Journal of Siberian Federal University. Mathematics & Physics 2024, 17(3), 448–454

EDN: BKHEBO
УДК 517.9; 515.124

On Cyclic Interpolative Kannan- Meir-Keeler Type Contraction in Metric Spaces

Mohamed Edraoui*
Faculty of Sciences Ben M'sik
Hassan II University
Casablanca, Morocco
Soukaina Semami†
Faculty of Science Ain Chock
University Hassan II
Casablanca, Morocco
Amine El'koufi‡
High School of Technology, Ibn Tofail University
BP 240, Kenitra, Morocco
Faculty of Sciences
Ibn Tofail University
BP 133, Kenitra, Morocco
Mohamed Aamri§
Faculty of Sciences Ben M'sik, Hassan II University
Casablanca, Morocco

Received 10.11.2023, received in revised form 15.12.2023, accepted 14.03.2024

Abstract. This paper introduces the idea of interpolative contractions within the category of Kannan-Meir-Keeler type cyclic contractions. Additionally, we provide a demonstration establishing the existence of a fixed point in a complete metric space.

Keywords: interpolative, cyclic contraction, fixed point, metric space, Kannan- Meir-Keeler.

Citation: M. Edraoui, S. Semami, A. El'koufi, M. Aamri, On Cyclic Interpolative Kannan-Meir-Keeler Type Contraction in Metric Spaces, J. Sib. Fed. Univ. Math. Phys., 2024, 17(4), 448–454. EDN: BKHEBO.

1. Introduction

Fixed point theorems play a crucial role in the examination of cyclic mappings as they establish conditions that ensure the existence and uniqueness of fixed points. These conditions may vary depending on the specific class of cyclic mappings being considered. For example, the Banach fixed point theorem guarantees the existence and uniqueness of a fixed point for contraction mappings defined on a complete metric space. Conversely, the Brouwer fixed point theorem guarantees the existence of at least one fixed point for continuous mappings defined on a compact, convex set.

It is important to note that while a fixed point theorem may guarantee the existence of fixed points for a given class of mappings, it does not necessarily provide a method for finding those

*edraoui.mohamed@gmail.com https://orcid.org/0000-0002-3324-2546
†soukaina.sema@gmail.com https://orcid.org/0000-0002-7056-5117
‡amine.elkoufi@uit.ac.ma https://orcid.org/0000-0001-6801-6672
§aamrimohamed9@yahoo.fr

fixed points. In practice, finding fixed points for specific mappings can be challenging or even impossible.

The uniqueness of fixed points is also significant in certain problems, and fixed point theorems that guarantee uniqueness become particularly useful in such cases. example that applies to contraction mappings see for example [13–20]

In 2003, Kirk et al. [13] introduced the concept of cyclical contractive mappings and extended the Banach fixed point theorem to this class of cyclic mappings. They generalized the notion of contractive mappings to cyclical contractive mappings and proved the existence of a a unique fixed point for such mappings. This extension expanded the class of mappings for which the existence and uniqueness of fixed points can be guaranteed, providing a new tool for studying fixed points in various mathematical contexts.

Regarding the "interpolative Kannan- Meir-Keeler type contractive mapping" and its generalization of Meir-Keeler's fixed point theorem, it is interesting to observe that Erdal Karapınar introduced this new type of mapping by incorporating the concept of interpolation into the Kannan- Meir-Keeler framework. This approach likely allows for the generation of intermediate points between known data points and expands the applicability of the original theorem.

The utilization of interpolation to generalize various forms of contractions is a common practice in mathematical research. By integrating interpolation techniques into contraction mappings, researchers can extend the scope of existing theorems and provide a more flexible framework for analyzing fixed points in metric spaces.

Furthermore, it appears that the interpolative method has been employed in other research as well to generalize different types of contractions. This demonstrates the versatility and effectiveness of the interpolation approach in expanding the theory of fixed points and providing new insights into the existence and uniqueness of solutions.

For a more in-depth exploration of Karapinar's work and the generalization of other forms of contractions using the interpolative method, I recommend referring to the cited papers [1–4, 6–14, 16] and exploring related research in the field. These sources should provide a comprehensive understanding of the interpolative Hardy-Rogers type contractive mapping and its applications in fixed point theory.

Definition 1.1. *Let (E, d) be a metric space and let X and Y be two nonempty subsets of E. A mapping $T : X \cup Y \to X \cup Y$ is said to be a cyclic mapping provided that*

$$T(X) \subseteq Y, \quad T(Y) \subseteq X. \tag{1}$$

A point $x \in X \cup Y$ is called a best proximity point if $d(x, Tx) = d(X, Y)$ where $d(X, Y) = \inf(d(x, y) : x \in X, y \in Y)$.

In 1969 A. Meir, E. Keeler [14] proved

Theorem 1.2. *Let (E, d) be a complete metric space and T be a self-mapping of E is said to be a Meir-Keeler contraction on E , if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leqslant d(x, y) < \epsilon + \delta \Rightarrow d(Tx, Ty) < \epsilon \tag{2}$$

for evrey $x, y \in E$. Then T has a unique fixed point in E.

Recently Karapinar [1] proposed a new Kannan-Meir-Keeler contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called "interpolative Kannan-Meir-Keeler type contractive mapping" is a generalization of Meir-Keeler 's fixed point theorem. The interpolative method has been used in other research to generalize other forms of contractions as well [1–4, 6–12] . This method is a powerful tool in the study of fixed point theory, as it allows for the construction of new classes of contractive mappings and the discovery of new fixed point theorems.

2. Main results

The interpolation method has been used to generalize the definition of interpolation Meir-Keeler contraction type cyclic contraction by incorporating the notion of interpolation. This leads to a more general definition of a Meir-Keeler-type cyclic contraction that allows for the construction of new classes of contractive mappings and the discovery of new fixed point theorems. The idea is that, by incorporating interpolation, the definition of a Meir-Keeler-type cyclic contraction can be expanded and new properties can be discovered.

Consequently, with the generalization of the definition of Meir-Keeler-type cyclic contraction via interpolation, it becomes reasonable to anticipate the establishment of a fixed point theorem for this newly introduced class of mappings.

Definition 2.1. *Let (E,d) be a metric space and let X and Y be nonempty subsets of E. A cyclic map $T : X \cup Y \to X \cup Y$ is said to be a interpolative Kannan- Meir-Keeler type cyclic contraction on E, if there exists $\alpha \in (0,1)$ such that*

1. *given $\epsilon > 0$, there exists $\delta > 0$ so that*

$$\epsilon < [d(Tx,x)]^{\alpha} \cdot [d(Ty,y)]^{1-\alpha} < \delta + \epsilon \Rightarrow d(Tx,Ty) \leqslant \epsilon \tag{3}$$

2. $$d(Tx,Ty) < [d(Tx,x)]^{\alpha} \cdot [d(Ty,y)]^{1-\alpha} \tag{4}$$

for all $(x,y) \in X \times Y$ with $x, y \notin Fix(T)$

The fixed point theorem for an interpolative Kannan- Meir-Keeler type cyclic contraction can be stated as: In a complete metric space if a mapping satisfies certain conditions such as being an interpolative Kannan- Meir-Keeler-type cyclic contraction, then it has a fixed point.

This theorem can be proved by using the properties of interpolative Kannan- Meir-Keeler-type cyclic contraction mappings and the Banach fixed point theorem. By using interpolation, we can construct a new class of contractive mappings with a unique fixed point.

Theorem 2.2. *Let (E,d) be a complete metric space and let X and Y be nonempty subsets of X and let $T : X \cup Y \to X \cup Y$ be interpolative Kannan- Meir-Keeler type cyclic contraction. Then T has a fixed point in $X \cap Y$.*

Proof. Suppose that x is an arbitrary point in X each positive integer n

From (7) it follows that which yields that

$$\begin{aligned} d\left(T^{n+2}x, T^{n+1}x\right) &= d\left(T\left(T^{n+1}x\right), T\left(T^{n}x\right)\right) < \\ &= < \left[d\left(T\left(T^{n+1}x\right), T^{n+1}x\right)\right]^{\alpha} \cdot \left[d\left(T\left(T^{n}x\right), T^{n}x\right)\right]^{1-\alpha} = \\ &= \left[d\left(T^{n+2}x, T^{n+1}x\right)\right]^{\alpha} \cdot \left[d\left(T^{n+1}x, T^{n}x\right)\right]^{1-\alpha} \end{aligned}$$

which yields that

$$d\left(T^{n+2}x, T^{n+1}x\right)^{1-\alpha} < . \left[d\left(T^{n+1}x, T^{n}x\right)\right]^{1-\alpha} \text{ for all } n \geqslant 0.$$

Then, the sequence $\left\{d\left(T^{n+1}x, T^{n}x\right)\right\}$ is strictly nonincreasing and since $d\left(T^{n+1}x, T^{n}x\right) > 0$, for $n \in \mathbb{N} \cup \{0\}$ it follows that the sequence $\left\{d\left(T^{n+1}x, T^{n}x\right)\right\}$ tends to a point $r \geqslant 0$. We claim that $r = 0$. Indeed, if we suppose that $r > 0$, we can find $N \in \mathbb{N},$, such that:

$$r < d\left(T^{n+1}x, T^{n}x\right) < r + \delta(r)$$

for any $n \geqslant N$. Then, since $r < d\left(T^{n+1}x, T^n x\right) < \left[d\left(T^{n+1}x, T^n x\right)\right]^{\alpha} \cdot \left[d\left(T^n x, T^{n-1}x\right)\right]^{1-\alpha}$ keeping in mind (3), it follows that $d\left(T^{n+1}x, T^n x\right) \leqslant r$, for any $n \geqslant N$. This is a contradiction, and that's why we get $r = 0$.

In order to show that $\{T^n x\}$ is a Cauchy sequence, let $\epsilon > 0$, be fixed and we can consider that $\delta(\epsilon)$ can be choose such that $\delta(\epsilon) < \epsilon$. Since $\lim_{n\to\infty} d\left(T^{n+1}x, T^n x\right) = 0$, we can find $l \in \mathbb{N}$ such that $d\left(T^{n+1}x, T^n x\right) < \frac{\epsilon}{2}$, for $n \geqslant l$, and we claim that $d(x_{n+1}, x_n) < \frac{\epsilon}{2}$, for $m > l$, and we claim that

$$d\left(T^{n+m}x, T^n x\right) < \epsilon \tag{5}$$

for any $m \in \mathbb{N}$. Of course, the above inequality holds for $m = 1$. Supposing that for some m, (5) holds, we will prove it for $m+1$. Indeed, using the triangle inequality, together with (7) we have

$$\begin{aligned} d\left(T^{n+m+1}x, T^n x\right) &\leqslant d\left(T^{n+m+1}x, T^{n+1}x\right) + d\left(T^{n+1}x, T^n x\right) = \\ &= d\left(T\left(T^{n+m}x\right), T\left(T^n x\right)\right) + d\left(T^{n+1}x, T^n x\right) < \\ &< \left[d\left(T^{n+m+1}x, T^{n+m}x\right)\right]^{\alpha} \cdot \left[d\left(T^n x, T^{n-1}x\right)\right]^{1-\alpha} + d\left(T^{n+1}x, T^n x\right) < \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, the sequence $\{T^n x\}$ is Cauchy. Since (E, d) is complete there exists $z \in E$ such that $T^n x \to z$.

Hence $\{T^n x\}$ is a Cauchy sequence. Then, there exists a $z \in X \cup Y$ such that $T^n x \to z$.

Notice that $\left\{T^{2n}x\right\}$ is a sequence in X and $\left\{T^{2n+1}x\right\}$ is a sequence in Y having the same limit z. As X and Y are closed, we conclude $z \in X \cap Y$, that is, $X \cap Y$ is nonempty.

We claim that $Tz = z$. Observe that:

$$\begin{aligned} d(z, Tz) &= \lim_{n\to\infty} d\left(Tz, T^{2n}x\right) = \lim_{n\to\infty} d\left(Tz, TT^{2n-1}x\right) \leqslant \\ &\leqslant \lim_{n\to\infty} \left[d(Tz, z)\right]^{\alpha} \cdot \left[d\left(T^{2n}x, T^{2n-1}x\right)\right]^{1-\alpha} \leqslant \\ &\leqslant \lim_{n\to\infty} \left[d(Tz, z)\right]^{\alpha} \cdot \left[d\left(T^{2n-1}x, T^{2n}x\right)\right]^{1-\alpha}. \end{aligned}$$

Taking $n \to \infty$ in the inequality above, we derive that $d(z; Tz) = 0$ that is $Tz = z$. □

Example 2.3. *Let $E = \mathbb{R}^2$ and $X = Y = \{\xi_1, \xi_2, \xi_3, \xi_4\}$, where $\xi_1 = (1, -1)$, $\xi_2 = (-1, 0)$, $\xi_3 = (2, -1)$, $\xi_4 = (2, 0)$. Let $d : E \times E \to [0, \infty)$ be defined as $d(\zeta, \zeta') = \sqrt{(x_1, y_1)^2 + (x_2 - y_2)^2}$ for any $\zeta, \zeta' \in E$, $\zeta = (\zeta_1, \zeta_2)$ and $\zeta' = (\zeta_1', \zeta_2')$, with $\zeta_1, \zeta_2,'_1, \zeta_2' \in \mathbb{R}$. Define the mapping $T : E \to E$ as follows*

$$T\xi_1 = T\xi_3 = T\xi_4 = \xi_3, \quad T\xi_2 = \xi_4, \ \textit{and} \ T\zeta = \zeta \ \textit{ for any } \ \zeta \in E \setminus \{\xi_1, \xi_2, \xi_3, \xi_4\}.$$

We choose $\alpha = \frac{1}{2}$.

Thus, we claim that T satisfies the conditions of Theorem.

Indeed, for $\epsilon < 1$, with $\delta = \sqrt{2} - \epsilon$

$$\epsilon < 1 = \sqrt{d(\xi_1, T\xi_1)\, d(\xi_4, T\xi_4)} = \sqrt{d(\xi_1, \xi_3)\, d(\xi_4, \xi_3)} < \sqrt{2} = \epsilon + \delta \Rightarrow$$

$$d(T\xi_1, T\xi_4) = d(\xi_3, \xi_3) = 0 < \epsilon$$

and also

$$d(T\xi_1, T\xi_4) < \sqrt{d(\xi_1, \xi_3)\, d(\xi_4, \xi_3)}.$$

For $\epsilon \geqslant 1$, choosing $\delta = 1$, we get

$$\epsilon < \sqrt{d(\xi_1, T\xi_1)\, d(\xi_2, T\xi_2)} = \sqrt{d(\xi_1, \xi_3)\, d(\xi_2, \xi_4)} < \sqrt{3} < \epsilon + \delta \Rightarrow$$
$$d(T\xi_1, T\xi_2) = d(\xi_3, \xi_4) = 1 < \epsilon$$

and

$$d(T\xi_1, T\xi_2) = 1 < \sqrt{3} = \sqrt{d(\xi_1, \xi_3)\, d(\xi_2, \xi_4)}.$$

Similar,

$$\epsilon < \sqrt{d(\xi_4, T\xi_4)\, d(\xi_2, T\xi_2)} = \sqrt{d(\xi_4, \xi_3)\, d(\xi_2, \xi_4)} < \sqrt{3} < \epsilon + \delta \Rightarrow$$
$$d(T\xi_4, T\xi_2) = d(\xi_3, \xi_4) = 1 < \epsilon$$

and

$$d(T\xi_4, T\xi_2) = 1 < \sqrt{3} = \sqrt{d(\xi_1, \xi_3)\, d(\xi_2, \xi_4)}$$

T satisfies all the conditions of the above theorem. Hence T has fixed points in $X \cap Y$. In fact $\xi_3 = (2, -1) \in X \cap Y$ is the fixed point. and all $P \in E \backslash \{\xi_1, \xi_2, \xi_3, \xi_4\}$.

Corollary 2.4. *Let X and Y be two non-empty closed subsets of a complete metric space (E, d). Let $T_1 : X \to Y$ and $T_2 : Y \to X$ be two functions. Assume that there exists $\alpha \in (0, 1)$ such that*

1 given $\epsilon > 0$, there exists $\delta > 0$ so that

$$\epsilon < [d(T_1 x, x)]^{\alpha} \cdot [d(T_2 y, y)]^{1-\alpha} \Rightarrow d(T_1 x, T_2 y) \leqslant \epsilon \tag{6}$$

2

$$d(T_1 x, T_2 y) < [d(T_1 x, x)]^{\alpha} \cdot [d(T_2 y, y)]^{1-\alpha} \tag{7}$$

for all $(x, y) \in X \times Y$ with $x, y \notin Fix(T_i)$ $i = 1, 2$. Then there exists a unique $z \in X \cap Y$ such that $T_1(z) = T_2(z) = z$.

Proof. Let $T : X \cup Y \to X \cup Y$ defined by

$$T(x) = T_1(x) \text{ if } x \in X T_2(x) \text{ if } x \in Y.$$

Then T be interpolative Kannan- Meir-Keeler type cyclic contraction on complete metric space (E, d), we can now apply Theorem 2 to deduce that T has a fixed point $z \in X \cap Y$ such that $T_1(z) = T_2(z) = z$. ☐

Dedicated to my student Hafsa Said.

References

[1] E.Karapınar, Interpolative Kannan-Meir-Keeler type contraction, *Adv. Theory Nonlinear Anal. Its App.*, **5**(2021), 611–614. DOI: 10.31197/atnaa.989389

[2] E.Karapınar, Andreea, S.S.Yeşilkaya, Interpolative Meir–Keeler Mappings in Modular Metric Spaces, *Mathematics*, **10**(2022), 2986. DOI: 10.3390/math1016298

[3] E.Karapinar, Revisiting the Kannan type contractions via interpolation, *Adv. Theory Nonlinear Anal. Appl.*, **2**(2018), no. 2, 85–87. DOI: 10.31197/atnaa.431135

[4] E.Karapınar, O.Alqahtani, H.Aydi, On Interpolative Hardy-Rogers Type Contractions, *Symmetry*, **11**(2018), 8. DOI: 10.3390/sym11010008

[5] Y.U.Gaba, E.Karapınar, A New Approach to the Interpolative Contractions, *Axioms*, **8**(2019), 110. DOI: 10.3390/axioms8040110

[6] E.Karapınar, R.P.Agarwal, Interpolative Rus-Reich-Ciric- type contractions via simulation functions, *Analele Univ. Ovidius Constanta-Ser. Mat.*, **27**(2019), 137–152. DOI: 10.2478/auom-2019-0038

[7] E.Karapınar, R.Agarwal, H.Aydi, Interpolative Reich–Rus–Cirić type contractions on partial metric spaces, *Mathematics*, **6**(2018), 256. DOI: 10.3390/math6110256

[8] E.Karapınar, A.Fulga, S.S.Yeşilkaya, New results on Perov-Interpolative contractions of Suzuki type mappings, *J. Funct. Spaces*, **2021**(2021), 9587604. DOI:10.1155/2021/9587604

[9] M.Edraoui, A.El koufi, M.Aamri, On interpolative Hardy-Rogers type cyclic contractions, *Appl. Gen. Topol.*, **25**(2024), no. 1, 117–124.

[10] E.Karapınar, Revisiting simulation functions via interpolative contractions, *Appl. Anal. Discret. Math.*, **13**(2019), 859–870.

[11] E.Karapınar, A.Fulga, Roldán López de Hierro, A.F. Fixed point theory in the setting of (α, β, ψ, ϕ)-interpolative contractions, *Adv. Differ. Equation*, **2021**(2021), 339.

[12] H.Aydi, C.M.Chen, E.Karapınar, Interpolative Cirić-Reich-Rus type contractions via the Branciari distance, *Mathematics*, **7**(2019), 84. DOI:10.3390/math7010084

[13] W.A.Kirk, P.S.Srinivasan, P.Veeramani, Fixed point fo mappings satisfyaing cyclical contractive conditions, *Fixed Point Theory*, **4**(2003), 79–89

[14] A.Meir, E.Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, 1969, 28, 326.

[15] M.Edraoui, A El koufi, M.Aamri "Best Proximity point theorems for proximal pointwise tricyclic contraction Adv, *Fixed Point Theory*, **13**(2023), no. 22. DOI: 10.28919/afpt/8194

[16] M.Edraoui, A.El koufi, S.Semami, Fixed points results for various types of int olative cyclic contraction, *Appl. Gen. Topol.*, **24**(2023), no. 2, 247–252. DOI: 10.4995/agt.2023.19515

[17] M.Edraoui, M.Aamri, S.Lazaiz, Fixed point theorem in locally K-convex space, *Journal of Mathematical Analysis*, **12**(2018), no. 10, 485–490. DOI: 10.12988/ijma.2018.8753

[18] Edraoui Mohamed, Aamri Mohamed, Lazaiz Samih, Fixed point theorem for pα-nonexpansive wrt orbits in locally convex space, *J. Math. Comput. Sci.*, **1**(2020), 1582–1589

[19] E.Mohamed, A.Mohamed, L.Samih, Relatively cyclic and noncyclic P-contractions in locally K-convex space, *Axioms*, **8**(2019), 96.

[20] M.Edraoui, A.Baiz, A.Faiz, J.Mouline, K.Bouzkoura, Best proximity point theorems for tricyclic diametrically contractive mappings, *Adv. Fixed Point Theory*, **14**(2024), Article ID 4.

О циклическом интерполяционном сжатии типа Каннана-Меира-Килера в метрических пространствах

Мохамед Эдрауи
Факультет естественных наук Бен М'сик
Университет Хасана II
Касабланка, Марокко

Соукаина Семами
Факультет естественных наук Айн Чок
Университет Хасана II
Касабланка, Марокко

Амин Элькуфи
Высшая школа технологий Университета Ибн Тофаила
BP 240, Кенитра, Марокко
Факультет естественных наук
Университет Ибн Тофаила
BP 133, Кенитра, Марокко

Мохамед Аамри
Факультет естественных наук Бен М'сик
Университет Хасана II
Касабланка, Марокко

Аннотация. В данной статье представлена идея интерполяционных сокращений в категории циклических сокращений типа Каннана-Меира-Килера. Кроме того, мы обеспечиваем демонстрацию, устанавливающую существование неподвижной точки в полном метрическом пространстве.

Ключевые слова: интерполяция, циклическое сжатие, неподвижная точка, метрическое пространство, Каннан-Меир-Килер.