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The Dirichlet Problem in the Class of sh_m-functions on a Stein Manifold X

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Abstract. The purpose of this paper is to introduce and study strongly m-subharmonic (sh_m) functions on complex manifolds $X \subset \mathbb{C}^N$, $dim X = n$, $n \leqslant N$. There are different ways to define sh_m-functions on complex manifolds: using local coordinates, using retraction $\pi: \mathbb{C}^N \to X$ or using Jensen measures (see for example [1, 8, 13]). In this paper we use the local coordinates. In Section 1 we present the definition and simplest properties of sh_m-functions in $\mathbb{C}^n$. In Section 2, we provide the definition of sh_m-functions in the domains $D \subset X$ of the complex manifold X and prove several of their potential properties. Section 3 introduces maximal functions and their properties, while Section 4 presents the main result of the work (Theorem 4.1) concerning the solvability of the Dirichlet problem in regular domains.

Keywords: sh_m-functions, plurisubharmonic functions, Stein manifolds, Dirichlet problem.

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The theory of strongly m-subharmonic (sh_m) functions plays an important role in the potential theory. It expands and develops the well-known pluripotential theory, introduced at the end of the last century, which at present is the main subject for studying analytic functions of several complex variables and plurisubharmonic functions.

The pluripotential theory is based on plurisubharmonic (psh) functions and is related to the Monge-Ampère operator $(dd^c u)^n$. Here, as usual $d = \partial + \overline{\partial}$ and $d^c = \dfrac{\partial - \overline{\partial}}{4i}$. This theory is based on research in numerous fundamental works of E. Bedford, A. Taylor, J. Siciak, A. Sadullaev and others (see, for example, [2, 10, 14]). sh_m-functions are related to the operator

$$(dd^c u)^m \wedge \beta^{n-m}, \quad 1 \leqslant m \leqslant n, \tag{1}$$

where $\beta = dd^c|z|^2$ is the standard volume form in the complex space $\mathbb{C}^n$.

Since $dd^c u \wedge \beta^{n-1} = \Delta u \beta^n$, operator (1) for $m = 1$ gives the Laplace operator, and for $m = n$ the Monge–Ampère operator. The operator (1) is called the complex operator in Hessians, because it is easy to calculate

$$(dd^c u)^m \wedge \beta^{n-m} = m!(n-m)! H_m(u) \beta^n,$$

where $H_m(u) = \sum\limits_{1 \leqslant j_1 < \ldots < j_m \leqslant n} \lambda_{j_1} \ldots \lambda_{j_m}$ is the Hessian of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the matrix $(u_{j,\bar{k}})$.

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With the help of Hessians, a class of sh_m-functions was defined (see Definition 2.1. below) in the works of Z. Blocki, S. Dinew, S.-Y. Li, H. Lu and others (see, for example, [3, 4, 6, 7]). Moreover, in their works sh_m-functions are also defined in the class $L^1_{loc}(D)$ and a number of their fundamental properties are proven. The potential theory in the class of sh_m-functions is developed in the work of A. Sadullaev and B. Abdullaev [9].

1. Hessians

Let $u \in C^2(D)$ be a twice differentiable function given in a domain $D \subset \mathbb{C}^n$. The second-order differential $dd^c u = \frac{i}{2}\sum_{j,k} u_{j,\bar{k}} dz_j \wedge d\bar{z}_k$ represents a Hermitian quadratic form, where $u_{j,\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$. Therefore, through an appropriate unitary transformation of coordinates, it can be reduced to a diagonal form $dd^c u = \frac{i}{2}[\lambda_1 dz_1 \wedge d\bar{z}_1 + \cdots + \lambda_n dz_n \wedge d\bar{z}_n]$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hermitian matrix $(u_{j,\bar{k}})$.

It is clear that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H_k(u)\beta^n, \quad k = 1, \ldots, n,$$

where $H_k(u) = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \lambda_{j_1} \ldots \lambda_{j_k}$ is the Hessian of dimension k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Definition 1.1 (see [9])**.** *A function $u \in C^2(D)$ is called sh_m in domain $D \subset \mathbb{C}^n$, if it satisfies the following condition*

$$(dd^c u)^k \wedge \beta^{n-k} \geqslant 0 \quad \forall\, k = 1, 2, \ldots, n-m+1.$$

It is known that for all twice differentiable sh_m-functions $u, v_1, \ldots, v_{n-m}$ the following inequality holds

$$dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geqslant 0. \tag{2}$$

Moreover, if a twice differentiable function u satisfies (2) for all twice differentiable sh_m-functions $v_1, \ldots, v_{n-m}$, then u is a sh_m-function. Using this, we can define sh_m-functions in the class L^1_{loc}.

Definition 1.2 (see [9])**.** *An upper semicontinuous function u in the domain $D \subset \mathbb{C}^n$ is called sh_m in D, if for any twice differentiable sh_m-functions $v_1, \ldots, v_{n-m}$ the current $dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as*

$$[dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1}](\omega) =$$

$$= \int u dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{(0,0)}(D)$$

is positive, where $F^{(0,0)}(D)$ is a space of test functions in D.

The set of sh_m-functions in D is denoted by $sh_m(D)$. It is clear that $psh = sh_1 \subset sh_2 \subset \cdots \subset sh_n = sh$ and we have the following important property.

Theorem 1.1. *If $u \in sh_m(D)$, then for any complex hyperplane $\Pi \subset \mathbb{C}^n$ restriction $u|_\Pi$ is a sh_m-function in $D \bigcap \Pi$, i.e.*

$$u|_\Pi \in sh_m(D \cap \Pi).$$

2. $\mathrm{sh_m}$-functions on a Stein manifold X.

Let us recall the definition of a Stein manifold. Let X be a complex manifold of complex dimension n and denote by $\mathcal{O}(X)$ the ring of holomorphic functions on X.

Definition 2.1 (see [16]). *A complex analytic manifold X of dimension n is called Stein manifold if*

1) X is holomorphic convex, i.e.

$$\hat{K} = \{z : \ z \in X, \ |f(z)| \leqslant \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$$

is a compact subset of X for every compact subset $K \subset X$;

2) If z_1 and z_2 are different points in X, then $f(z_1) \neq f(z_2)$ for some $f \in \mathcal{O}(X)$;

3) For every $z \in X$, one can find functions $f_1, \ldots, f_n \in \mathcal{O}(X)$ which form a coordinate system at z.

It is well-known that the Stein manifold X can always be embedded in some space of higher dimension, $X \subset \mathbb{C}^N, \ N \geqslant n.$

We define sh_m-functions on a Stein manifold $X \subset \mathbb{C}^N$, $\dim X = n$, for $1 \leqslant m \leqslant n$ by restricting $\beta = dd^c||z||^2$, $z = (z_1, \ldots, z_N)$ to X. In local coordinates $\phi(\xi) : \ B \to U, \ B \subset \mathbb{C}^n, \ U \subset X, \ \xi = (\xi_1, \ldots, \xi_n)$ the differential form $\beta|_X$ has the following form

$$\beta|_X = \beta|_U = \alpha(\xi) = \frac{i}{2}[d\phi_1(\xi) \wedge d\bar{\phi}_1(\xi) + \cdots + d\phi_N(\xi) \wedge d\bar{\phi}_N(\xi)].$$

Definition 2.2 (see [15]). *A function $u \in C^2(D)$ is called sh_m-function in the domain $D \subset X$ if*

$$[(dd^c u)|_X]^k \wedge [\beta|_X]^{n-k} \geqslant 0, \ \ k = 1, 2, \ldots, n - m + 1,$$

or, equivalently, in local coordinates of D the following holds

$$(dd^c u(\varphi(\xi)))^k \wedge \alpha^{n-k}(\xi) \geqslant 0, \ \ k = 1, 2, \ldots, n - m + 1. \tag{3}$$

It is clear that if $U_1 \bigcap U_2 \neq \varnothing$ are two open sets on X, then from $\beta|_{U_j} = \beta|_{U_k} \circ \phi_k^{-1} \circ \phi_j$ it is easy to obtain that the positivity of the forms in (3) does not depend on the choice of the local coordinates, i.e. Definition 2.2 is correct.

From the definition of sh_m-function, it obviously follows that if $u, v_1, \ldots, v_{n-m} \in sh_m(D) \bigcap C^2(D)$, then

$$dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge [\beta|_X]^{m-1} \geqslant 0. \tag{4}$$

Conversely, if a twice differentiable function u satisfies (4) for all $v_1, \ldots, v_{n-m} \in sh_m(D) \bigcap C^2(D)$, then u is a sh_m-function in D. This conclusion can be proved in the same way as in the case $X = \mathbb{C}^n$ since the differential forms $\beta|_X$ in local coordinates is a strictly positive $(1,1)$ form and by using suitable linear mapping it can be reduced to a diagonal form $\lambda_1 d\xi_1 \wedge d\bar{\xi}_1 + \cdots + \lambda_n d\xi_n \wedge d\bar{\xi}_n$.

As above, we can define sh_m-functions in the class of functions L^1_{loc}.

Definition 2.3 (see [5]). *A function $u \in L^1_{loc}(D)$ is called sh_m in a domain $D \subset X$ if it is upper semicontinuous and for any twice differentiable sh_m-functions $v_1, \ldots, v_{n-m}$ the current $dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge [\beta|_X]^{m-1}$ which is defined as*

$$\begin{gathered}\left[dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge (\beta|_X)^{m-1}\right](\omega) = \\ = \int u \, dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge (\beta|_X)^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0}(D)\end{gathered} \tag{5}$$

is positive.

The class of sh_m-functions in a domain D is denoted by $sh_m(D)$. Usually a trivial function $u(z) \equiv -\infty$ is also included in $sh_m(D)$.

The following properties of $sh_m(D)$ follow easily from definitions of sh_m-function.

1) *A linear combination of sh_m-functions with non-negative coefficients also is a sh_m-function, i.e.*

$$u_k(z) \in sh_m(D), \; a_k \in \mathbb{R}^+ \; (k = 1, 2, \ldots, p) \quad \Rightarrow \quad a_1 u_1(z) + a_2 u_2(z) + \cdots + a_p u_p(z) \in sh_m(D);$$

2) *We have the following relation*

$$sh_1(D) \subset \cdots \subset sh_m(D) \subset \cdots \subset sh_n(D).$$

3) *The limit of a uniformly converging or monotonically decreasing sequence of sh_m-functions is also sh_m-function:*

$$u_j(z) \in sh_m(D), \; u_j(z) \rightrightarrows u(z) \quad \Rightarrow \quad u(z) \in sh_m(D);$$

$$u_j(z) \geqslant u_{j+1}(z) \; (j = 1, 2, \ldots) \;\Rightarrow\; \lim_{j\to\infty} u_j(z) \in sh_m(D).$$

The above properties 1)–3) follow directly from Definition 2.3 and from the Lebesgue–Levi theorem on monotone convergence.

Let us now state properties whose proofs are more complicated.

4) *The maximum of a finite number of sh_m-functions is also a sh_m-function, i.e.,*

$$u_1(z), u_2(z), \ldots, u_p(z) \in sh_m(D) \quad \Rightarrow \quad \max\{u_1(z), u_2(z), \ldots, u_p(z)\} \in sh_m(D).$$

Proof. We fix $v_1, \ldots, v_{m-1} \in sh_m(D) \bigcap C^2(D)$ and put $\alpha = dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge [\beta|_X]^{m-1}$. According to (4), the differential form α is positive. For small positive number $\varepsilon > 0$, considering the differential form $\alpha + \varepsilon (dd^c \beta|_X)^{n-1}$, without loss of generality, we can assume that it is strictly positive. Then the operator

$$dd^c u \wedge \alpha = dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge [\beta|_X]^{m-1}$$

is an elliptic operator. If the function $u(z)$ is sh_m-function in D, then from the positivity in the generalized sense of the form $dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge [\beta|_X]^{m-1}$ we have the positivity of the form $dd^c u \wedge \alpha$, which means α-subharmonicity (see, for example, [11, 12]) of function u in local coordinates, defined by formula (3).

Let us take functions $u_1(z), u_2(z), \ldots, u_p(z) \in sh_m(D)$. Since they are α-subharmonic in the local coordinate, the maximum function $u = \max\{u_1(z), u_2(z), \ldots, u_p(z)\}$ is also α-subharmonic. This means that $dd^c u \wedge \alpha \geqslant 0$ in the generalized sense. So, we have $dd^c u \wedge \alpha \geqslant 0$ for every $v_1, \ldots, v_{m-1} \in sh_m(D) \bigcap C^2(D)$ and $\alpha = dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge [\beta|_X]^{m-1}$, i.e.

$$\left[dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge (\beta|_X)^{m-1}\right](\omega) =$$

$$= \int u \, dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge (\beta|_X)^{m-1} \wedge dd^c \omega \geqslant 0, \quad \forall \omega \in F^{0,0}(D), \; \omega \geqslant 0.$$

According to Definition 2.3, u is a sh_m-function. The proof is complete. □

5) *For any locally uniformly bounded family* $u_t(z) \in sh_m(D)$, $t \in T$, *we have*

$$\left[\sup_t u_t(z)\right]^* \in sh_m(D).$$

Similarly, the regularization of the upper limit of locally uniformly bounded sequence $u_j(z) \in sh_m(D)$ *is a* sh_m*-function, i.e.,* $\left[\overline{\lim_{j\to\infty}} u_j(z)\right]^* \in sh_m(D)$. *In particular, the regularization of the limit of a monotonically increasing, locally uniformly bounded sequence of* sh_m*-functions is again* sh_m*-function.*

Proof. Let us deal with the supremum, assuming without loss of generality that there exists $M > 0 :\ u_t(z) \leqslant M$. We fix $v_1, \dots, v_{m-1} \in sh_m(D) \bigcap C^2(D)$ and put it as above $\alpha = dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge [\beta|_X]^{m-1}$, assuming without loss of generality that α is a strictly positive $(n-1, n-1)$-form. Since $dd^c u_j \wedge \alpha \geqslant 0$, then u_j are α-subharmonic functions for the elliptic operator $dd^c u_j \wedge \alpha$. Then, just as for the Laplace operator $dd^c u_j \wedge \beta^{n-1}$ in $\mathbb{C}^n$ (see [14]), we can show that $\left[\sup_t u_t(z)\right]^* \wedge \alpha \geqslant 0$. *The proof is complete.* □

6) *Let* $u_j(z) \in sh_m(D)$ *be a sequence of* sh_m*-functions satisfying* $u_j(z) \leqslant M_j (j = 1, 2, \dots)$ *where* $\sum_{j=1}^{\infty} M_j$ *converges. Then* $\sum_{j=1}^{\infty} u_j(z)$ *is a* sh_m*-function.*

Proof. The functions $u_j(z) - M_j$ $(j = 1, 2, \dots)$ are not positive. Therefore, the sequence $v_k(z) = \sum_{j=1}^{k} [u_j(z) - M_j]$ is monotonically decreasing. By property 3) we have $\sum_{j=1}^{\infty} (u_j(z) - M_j) \in sh_m(D)$. Since the series $\sum_{j=1}^{\infty} M_j$ converges, then $\sum_{j=1}^{\infty} u_j(z) \in sh_m(D)$. The proof is complete. □

7) *Let* $\gamma(t) : \mathbb{R} \to \mathbb{R}$ *be a convex and non-decreasing function, and* $u(z) \in sh_m(D)$. *Then* $\gamma \circ u \in sh_m(D)$.

3. Maximal functions.

Maximal functions are analogous of harmonic functions in the class of sh_m-functions, they are studied by the A. Sadullaev, B. Abdullaev [9] in $\mathbb{C}^n$. Let us give the definition of a maximal sh_m-function on a Stein manifold X.

Definition 3.1. *A function* $u(z) \in sh_m(D)$, $D \subset X$ *is called maximal in the domain* $D \subset X$ *if for any function* $v(z) \in sh_m(D)$ *for which* $\underline{\lim}_{z\to\partial D} (u(z) - v(z)) \geqslant 0$ *holds* $u(z) \geqslant v(z)$ *in* D.

The condition $\underline{\lim}_{z\to\partial D} (u(z) - v(z)) \geqslant 0$ for arbitrary sh_m-functions $u(z), v(z)$ can be understood as follows: for any $\varepsilon > 0$ there exists a compact subset $F \subset D$ outside of which $v(z) \leqslant u(z) + \varepsilon$. In particular, $v(z) = -\infty$ if $u(z) = -\infty$.

Let us formulate the following theorem, which allows us to define maximal functions in convenient forms

Theorem 3.1. *The following statements are equivalent*

1) $u(z)$ *is a maximal function in* D;

2) for any subdomain $G \subset\subset D$ *the inequality* $u(z) \geqslant v(z)$, $\forall z \in G$ *holds for all functions* $v(z) \in sh_m(G)$ *satisfying* $\underline{\lim}_{z\to\partial G} (u(z) - v(z)) \geqslant 0$;

3) for any subdomain $G \subset\subset D$ the inequality $u(z) \geqslant v(z)$, $\forall z \in G$ holds for all functions $v(z) \in sh_m(D)$ for which

$$u|_{\partial G} \geqslant v|_{\partial G}.$$

4. The Dirichlet problem in the class of $\mathbf{sh_m}$-functions on a Stein manifold $\mathbf{X}$.

In this section we will discuss the solvability of the Dirichlet problem in the class of sh_m-functions on a Stein manifold $X \subset \mathbb{C}^N, \dim X = n$.

Definition 4.1. *A domain $D \subset X$ is called strictly m-convex if $D = \{\rho(z) < 0\}$ for some strictly sh_m-function $\rho(z)$ in some neighborhood D^+ of $\bar{D}$. Strictly of the sh_m-function $\rho(z)$ means that there is a $\delta > 0$ such that $\rho(z) - \delta \cdot \left(\|z\|^2\right)_X$ is a sh_m-function in D^+.*

Remark 4.1. *If the domain $D \subset X$ is a strictly m-convex, then any point $\zeta_0 \in \partial D$ is a peak point, i.e. there is a peak function $q(z) \in sh_m(D) \bigcap C\left(\overline{D}\right)$: $q\left(\zeta^0\right) = 0$, $q|_{\bar{D}\setminus\{\zeta^0\}} < 0$.*

In fact, by Definition 4.1, there is $\delta > 0$ such that the function

$$q(z) = \rho(z) - \delta \cdot \left(\left\|z - \zeta^0\right\|^2\right)_X$$

is a sh_m-function in D, which will be continuous on $\overline{D}$ and $q\left(\zeta^0\right) = 0$, $q|_{\bar{D}\setminus\zeta^0} < 0$.

Let $D \subset X$ be a strictly m-convex domain and given a continuous function $\varphi(\zeta) \in C(\partial D)$. We consider the following Dirichlet problem: find a function satisfying the following conditions

a) $u \in sh_m(D)$;

b) $\lim\limits_{z\to\zeta} u(z) = \varphi(\zeta)$, $\forall\zeta \in \partial D$;

c) *u is maximal function in D.*

In order to solve the Dirichlet problem, we will use the Perron method. Let us define the following class

$$\mathcal{U}(\varphi, D) = \left\{v \in sh_m(D) : \varlimsup_{z\to\partial D} v(z) \leqslant \varphi(\zeta)\right\}$$

and put

$$\omega(z) = \sup_{v\in U(\varphi,D)} v(z).$$

Theorem 4.1. *The upper regularization $\omega^*(z)$ of $\omega(z)$ is a solution to the Dirichlet problem, i.e. $\omega^*(z)$ satisfies the conditions a), b) and c).*

Proof. First we prove that $\omega^*(z)$ is a sh_m-function in D. Since φ is continuous and by the maximum principle we deduce that the class of functions of $\mathcal{U}(\varphi, D)$ is uniformly bounded from above. By property 5 of Section 2 its regularization is a sh_mf-unction in D.

Now we prove the continuity of the function $\omega^*(z)$ on ∂D. First, we show that $\varliminf\limits_{z\to\zeta^0} \omega(z) \geqslant \varphi(\zeta^0)$ for any fixed point $\zeta^0 \in \partial D$. Set $M = \|\varphi\|_{\partial D}$ and fix $\varepsilon > 0$. Then from the continuity of the function $\varphi(\zeta) \in C(\partial D)$ there is $r > 0$ such that

$$\left|\varphi(\zeta) - \varphi\left(\zeta^0\right)\right| < \varepsilon \quad \forall\zeta \in \partial D \bigcap B\left(\zeta^0, r\right),$$

where $B\left(\zeta^0, r\right) \subset \mathbb{C}^N$.

Since the point ζ^0 is a peak point, then there is a peak function $q(z) \in sh_m(D)$ such that

$$q\left(\zeta^0\right) = 0, \quad \sup_{\|z-\zeta^0\| \geqslant \varepsilon,\ z\in D} q(z) = q_\varepsilon < 0.$$

Let us estimate the boundary values of the following function

$$v_\varepsilon(z) = -\varepsilon + \varphi\left(\zeta^0\right) + \frac{q(z)}{|q_\varepsilon|}\left(M + \varphi\left(\zeta^0\right)\right).$$

If $\zeta \in \partial D \bigcap B\left(\zeta^0, r\right)$, then

$$\overline{\lim_{z\to\zeta}}\, v_\varepsilon \leqslant -\varepsilon + \varphi\left(\zeta^0\right) \leqslant \varphi(\zeta);$$

if $\zeta \in \partial D \backslash B\left(\zeta^0, r\right)$, then

$$\overline{\lim_{z\to\zeta}}\, v_\varepsilon \leqslant -\varepsilon + \varphi\left(\zeta^0\right) - M - \varphi\left(\zeta^0\right) \leqslant \varphi(\zeta).$$

Hence, $\overline{\lim\limits_{z\to\zeta}}\, v_\varepsilon \leqslant \varphi(\zeta)$ for all $\zeta \in \partial D$ and $v_\varepsilon \in \mathcal{U}(\varphi, D)$. Consequently, we get that $v_\varepsilon(z) \leqslant \omega(z)$ and $\underline{\lim\limits_{z\to\zeta^0}}\, \omega(z) \geqslant \underline{\lim\limits_{z\to\zeta^0}}\, v_\varepsilon(z) = -\varepsilon + \varphi\left(\zeta^0\right)$. Since $\varepsilon > 0$ is arbitrary, we have

$$\underline{\lim_{z\to\zeta^0}}\, \omega(z) \geqslant \varphi\left(\zeta^0\right).$$

Now we will show that $\overline{\lim\limits_{z\to\zeta^0}}\, \omega(z) \leqslant \varphi\left(\zeta^0\right)$. To prove this inequality we fix the function $u(z) \in \mathcal{U}(\varphi, D)$ and consider the sum $u(z) + g_\varepsilon(z)$, where

$$g_\varepsilon(z) = -\varepsilon - \varphi\left(\zeta^0\right) + \frac{q(z)}{|q_\varepsilon|}\left(M - \varphi\left(\zeta^0\right)\right).$$

It's clear that $u(z) + g_\varepsilon(z) \in sh_m(D)$. Now let's estimate the boundary values of the function $g_\varepsilon(z)$: If $\zeta \in \partial D \bigcap B\left(\zeta^0, r\right)$, then

$$\overline{\lim_{z\to\zeta}}\, g_\varepsilon(z) \leqslant -\varepsilon - \varphi\left(\zeta^0\right) \leqslant \varphi(\zeta).$$

Similarly, if $\zeta \in \partial D \backslash B\left(\zeta^0, r\right)$, then

$$\overline{\lim_{z\to\zeta}}\, g_\varepsilon(z) \leqslant -\varepsilon - \varphi\left(\zeta^0\right) + \overline{\lim_{z\to\zeta}}\, \frac{q(z)}{|q_\varepsilon|}\left(M - \varphi\left(\zeta^0\right)\right) =$$
$$= -\varepsilon - \varphi\left(\zeta^0\right) + \frac{q(\varepsilon)}{|q_\varepsilon|}\left(M - \varphi\left(\zeta^0\right)\right) = -\varepsilon - M \leqslant -\varphi(\zeta).$$

Consequently, we have

$$\overline{\lim_{z\to\zeta}}\, \left[u(z) + g_\varepsilon(z)\right] \leqslant \overline{\lim_{z\to\zeta}}\, u(z) + \overline{\lim_{z\to\xi}}\, g_\varepsilon(z) \leqslant \overline{\lim_{z\to\zeta}}\, u(z) - \varphi(\zeta) \leqslant 0$$

for any $\zeta \in \partial D$. Thus thanks to the maximum principle, $u(z) + g_\varepsilon(z) \leqslant 0$ in D, i.e. $u(z) \leqslant -g_\varepsilon(z)$, $\forall z \in D$. Since the function $u(z) \in \mathcal{U}(\varphi, D)$ is arbitrary, we get $\omega(z) \leqslant -g_\varepsilon(z)$, $z \in D$. As a consequence we deduce that

$$\overline{\lim_{z\to\zeta^0}}\, \omega(z) \leqslant \overline{\lim_{z\to\zeta^0}}\, \left(-g_\varepsilon(z)\right) = -\varepsilon + \varphi\left(\zeta^0\right).$$

Since $\varepsilon > 0$ is arbitrary, by letting $\varepsilon \to 0$ we get $\overline{\lim_{z\to\zeta^0}}\, \omega(z) \leqslant \varphi(\zeta^0)$.

Combining $\underline{\lim}_{z\to\zeta^0}\, \omega(z) \geqslant \varphi(\zeta^0)$ with $\overline{\lim}_{z\to\zeta^0}\, \omega(z) \leqslant \varphi(\zeta^0)$ we get the continuity $\lim_{z\to\zeta^0} \omega(z) = \varphi(\zeta^0)$ at every point $\zeta^0 \in \partial D$. This means that $\lim_{z\to\zeta} \omega(z) = \varphi(\zeta)$ is true in ∂D, i.e. $\omega(z)$ is continuous on ∂D. It is not difficult to see that the regularization $\omega^*(z)$ is continuous at the boundary i.e., $\lim_{z\to\zeta} \omega^*(z) = \varphi(\zeta)$, $\forall\zeta \in \partial D$.

Let us now prove that the function $\omega^*(z)$ is maximal in D. We will prove this by contrary, assume there is a domain $G \subset\subset D$ and a function $\vartheta(z) \in sh_m(D)$: $\vartheta|_{\partial G} \leqslant \omega|_{\partial G}$, but $\vartheta(z^0) > \omega(z^0)$ at some point $z^o \in G$. It's easy to see that function

$$v(z) = \begin{cases} \max\{\vartheta(z), \omega(z)\}, & z \in G \\ \omega(z), & z \in D\backslash G \end{cases}$$

is a sh_m-function and $v|_{\partial D} = \omega|_{\partial D} = \varphi$. Therefore, $v(z) \leqslant \omega(z)$ and hence $\vartheta(z^o) \leqslant \omega(z^o)$. This leads to contradiction. The proof is complete. □

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Задача Дирихле в классе $\mathbf{sh_m}$-функций на многообразии Штейна X

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Аннотация. Целью данной работы является введение и изучение sh_m-функций на комплексных многообразиях $X \subset \mathbb{C}^N$, $dimX = n$, $n \leqslant N$. Имеются разные способы определения sh_m-функций на комплексных многообразиях: при помощи локальных координат, при помощи ретракции $\pi : \mathbb{C}^N \to X$, при помощи мер Иенсена (см. [1, 8, 13]). Для определения sh_m-функций на комплексном многообразии X мы пользуемся локальными координатами. В разделе 1 мы приводим определение и простейшие свойства sh_m-функций в пространстве $\mathbb{C}^n$. В разделе 2 дается определение sh_m-функций в областях $D \subset X$ комплексного многообразия X и доказывается ряд их потенциальных свойств. В разделе 3 определяются максимальные функции и их свойства, и в разделе 4 мы докажем основной результат работы (Теорема 4.1.) о разрешимости задачи Дирихле в регулярных областях.

Ключевые слова: sh_m-функции, плюрисубгармонические функции, многообразие Штейна, задача Дирихле.