Some Classes of Sets Sufficient for Holomorphic Continuation of Integrable Functions

Bakhodir A. Shoimkulov[*]
National University of Uzbekistan
Tashkent, Uzbekistan

Baymurat J. Kutlimuratov[†]
Karakalpak State University
Nukus, Uzbekistan

Abstract. In the present work we considers integrable functions defined on a boundary of a bounded domain D in $\mathbb{C}^n$, $n > 1$, and possessing a generalized Morera boundary property. We show that such function possesses a holomorphic continuation into the domain D for some sufficient sets Γ of complex lines.

Keywords: holomorphic continuation, Morera boundary condition, Bochner-Martinelli kernel.

Introduction

This article contains some results related to the holomorphic extension of functions integrable on the boundary of a bounded domain into this domain. We will say about functions that satisfy the multidimensional Morera boundary condition. It consists in the equality to zero of the integrals of a given function over the intersection of the boundary of the domain with complex lines. E. Grinberg [1] studied functions with the Morera property in a ball (in fact, this result was contained in the article by M. L. Agranovsky and R. E. Valsky [2]). I. Globevnik and E. L. Stout [3] obtained Morera's boundary theorem for an arbitrary bounded domain with a twice smooth boundary. A local version of Morera's theorem was considered by I. Globevnik [4], D. Govekar-Leban [5]. In the work of S. G. Myslivets [6] considered are functions with the Morera property along complex curves. In the works [7–11] and [12] have given some families of complex lines sufficient for holomorphic continuation functions. The monographs [13] and [14] present some results related to this issue.

This article considers integrable functions defined on the boundary of a bounded domain D in $\mathbb{C}^n$, $n > 1$, and possessing the generalized Morera boundary property.

[*]shoimkba@rambler.ru
[†]bayko-2020@mail.ru   https://orcid.org/0000-0001-6683-5199

1.   Basic notations and definitions

We consider the set of complex lines intersecting the germ of a smooth manifold of real dimension $(2n - 2)$. Let $D \subset \mathbb{C}^n$ $(n > 1)$ be a bounded domain with a connected boundary of class C^1 of form

$$D = \{z \in \mathbb{C}^n : \ \rho(z) < 0\},$$

where $\rho(z)$ is a smooth function of class C^1 being real in the vicinity of the set $\overline{D}$ such that $d\rho \, |_{\partial D} \neq 0$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ as follows: $z = (z_1, \ldots, z_n)$, where $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $j = 1, \ldots, n$.

We consider complex lines $l_{z,b}$ of the form

$$l_{z,b} = \{\zeta \in \mathbb{C}^n : \zeta_j = z_j + b_j t, \ j = 1, 2, \ldots, n, \ t \in \mathbb{C}\} \tag{1}$$

passing through the point $z \in \mathbb{C}^n$ along the vector $b = \{b_1, \ldots, b_n\} \in \mathbb{C}P^{n-1}$ (the direction b is defined up to a multiplication of a complex number $\lambda \neq 0$).

Definition 1. *An integrable function f on ∂D ($f \in L^p(\partial D)$, $p \geqslant 1$) satisfies Morera property along complex planes l of dimension k, $(1 \leqslant k \leqslant n - 1)$, if*

$$\int\limits_{\partial D \cap l} f(\zeta) \, \beta(\zeta) = 0$$

for each differential form β of type $(k, k - 1)$ with constant coefficients.

The plane l is assumed to intersect transversally the boundary of the domain D.

If $l_{z,b}$ is a complex line intersecting ∂D transversally, then the Morera property along the planes $l_{z,b}$ is

$$\int\limits_{\partial D \cap l_{z,b}} f(z + bt) dt = \int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t) dt = 0 \tag{2}$$

for a given parameterization $\zeta = z + bt$ along a complex line $l_{z,b}$.

For complex lines we consider a more general condition.

Let m be a fixed nonnegative integer, then the condition

$$\int\limits_{\partial D \cap l_{z,b}} f(z + bt) t^m dt = \int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t) t^m dt = 0 \tag{3}$$

we will call *the generalized Morera property*. As $m = 0$, conditions (3) become the Morera boundary condition (2) (see [3]).

Let Γ be the germ of a C^1 manifold of real dimension $(2n - 2)$. We assume that $0 \in \Gamma$ and in some neighborhood of origin the manifold Γ is of the form

$$\Gamma = \{\zeta \in \mathbb{C}^n : \Phi(\zeta) + i\psi(\zeta) = 0\},$$

where Φ, ψ are C^1 smooth, real-valued functions in the neighborhood of the point zero. Here $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\zeta_j = \xi_j + i\eta_j$, $\xi_j, \eta_j \in \mathbb{R}$, $j = 1, \ldots, n$. The smoothness condition of the manifold Γ is that

$$rang \, A = rang \begin{pmatrix} \dfrac{\partial \Phi}{\partial \xi_1} \cdots \dfrac{\partial \Phi}{\partial \xi_n} & \dfrac{\partial \Phi}{\partial \eta_1} \cdots \dfrac{\partial \Phi}{\partial \eta_n} \\ \dfrac{\partial \psi}{\partial \xi_1} \cdots \dfrac{\partial \psi}{\partial \xi_n} & \dfrac{\partial \psi}{\partial \eta_1} \cdots \dfrac{\partial \psi}{\partial \eta_n} \end{pmatrix} = 2$$

at each point $\zeta \in \Gamma$.

We consider complex lines of form (1) and recall the following lemmas.

Lemma 1. *Let a vector $b^0 = (b_1^0, \ldots, b_n^0) \in \mathbb{C}P^{n-1}$ be such that $D \cap l_{0,b^0} \neq \varnothing$. Then there exists $\varepsilon > 0$ such that for all z such that $|z| < \varepsilon$ and for all b such that $\left|b - b^0\right| < \varepsilon$, the following intersections are non-empty: $D \cap l_{z,b} \neq \varnothing$ and $\Gamma \cap l_{z,b} \neq \varnothing$.*

Lemma 2. *Let for some z and for all ζ, b such that $\Gamma \cap l_{z,b} \neq \varnothing$ for $\zeta \in \partial D \cap l_{z,b}$, the function ρ defining the domain D satisfies the conditions*

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j} b_j \neq 0, \tag{4}$$

then the curves $\partial D \cap l_{z,b}$ are smooth.

For instance, the assumptions of Lemma 2 are satisfied by domains in $\mathbb{C}^n$ strongly star-shaped with respect to a point $z \in D$, strongly convex, and strongly linear convex.

Lemma 3. *If $f \in L^p(\partial D)$, $(p \geqslant 1)$ then a function $f \in L^p(\partial D \cap l_{z,b})$ for almost all $b \in \mathbb{C}P^{n-1}$.*

Proof. Let $f \in L^p(\partial D)$, $(p \geqslant 1)$ and $l_{z,b}$ be one-dimensional complex lines of the form (1) passing through the point z. Consider an open set $W \subset \mathbb{C}P^{n-1}$ such that $\partial D \cap l_{z,b}$ be smooth curves for $b \in W$. We denote the open set by $S = \bigcup_{b \in W} \partial D \cap l_{z,b}$. Then by Fubini's theorem we have

$$\int_S |f(\zeta)|^p d\sigma(\zeta) = \int_W d\sigma(b) \int_{\partial D \cap l_{z,b}} |f((z+bt)|^p \left| \frac{d\sigma(\zeta)}{d\sigma(b)} \right| dt,$$

where $d\sigma(\zeta)$, $\sigma(b)$ and dt are Lebesgue measures, respectively, on S, W and $\partial D \cap l_{z,b}$, and $\left| \frac{d\sigma(\zeta)}{d\sigma(b)} \right|$ be module of the Jacobian arising when passing to the iterated integral. The inner integral converges for almost all $b \in W$. And since

$$0 < c_1 \leqslant \left| \frac{\sigma(\zeta)}{\sigma(b)} \right| \leqslant c_2 < \infty$$

then the assessment is fair

$$0 < c_1 \int_{\partial D \cap l_{z,b}} |f|^p dt \leqslant \int_{\partial D \cap l_{z,b}} |f|^p \left| \frac{\sigma(\zeta)}{\sigma(b)} \right| dt \leqslant c_2 \int_{\partial D \cap l_{z,b}} |f|^p dt < \infty$$

The lemma is proven. □

2. Main result

Theorem 1. *Let a domain $D \subset \mathbb{C}^n$ satisfy conditions (4) for the points z lying in the vicinity of the manifold Γ such that $\partial D \cap \Gamma = \varnothing$. Let a function $f \in L^p(\partial D)$, $(p \geqslant 2)$ satisfy generalized Morera conditions (3), that is,*

$$\int_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t) t^m dt = 0$$

for each $z \in \Gamma$, $b \in \mathbb{C}P^{n-1}$ and a fixed nonnegative integer number m. Then the function f possesses a holomorphic continuation into the domain D.

Proof. We consider Bochner–Martinelli kernel

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\overline{\zeta}_k - \overline{z}_k}{|\zeta - z|^{2n}} d\overline{\zeta}[k] \wedge d\zeta.$$

As it is known, in coordinates b and t (see [13], Lemma 3.2.1), the kernel $U(\zeta, z)$ reads as:

$$U(\zeta, z) = \lambda(b) \wedge \frac{dt}{t}, \tag{5}$$

where $\lambda(b)$ is a differential form of type $(n-1, n-1)$ in $\mathbb{C}P^{n-1}$, independent of t, while $z \notin \partial D$.

We consider an integral

$$M_\alpha f(z) = \int_{\partial D_\zeta} (\zeta - z)^\alpha f(\zeta) U(\zeta, z),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an arbitrary multi-index such that

$$\|\alpha\| = \alpha_1 + \cdots + \alpha_n = m + 1$$

and

$$(\zeta - z)^\alpha = (\zeta_1 - z_1)^{\alpha_1} \ldots (\zeta_n - z_n)^{\alpha_n}.$$

By the Fubini theorem and the form of the kernel (5) we obtain:

$$M_\alpha f(z) = \int_{\mathbb{C}\mathbf{P}^{n-1}} b^\alpha \lambda(b) \int_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t) t^m dt.$$

By the conditions of Theorem 1 and Lemma 1, the identities

$$\int_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t) t^m dt = 0$$

hold for all sufficiently small z, nearby to b^0, and any b. Then

$$M_\alpha f(z) = \int_{\partial D_\zeta} (\zeta - z)^\alpha f(\zeta) U(\zeta, z) \equiv 0 \tag{6}$$

for all z such that $|z| < \varepsilon$.

We rewrite a function $M_\alpha f(z)$ in another form. We consider differential forms $U_s(\zeta, z)$:

$$U_s(\zeta, z = \frac{(-1)^s (n-2)!}{(2\pi i)^n} \left(\sum_{j=1}^{s-1} (-1)^j \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n-2}} d\overline{\zeta}[j, s] + \right.$$

$$\left. + \sum_{j=s+1}^{n} (-1)^{j-1} \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n-2}} d\overline{\zeta}[s, j] \right) \wedge d\zeta.$$

Here

$$\overline{\partial} \left(\frac{1}{\zeta_s - z_s} U_s(\zeta, z) \right) = \frac{\partial}{\partial \overline{z}_s} \left(\frac{1}{\zeta_s - z_s} U_s(\zeta, z) \right) =$$

$$= \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n}} d\overline{\zeta}[j] \wedge d\zeta = U(\zeta, z)$$

at $\zeta_s \neq z_s$, $s = 1, \ldots, n$. Then condition (6) can be rewritten as

$$\int_{\partial D_\zeta} f(\zeta)\overline{\partial}\left((\zeta - z)^\beta U_s(\zeta, z)\right) \equiv 0, \tag{7}$$

for z such that $|z| < \varepsilon$ and for all monomials $(\zeta - z)^\beta$ with $\|\beta\| = m$.

We are going to show that condition (7) holds for monomials $(\zeta - z)^\gamma$ with $\|\gamma\| = m$. Indeed, we consider a monomial $(\zeta - z)^\gamma$ with $\|\gamma\| = m - 1$. Then condition (7) holds for the monomials of form:

$$(\zeta - z)^\beta (\zeta_k - z_k), \quad k = 1, \ldots, n,$$

since the degree of these monomials is equal to m.

The identity holds:

$$\frac{\partial}{\partial \zeta_k}\left((\zeta - z)^\gamma (\zeta_k - z_k)U_s(\zeta, z)\right) = (\gamma_k + 1)(\zeta - z)^\gamma U_s(\zeta, z) -$$
$$- (n - 1)(\zeta - z)^\gamma \frac{(\zeta_k - z_k)(\overline{\zeta}_k - \overline{z}_k)}{|\zeta - z|^2} U_s(\zeta, z). \tag{8}$$

Summing up identities (8) over k, we obtain:

$$\sum_{k=1}^n \frac{\partial}{\partial \zeta_k}\left((\zeta - z)^\gamma (\zeta_k - z_k)U_s(\zeta, z)\right) = (\|\gamma\| + 1)(\zeta - z)^\gamma U_s(\zeta, z). \tag{9}$$

Since condition (7) can be differentiated in z as $|z| < \varepsilon$, and the derivatives of expressions (9) in z and ζ differ only by the sign, it follows from (9) that the degree of the monomial in (7) can be lessen by one. By successively decreasing this degree, we arrive at the following conditions

$$\int_{\partial D_\zeta} f(\zeta)\overline{\partial} U_s(\zeta, z) \equiv 0$$

for $|z| < \varepsilon$ and $s = 1, \ldots, n$, i.e.

$$\int_{\partial D_\zeta} (\zeta_s - z_s)f(\zeta)U(\zeta - z) \equiv 0 \tag{10}$$

for $|z| < \varepsilon$ and $s = 1, \ldots, n$.

Applying the Laplace operator with respect to z to the left side of equality (10)

$$\Delta = \frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \cdots + \frac{\partial^2}{\partial z_n \partial \overline{z}_n}$$

we obtain that

$$\frac{\partial}{\partial \overline{z}_s} \int_{\partial D_\zeta} f(\zeta)U(\zeta, z) \equiv 0$$

for $|z| < \varepsilon$ and $s = 1, \ldots, n$. Here we used the harmonicity of the kernel $U(\zeta, z)$ and the identity

$$\Delta(gh) = h\Delta g + g\Delta h + \sum_{j=1}^n \frac{\partial g}{\partial \overline{z}_j} \cdot \frac{\partial h}{\partial z_j} + \sum_{j=1}^n \frac{\partial g}{\partial z_j} \frac{\partial h}{\partial \overline{z}_j}.$$

Hence, the Bochner–Martinelli integral of the function $f \in L^p(\partial D)$, namely,

$$Mf(z) = \int\limits_{\partial D_\varsigma} f(\varsigma)U(\varsigma, z)$$

is a function holomorphic in the vicinity of the origin.

If $\Gamma \subset \mathbb{C}^n \backslash \overline{D}$, then $Mf(z) \equiv 0$ outside $\overline{D}$ since the boundary is connected and then the function f is continued holomorphically in the domain D (see [13]).

If $\Gamma \subset D$, then the function Mf is holomorphic in D and the boundary values of Mf coincide with f. $\square$

Corollary 1. *Let a domain D satisfy the conditions of Theorem 1 and the function $f \in L^p(\partial D)\,(p \geqslant 2)$ satisfies condition (2) for all $z \in \Gamma$ and $b \in \mathbb{C}P^{n-1}$. Then the function f can be holomorphically continued into D.*

References

[1] E.A.Grinberg, A boundary analogue of Morera's theorem in the unit ball of C^n, *Proceedings of the American Society*, **102**(1988), no, 1, 114–116.

[2] M.L.Agranovskii, R.E.Valskii, Maximality of invariant algebras of functions, *Siberian Mathematical Journ*, **12**(1971), no. 1, 1–7.

[3] J.Globevnik, E.L.Sto, Boundary Morera theorems for holomorphic functions of several complex variables, *Duke Mathematical Journal*, **64**(1991), no. 3, 571–615.

[4] J.Globevnik, A boundary Morera theorem, *Jounal of Geometric Analysis*, **3**(1993), no. 3, 269–277.

[5] D.Govekar-Leban, Local boundary Morera theorems, *Mathematische Zeitschrift*, **233**(2000), 265-286. DOI: 10.1007/PL00004796

[6] S.G.Myslivets, One boundary version of Morera's theorem, *Siberian Mathematical Journal*, **42**(2001), no. 5, 952–960. DOI: 10.1023/A:1011923929133

[7] A.M.Kytmanov, S.G.Myslivets, On families of complex lines sufficient for holomorphic extension, *Mathematical Notes*, **83**(2008), no. 4, 500–505. DOI: 10.1134/S0001434608030231

[8] A.M.Kytmanov, S.G.Myslivets, Some families of complex lines sufficient for holomorphic extension of functions, *Russian Mathematic (Iz. VUZ)*, **55**(2011), no. 4, 60–66. DOI: 10.3103/S1066369X11040074

[9] A.M.Kytmanov, S.G.Myslivets, On family of complex straight lines sufficient for existence of holomorphic continuation of continuous functions on boundary of domain, *Ufa Mathematical Journal*, **12**(2020), no. 3, 45–50

[10] A.M.Kytmanov, S.G.Myslivets, On some sets sufficient for holomorphic continuation of functions with generalized boundary Morera property, *Udmurd University Journal*, **33**(2023), no. 3, 483–496 (in Russian).

[11] B.P.Otemuratov, On functions with one-dimensional holomorphic extension property *Vestnik KrasGU*, (2006), no. 9, 95–100.

[12] B.T.Kurbanov, Morera's Boundary Theorem in Siegel Domain of the First Kind, *J. Sib. Fed. Univ. Math. Phys.*, **15**(2022), no. 2, 253–260. DOI: 10.17516/1997-1397-2022-15-2-255-262

[13] A.M.Kytmanov, S.G.Myslivets, Multidimensional Integral Representations, Problems of Analytic Continuation, Springer Verlag, Basel, Boston, 2015.

[14] K.Ghudayberganov, A.M.Kytmanov, B.A.Shaymkulov, Analysis in matrix domains, Siberian Federal University, Institute of Mathematics and Fundamental Informatics, Krasnoyarsk, SFU, 2017 (in Russian).

О некоторых классах множеств, достаточных для голоморфного продолжения интегрируемых функций

Баходир А. Шоимкулов
Национальный университет Узбекистана
Ташкент, Узбекистан

Баймурат Ж. Кутлымуратов
Каракалпакский госуниверситет
Нукус, Узбекистан

Аннотация. В данной статье рассматриваются интегрируемые функции, заданные на границе ограниченной области D в $\mathbb{C}^n$, $n > 1$, и обладающие обобщенным граничным свойством Морера. Исследуется вопрос о существовании голоморфного продолжения таких функций в область D для некоторых достаточных множеств Γ комплексных прямых.

Ключевые слова: голоморфное продолжение, граничное условие Морера, ядро Бохнера-Мартинелли.