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Oscillatory Integrals for Mittag-Leffler Functions

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Abstract. Variations of the van der Corput lemmas that involve Mittag-Leffler functions are studied in this paper. The extension involves replacing the exponential function with a Mittag–Leffler-type function. It allows one to analyze oscillatory integrals encountered in the study of time-fractional partial differential equations. Several generalizations of both the first and second van der Corput lemmas are established. Optimal estimates for decay orders in specific cases of Mittag–Leffler functions are also derived.

Keywords: Mittag–Leffler functions, phase function, amplitude.

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1. Introduction and preliminaries

The Mittag–Leffler function $E_\alpha(z)$ is named after the Swedish mathematician Gösta Magnus Mittag–Leffler (1846–1927) who defined it by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0,$$

and studied its properties in 1902–1905 [13–16] in connection with his summation method for divergent series. It was also studied independently by Humbert and Agarval [1, 6, 7] and by Dzherbashyan [2–4] (see also [5] and the references therein).

In this paper, a special case of the generalized Mittag–Leffler function is considered. It is defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{R}.$$

Obviously,

$$E_{1,1}(x) = e^x. \tag{1}$$

Let us consider the following integral with phase ϕ and amplitude ψ

$$I_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx, \tag{2}$$

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where $0 < \alpha \leqslant 1$, $\beta > 0$ and $\lambda > 0$.

If $\alpha = \beta = 1$ in integral (2) then integral $I_{1,1}$ is called classical oscillatory integral. In harmonic analysis, the most important estimate for oscillatory integrals is van der Corput lemma [18,19,28]. Estimates for oscillatory integrals with polynomial phase can be bound [8–12] and also [22–27]. In this paper exponential function is replaced with the Mittag–Leffler type function and oscillatory type integrals (2) are studied. Analogues of the van der Corput lemmas involving Mittag–Leffler functions for one dimensional integrals were considered [20] and [21]. Result of [20] is extended for the case where amplitude is more general.

The main results of the paper are the following

Theorem 1. *Let $-\infty \leqslant a < b \leqslant +\infty$. Let $\phi : [a, b] \to \mathbb{R}$ be a measurable function and let $\psi \in L^p[a, b]$, $p \geqslant 1$. If $0 < \alpha < 1$, $\beta > 0$ and $m = \operatorname{ess\,inf}_{x\in[a,b]} |\phi(x)| > 0$ then there is estimate*

$$|I_{\alpha,\beta}(\lambda)| \leqslant \frac{C\|\psi\|_{L^p[a,b]}}{1 + m\lambda}, \tag{3}$$

where C does not depend on ϕ, ψ and $\lambda > 0$.

Theorem 2. *Let $-\infty \leqslant a < b \leqslant +\infty$ and $0 < \alpha < 1$, $\beta > 0$. Let $\phi \in L^\infty[a, b]$ be a real-valued differentiable monotonic function on $[a, b]$ with $m = \inf_{x\in[a,b]} |\phi'(x)| > 0$, and let $\psi \in L^p[a, b]$, $1 < p \leqslant \infty$. Assume that ϕ has finitely many zeros $\{c_j\} \subset [a, b]$ then*

(i): *If $1 < p < \infty$ then we have*

$$|I_{\alpha,\beta}(\lambda)| \leqslant \frac{C_p\|\psi\|_{L^p[a,b]}}{(1 + m\lambda)^{1-\frac{1}{p}}}, \quad \lambda \geqslant 0. \tag{4}$$

(ii): *If $p = \infty$ then we have*

$$|I_{\alpha,\beta}(\lambda)| \leqslant \frac{C\|\psi\|_{L^p[a,b]}}{1 + m\lambda} \log(2 + \lambda\|\phi\|_{L^p[a,b]}) \ , \ \lambda \geqslant 0, \tag{5}$$

here C does not depend on λ and C_p depends only on p.

Theorem 3. *Let $-\infty < a < b < +\infty$ and $0 < \alpha < 1$, $\beta > 0$, $p > 1$. Let ϕ is a real valued function such that $\phi \in C^k[a, b]$ and let $\psi \in L^p[a, b]$, $1 < p \leqslant \infty$. If ϕ has finitely many zeros on [a,b] and $\left|\phi^{(k)}(x)\right| \geqslant 1$, $k \geqslant 2$ for all $x \in [a, b]$ then*

(i): *If $1 < p < \infty$ then*

$$\left|\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx\right| \leqslant \frac{C_k\|\psi\|_{L^p[a,b]}}{(1 + \lambda)^{\frac{1}{k}-\frac{1}{pk}}}, \qquad \lambda \geqslant 0. \tag{6}$$

(ii): *If $p = \infty$, then*

$$\left|\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)\,dx\right| \leqslant \frac{C_k\|\psi\|_{L^p[a,b]}}{(1 + \lambda)^{\frac{1}{k}-\frac{1}{pk}}} \log\left(2 + \lambda\|\phi\|_{L^p[a,b]}\right), \tag{7}$$

here C_k does not depend on $\lambda \geqslant 0$.

2. Proof of main results

In this section, some auxiliary statements are briefly reviewed for the sake of the rest of the paper and results are proved.

Proposition 1. ([17]) *If* $0 < \alpha < 2$, β *is an arbitrary real number,* μ *is such that* $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ *then there is* $C > 0$ *such that*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \; z \in \mathbb{C}, \; \mu \leqslant |\arg(z)| \leqslant \pi. \tag{8}$$

Proposition 2. ([1]) *Let* $\alpha, \beta > 0$ *and* $\phi : [a,b] \to \mathbb{C}$. *Then for all* $\lambda \in \mathbb{C}$

$$E_{\alpha,\beta}(i\lambda\phi(x)) = E_{2\alpha,\beta}\left(-\lambda^2\phi^2(x)\right) + i\lambda\phi(x)\,E_{2\alpha,\beta+\alpha}\left(-\lambda^2\phi^2(x)\right). \tag{9}$$

Proof of Theorem 1. For small λ integral (2) is just bounded. Let us consider the case $\lambda \geqslant 1$. Let ϕ: $[a,b] \to \mathbb{R}$ be a measurable function and let $\psi \in L^p[a,b]$. Then

$$|I_{\alpha,\beta}(\lambda)| = \left|\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx\right| \leqslant \int_a^b |E_{\alpha,\beta}(i\lambda\phi(x))|\,|\psi(x)|\,dx. \tag{10}$$

Using formula (9) and estimate (8) we have that

$$\begin{aligned} |E_{\alpha,\beta}(i\lambda\phi(x))| &\leqslant \left|E_{2\alpha,\beta}\left(-\lambda^2\phi^2(x)\right)\right| + \lambda\,|\phi(x)|\,\left|E_{2\alpha,\beta+\alpha}\left(-\lambda^2\phi^2(x)\right)\right| \leqslant \\ &\leqslant \frac{C}{1+\lambda^2\phi^2(x)} + \frac{C\lambda\,|\phi(x)|}{1+\lambda^2\phi^2(x)} \leqslant \\ &\leqslant C\frac{1+\lambda\,|\phi(x)|}{1+\lambda^2\phi^2(x)}. \end{aligned} \tag{11}$$

Using inequality (11) in integral (10), we obtain

$$\begin{aligned} |I_{\alpha,\beta}(\lambda)| &\leqslant \int_a^b |E_{\alpha,\beta}(i\lambda\phi(x))|\,|\psi(x)|\,dx \leqslant \\ &\leqslant C\int_a^b \frac{1+\lambda\,|\phi(x)|}{1+\lambda^2\phi^2(x)}\,|\psi(x)|\,dx \leqslant \\ &\leqslant 2C\int_a^b \frac{1+\lambda\,|\phi(x)|}{(1+\lambda\,|\phi(x)|)^2}\,|\psi(x)|\,dx \leqslant \\ &\leqslant C\int_a^b \frac{|\psi(x)|}{1+\lambda\,|\phi(x)|}dx. \end{aligned} \tag{12}$$

Then using the Hölder inequality and $m = \operatorname{ess\,inf}_{x\in[a,b]} |\phi(x)|$ for the last integral, we establish

$$|I_{\alpha,\beta}(\lambda)| \leqslant C\left(\int_a^b |\psi(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b \frac{dx}{(1+\lambda\,|\phi(x)|)^q}\right)^{\frac{1}{q}} \leqslant \frac{C\|\psi\|_{L^p[a,b]}}{1+m\lambda},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q \in [1, \infty]$. The proof is complete. □

Proof of Theorem 2. Since $I_{\alpha,\beta}(\lambda)$ is bounded for small λ it is assumed that $\lambda \geqslant 1$. Without loss of generality, suppose that function ϕ has one zero at $c \in [a,b]$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let us assume that $p \neq \infty$ so that $q > 1$. Then using the Hölder inequality in integral (12), we obtain

$$|I_{\alpha,\beta}(\lambda)| \leqslant C\left(\int_a^b |\psi(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b \frac{dx}{(1+\lambda\,|\phi(x)|)^q}\right)^{\frac{1}{q}}.$$

Here and in what follows it is assumed that M is an arbitrary constant independent of λ. Without loss of generality one can assume that ϕ increases.

If $a=-\infty$ then $L^p[a,b]$ is understanded as $L^p(-\infty,b]$. If $b=+\infty$ then $L^p[a,b]$ is understanded as $L^p[a,\infty)$. Similarly, if $a=-\infty$ and $b=+\infty$ then $L^p[a,b]$ is understanded as $L^p(\mathbb{R})$. Since $\phi\in L^q[a,b]$ is differentiable function with $m=\inf_{x\in[a,b]}|\phi'(x)|>0$ we replace $\phi(x)$ by y and obtain

$$|I_{\alpha,\beta}(\lambda)|\leqslant C\|\psi\|_{L^p[a,b]}\left(\int_a^b\frac{dx}{(1+\lambda|\phi(x)|)^q}\right)^{\frac{1}{q}}\leqslant$$

$$\leqslant C\|\psi\|_{L^p[a,b]}\left(\int_{\phi(a)}^{\phi(b)}\frac{1}{(1+\lambda|y|)^q}\frac{dy}{\phi'(\phi^{-1}(y))}\right)^{\frac{1}{q}}\leqslant$$

$$\leqslant\frac{C\|\psi\|_{L^p[a,b]}}{m^{\frac{1}{q}}}\left(\int_{\phi(a)}^{\phi(b)}\frac{dy}{(1+\lambda|y|)^q}\right)^{\frac{1}{q}}.$$

Since ϕ increases one can define $\phi(a)$ and $\phi(b)$ as the limit at a and b if $a=-\infty$ or $b=+\infty$. Since $\phi\in L^q[a,b],\ q>1$ we have $-\infty<\ \phi(a)\leqslant\ \phi(b)<+\infty$.
Replacing λy by u in the above inequality, we obtain

$$|I_{\alpha,\beta}(\lambda)|\leqslant\frac{C\|\psi\|_{L^p[a,b]}}{m^{\frac{1}{q}}}\left(\int_{\phi(a)}^{\phi(b)}\frac{dy}{(1+\lambda|y|)^q}\right)^{\frac{1}{q}}=$$

$$=\frac{C\|\psi\|_{L^p[a,b]}}{(m\lambda)^{1/q}}\left(\int_{\lambda\phi(a)}^{\lambda\phi(b)}\frac{du}{(1+|u|)^q}\right)^{\frac{1}{q}}=$$

$$=\frac{C\|\psi\|_{L^p[a,b]}}{(m\lambda)^{1/q}}\left(\int_{\lambda\phi(a)}^{0}\frac{du}{(1-u)^q}+\int_0^{\lambda\phi(b)}\frac{du}{(1+u)^q}\right)^{\frac{1}{q}}=$$

$$=\frac{C\|\psi\|_{L^p[a,b]}}{(1+m\lambda)^{1/q}}\left(\frac{1}{q-1}\left[2-\frac{1}{(1+\lambda(-\phi(a)))^{q-1}}-\frac{1}{(1+\lambda(\phi(b)))^{q-1}}\right]\right)^{\frac{1}{q}}\leqslant$$

$$\leqslant\frac{C_q\|\psi\|_{L^p[a,b]}}{(1+m\lambda)^{1/q}},$$

where C_q is some coefficient that depends only on q and hence only on p.

Let us consider now the case $q=1$. Notice that coefficient $C_q\to+\infty$ as $q\to 1$. Therefore one cannot directly obtain the required estimate from the estimate for $q>1$. As $q=1$, we have $p=\infty$ and $\psi\in L^\infty$. In view of (12), first we estimate the integral as

$$|I_{\alpha,\beta}(\lambda)|\leqslant C\int_a^b\frac{|\psi(x)|}{1+\lambda|\phi(x)|}dx\leqslant C\sup_{x\in[a,b]}|\psi(x)|\int_a^b\frac{dx}{1+\lambda|\phi(x)|}\leqslant$$

$$\leqslant C\|\psi\|_{L^\infty[a,b]}\int_a^b\frac{dx}{1+\lambda|\phi(x)|}.$$

Since $\phi\in L^\infty[a,b]$ is differentiable function with $m=\inf_{x\in[a,b]}|\phi'(x)|>0$ we replace $\phi(x)$ by y and obtain

$$|I_{\alpha,\beta}(\lambda)|\leqslant C\|\psi\|_{L^\infty[a,b]}\int_{\phi(a)}^{\phi(b)}\frac{1}{1+\lambda|y|}\frac{dy}{\phi'(\phi^{-1}(y))}\leqslant\frac{C\|\psi\|_{L^\infty[a,b]}}{m}\int_{\phi(a)}^{\phi(b)}\frac{dy}{1+\lambda|y|}.$$

Replacing λy by u in the above inequality, we obtain

$$|I_{\alpha,\beta}(\lambda)| \leqslant \frac{C\|\psi\|_{L^\infty[a,b]}}{m\lambda}\int_{\lambda\phi(a)}^{\lambda\phi(b)}\frac{du}{1+|u|}=$$

$$=\frac{C\|\psi\|_{L^\infty[a,b]}}{m\lambda}\left(\int_{\lambda\phi(a)}^{0}\frac{du}{1-u}+\int_{0}^{\lambda\phi(b)}\frac{du}{1+u}\right)\leqslant$$

$$\leqslant \frac{C\|\psi\|_{L^\infty[a,b]}}{1+m\lambda}\left[\log(1+\lambda(-\phi(a))\ +\log(1+\lambda(\phi(b))\ \right]\leqslant$$

$$\leqslant \frac{C\|\psi\|_{L^p[a,b]}}{1+m\lambda}\log(2+\lambda\|\phi\|_{L^\infty[a,b]})\ .$$

In the case when ϕ has several zeros in $[a,b]$ estimates (4) and (5) can be obtained using the given above calculations. The proof is complete. □

Proof of Theorem 3. For small λ there is bounded estimate for the integral $I_{\alpha,\beta}(\lambda)$. Let $\lambda \geqslant 1$ and $k = 2$. Let $c \in [a,b]$ be a point where $|\phi'(c)| \leqslant |\phi'(x)|$ for all $x \in [a,b]$. As $\phi''(x)$ is non-vanishing, it cannot be the case that c is the interior local minimum/maximum of $\phi'(x)$. Therefore, either $\phi'(c) = 0$ or c is one of the endpoints a or b. One can assume that $\phi'' \geqslant 1$.

Let $\phi'(c) = 0$. If $x \in [c+\varepsilon, b]$ then

$$\phi'(x) = \phi'(x) - \phi'(c) = \int_c^x \phi''(s)ds \geqslant x-c \geqslant \varepsilon.$$

There is similar estimate for $x \in [a, c-\varepsilon]$. Now, one can write

$$\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx = \left(\int_a^{c-\varepsilon}+\int_{c-\varepsilon}^{c+\varepsilon}+\int_{c+\varepsilon}^{b}\right)E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)\,dx.$$

First, applying the results of Theorem 2 with $m=\varepsilon$ and estimate $\frac{1}{\varepsilon\lambda} \geqslant \frac{1}{1+\varepsilon\lambda}$ for $p \neq \infty,\ \ \lambda \geqslant 1$, one can obtain

$$\left|\int_a^{c-\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx\right| \leqslant \frac{C_p\|\psi\|_{L^p[a,b]}}{(\varepsilon\lambda)^{1-\frac{1}{p}}},$$

and

$$\left|\int_{c+\varepsilon}^{b} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx\right| \leqslant \frac{C_p\|\psi\|_{L^p[a,b]}}{(\varepsilon\lambda)^{1-\frac{1}{p}}}.$$

As

$$\left|\int_{c-\varepsilon}^{c+\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx\right| \leqslant 2\,\varepsilon\|\psi\|_{L^p[a,b]}$$

then

$$\left|\int_a^{b} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx\right| \leqslant \frac{2C_p\|\psi\|_{L^p[a,b]}}{(\varepsilon\lambda)^{1-\frac{1}{p}}}+2\,\varepsilon\|\psi\|_{L^p[a,b]}.$$

Taking $\varepsilon = \frac{1}{\sqrt{\lambda}}$, we obtain estimate

$$\left|\int_a^{b} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx\right| \leqslant \frac{2C_p\|\psi\|_{L^p[a,b]}}{\lambda^{\frac{1}{2}-\frac{1}{2p}}}+\frac{2\|\psi\|_{L^p[a,b]}}{\lambda^{\frac{1}{2}}} \leqslant \frac{C_p\|\psi\|_{L^p[a,b]}}{(1+\lambda)^{\frac{1}{2}-\frac{1}{2p}}}.$$

This gives inequality (6) for $k = 2$. The case $c = a$ or $c = b$ can be proved similarly.

Let $k \geqslant 3$ and $\lambda \geqslant 1$. Let us prove estimate (6) by induction method with respect to k. Let us assume that (6) is true for $k \geqslant 3$. Assuming $\phi^{(k+1)}(x) \geqslant 1$ for all $x \in [a, b]$, we prove estimate (6) for $k + 1$. Let $c \in [a, b]$ be a unique point where $\left|\phi^{(k)}(c)\right| \leqslant |\phi^{(k)}(x)|$ for all $x \in [a, b]$. If $\phi^{(k)}(c) = 0$ then we obtain $\phi^{(k)}(x) \geqslant \varepsilon$ on interval $[a, b]$ outside $(c - \varepsilon, c + \varepsilon)$. Now, we obtain $\int\limits_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx$ as

$$\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx = \left(\int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} + \int_{c+\varepsilon}^b\right) E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)\,dx.$$

By inductive hypothesis

$$\left|\int_a^{c-\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx\right| \leqslant \frac{C_k\|\psi\|_{L^p[a,b]}}{(1+\varepsilon\lambda)^{\frac{1}{k}-\frac{1}{pk}}} \leqslant$$

$$\leqslant \frac{C_p\|\psi\|_{L^p[a,b]}}{(\varepsilon\lambda)^{\frac{1}{k}-\frac{1}{pk}}}$$

and

$$\left|\int_{c+\varepsilon}^{b} E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx\right| \leqslant \frac{C_k\|\psi\|_{L^p[a,b]}}{(1+\varepsilon\lambda)^{\frac{1}{k}-\frac{1}{pk}}} \leqslant$$

$$\leqslant \frac{C_p\|\psi\|_{L^p[a,b]}}{(\varepsilon\lambda)^{\frac{1}{k}-\frac{1}{pk}}}.$$

As

$$\left|\int_{c-\varepsilon}^{c+\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx\right| \leqslant 2\,\varepsilon\|\psi\|_{L^p[a,b]}$$

then

$$\left|\int_a^{b} E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx\right| \leqslant \frac{2C_p\|\psi\|_{L^p[a,b]}}{(\varepsilon\lambda)^{\frac{1}{k}-\frac{1}{pk}}} + 2\,\varepsilon\|\psi\|_{L^p[a,b]}.$$

Taking $\varepsilon = \lambda^{-\frac{1}{k+1}}$, we obtain estimate (6) for $k + 1$. This proves the result. The cases when $c = a$ or $c = b$ can be proved similarly.
Second, by induction method for case $p = \infty$, one can obtain estimate (7). □

3. Decay estimates for the time-fractional PDE

Let us consider the time-fractional Schrödinger-type equation

$$\mathcal{D}^{\alpha}_{0+,t}u(t,x) - \lambda\mathcal{D}^{\alpha}_{0+,t}u_{xx}(t,x) + iu_{xx}(t,x) - i\mu u_{xx}(t,x) = 0, \quad t > 0,\ x \in \mathbb{R} \tag{13}$$

with Cauchy data

$$u(0,x) = \psi(x),\ x \in \mathbb{R} \tag{14}$$

where $\lambda,\ \mu > 0$ and $\mathcal{D}^{\alpha}_{0+,t}u(t,x) = \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_a^t (t-s)^{-\alpha}u_s(s,x)ds$ is the Caputo fractional derivative of order $0 < \alpha < 1$.

Using the direct and inverse Fourier and Laplace transforms, one can obtain a solution of problem (13)–(14) in the form

$$(t,x) = \int_{\mathbb{R}} e^{ix\xi} E_{\alpha,1}\left(i\frac{\xi^2+\mu}{1+\lambda\xi^2}t^{\alpha}\right)\widehat{\psi}(\xi)d\xi, \tag{15}$$

where $\widehat{\psi}\,(\xi) = \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} e^{-iy\xi}\psi(y)dy$. Suppose that $\psi \in L^1(\mathbb{R})$ and $\widehat{\psi} \in L^1(\mathbb{R})$. As

$$\inf_{\xi\in\mathbb{R}} \frac{\xi^2+\mu}{1+\lambda\xi^2} = \min\left\{\mu, \frac{1}{\lambda}\right\} > 0,$$

and using Theorem 1, the following dispersive estimate is obtained

$$\|u(t,\cdot)\|_{L^{\infty}(\mathbb{R})} \leqslant C(1+t)^{-\alpha}\left\|\widehat{\psi}\right\|_{L^p(\mathbb{R})}, \quad t \geqslant 0,\ p \geqslant 1.$$

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Осциллирующие интегралы для функций Миттаг-Леффлера

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Аннотация. В данной статье изучаются аналоги лемм Ван дер Корпута, связанные с функциями Миттаг–Леффлера. Обобщение состоит в том, что мы заменяем показательную функцию функцией типа Миттаг–Леффлера для изучения интегралов осциллирующего типа, появляющихся при анализе дробных по времени уравнений в частных производных. Доказаны некоторые обобщения первой и второй лемм Ван дер Корпута. Получены также оптимальные оценки порядков убывания для частных случаев функций Миттаг–Леффлера.

Ключевые слова: функции Миттаг–Леффлера, фазовая функция, амплитуда.