho_k+\rho)\Big)+\frac{1}{8}\cdot\frac{N}{\rho_k-\rho}+\frac{1}{8}\cdot\frac{N}{\rho_k-\rho}e^{\rho_k-\rho}\cdot\Big(\sin\sqrt{3}\,(\rho_k-\rho)-$$
$$-\cos\sqrt{3}\,(\rho_k+\rho)\Big)\Big]\Big)-\Big(2\cos\sqrt{3}\rho+e^{3\rho}\Big)\cdot\Bigg(\sum_{k=1}^{\infty}\alpha_k\Bigg[\frac{\sqrt{3}\rho}{\left(2\rho_k^2+\rho\right)^2+3\rho^2}-\frac{1}{4}\cdot\frac{\sqrt{3}}{\rho_k-\rho}+$$
$$+\frac{1}{4}\cdot\frac{\sqrt{3}\,(\rho_k+\rho)}{\rho_k+2\rho}-\frac{N}{8}\cdot\frac{1}{\rho_k-\rho}+\frac{1}{2}\cdot\frac{\rho_k-\rho}{3(\rho_k+\rho)^2+(\rho_k-\rho)^2}-\frac{e^{-(2\rho_k+\rho)}}{\left(2\rho_k^2+\rho\right)^2+3\rho^2}\times$$
$$\times\Big(\sqrt{3}\rho\cos\sqrt{3}\rho+(2\rho_k+\rho)\sin\sqrt{3}\rho\Big)+\frac{1}{4}\cdot\frac{e^{\rho_k-\rho}}{\rho_k-\rho}\Big(\sqrt{3}\cos\sqrt{3}\,(\rho_k-\rho)+\sin\sqrt{3}\,(\rho_k-\rho)\Big)+$$
$$+\frac{1}{4}\cdot\frac{e^{\rho_k-\rho}}{\rho_k+2\rho}\Big(\sin\sqrt{3}\,(\rho_k+\rho)-\sqrt{3}\,(\rho_k+\rho)\cos\sqrt{3}\,(\rho_k+\rho)\Big)+\frac{1}{8}N\frac{e^{\rho_k-\rho}}{\rho_k-\rho}\Big(\sqrt{3}\sin\sqrt{3}\,(\rho_k-\rho)+$$
$$+\cos\sqrt{3}\,(\rho_k-\rho)\Big)-\frac{1}{2}\cdot\frac{\rho_k-\rho}{3(\rho_k+\rho)^2+(\rho_k-\rho)^2}\Big((\rho_k-\rho)\cos\sqrt{3}\,(\rho_k+\rho)+$$
$$+\sqrt{3}\,(\rho_k+\rho)\sin\sqrt{3}\,(\rho_k+\rho)\Big)\Big]\Bigg), \tag{14}$$

where $N=\dfrac{\cos\sqrt{3}\rho_k-e^{3\rho_k}}{\sin\sqrt{3}\rho_k}$. This determinant is an entire analytical function of variable ρ because it has poles of first order at the points $\rho=\rho_k$. Determinant $\Delta_0\,(\rho)$ is equal to zero at these points.

Therefore, the following statement is proved

Theorem 1. *The characteristic determinant of "perturbed" spectral problem (1)—(2) is represented in form (14), where determinant $\Delta_0\,(\rho)$ is the characteristic determinant of "unperturbed" spectral problem (1)-(6), α_k are Fourier coefficients of bi-orthogonal expansion (13) of functions $P\,(x)$ in terms of the system of eigenfunctions $\{v_k\,(x)\}$ of conjugate unperturbed spectral problem (1)–(2).*

Remark 1. The "perturbed" spectral problem (1)–(2) is reduced to the study of zeros of entire analytic function $\Delta_1\,(\rho)$. The issue of finding zeros of this function remains open.

The research has been funded by the grant (no. AP23485279) of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

References

[1] G.D.Birkhoff, On the asymptotic character of the solutions of certain linear differential equations containing a parameter, *Transactions of the American Mathematical Society*, **9**(1908), no. 2, 219–231

[2] G.D.Birkhoff, Boundary value and expansion problems of ordinary linear differential equations, *Transactions of the American Mathematical Society*, **9**(1908), no. 4, 373–395

[3] M.A.Naimark, Linear Differential Operators, Moscow, Nauka, 1969 (in Russian)

[4] N.S.Imanbaev, B.E.Kangushin, B.T.Kalimbetov, On zepos of the characteristic determinant of the spectral problem for a third-order differential operator on a segment with nonlocal boundary conditions, *Advances in Difference Equations*, **2013**(2013), article 110. DOI: 10.1186/1687-1847-2013-110.

[5] N.S.Imanbaev, Distribution of eigenvalues of a third-order differential operator with strongly regular nonlocal boundary conditions, *AIP Conference Proceedings*, **1997**(2018), 020027. DOI: 10.1063/1/5049021

[6] N.S.Imanbaev, Ye.Kurmysh, On computation of eigenfunctions of composite type equations with regular boundary value conditions, *International journal of Applied Mathematics*, **34**(2021), no. 4, 681–692. DOI: 10.12732/ijam.v34i4.7

[7] A.I.Kozhanov, S.V.Potapova, Conjugation problem for a third-order equation with multiple characteristics, with an alternating function at the highest derivative, *Journal of Mathematical Sciences*, **215**(2016), no. 4, 510–516. DOI: 10.1007/s10958-016-2855-5

[8] V.P.Mikhailov, On Riesz basis in, *Doklady Mathematics*, **144**(1962), no. 5, 981–984 (in Russian)

[9] G.M.Kesel'man, About unconditional convergence of expansion by eigenfunctions of some differential operators, *Izv. Vyssh. Ucheb. Zaved. Math.*, **2**(1964), 82 (in Russian)

[10] A.A.Shkalikov, On the basis problem of the eigenfunctions of an ordinary differential operator, *Russian Mathematical Surveys*, **34**(1979), no. 5, 249–250.
DOI: 10.1070/RM1979v034n05ABEH003901

[11] A.M.Minkin, Resolvent growth and Birkhoff-regularity, *Journal of Mathematical Analysis and Applications*, **323**(2006), no. 1, 387–402

[12] M.Kandemir, O.Sh.Mukhtarov, A method on solving irregular boundary value problems with transmission conditions, *Kuwait journal of Science and Eingeneering*, **36**(2A), 2009, 79–98

[13] N.S.Imanbaev, M.A.Sadybekov, Stability of basis property of a type of problems on eigenvalues with nonlocal perturbation of boundary conditions, *Ufa mathematical journal*, **3**(2011), no. 2, 27–32.

[14] N.S.Imanbaev, M.A.Sadybekov, Stability of basis property of a periodic problem with nonlocal perturbation of boundary conditions, *AIP Conference Proceedings*, **1759**(2016), 020080. DOI: 10.1063/1.4959694

[15] N.S.Imanbaev, On a problem that does not habe basis property of root vectors, associated with a perturbed regular operator of multiple differentiation, *Journal of Siberian Federal University. Mathematics and Physics*, **13**(2020), no. 5, 568–573.
DOI: 10.17516/1997-1397-2020-13-5-568-573

[16] M.A.Sadybekov, N.S.Imanbaev, On system of root vectors of perturbed regular second-order differential operator not possessing basis property, *Mathematics*, **11**(2023), 4364.
DOI: 10.3390/math11204364

[17] A.M.Sarsenbi, Criteria for the Riesz basis property of systems of eigen-and associated functions for higner-order differential operators on an interval, *Doklady Mathematics*, **77**(2008), no. 2, 290–292

[18] T.Dzh.Dzhuraev, On spectral problems for third-order equations of composite type, *Reports of the Academy of Sciences of the Republic of Uzbekistan*, **2**(2006), 5–8 (in Russian)

[19] T.Sh.Kal'menov, Selfadjoint boundare value problems for the Tricomi equation, *Differentsial'nye Uravneniya*, **19**(1983), no. 1, 66–75 (in Russian)

Характеристический определитель возмущенного регулярного дифференциального оператора третьего порядка на отрезке

Нурлан С. Иманбаев
Южно-Казахстанский педагогический университет имени Жанибекова
Шымкент, Казахстан
Институт математики и математического моделирования
Алматы, Казахстан

Аннотация. В работе рассматривается спектральная задача для оператора третьего порядка при интегральном возмущении одного из краевых условий, являющихся регулярными, одновременно усиленно регулярными, где особенностью задачи является, что сопряженным оператором будет нагруженный дифференциальный оператор третьего порядка с регулярными (усиленно регулярными) краевыми условиями. Построен характеристический определитель спектральной задачи, на основании которой сделаны выводы о собственных значениях возмущенного оператора.

Ключевые слова: дифференциальный оператор, интегральное возмущение, сопряженный оператор, нагруженный, собственные значение, собственные функции, характеристический определитель, целая функция.