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Research of the Equations of a Viscous Inhomogeneous Fluid in a Hele-Shaw Cell by Group Analysis Method

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Abstract. In this paper, we have constructed the main group of transformations allowed by the system of differential equations for the flow of a viscous inhomogeneous fluid in a Hele–Shaw cell. A classifying equation for the viscosity function was obtained, and a basis of operators was constructed that preserved the form of the original equations. The basis of the space of solutions of the defining equations is described. Invariants of operators are found and invariant solutions of the equations are obtained.

Keywords: group analysis, fluid equations, infinitesimal operator, invariant, invariant solution, Hele–Shaw cell.

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Introduction

Systems of differential equations are important in the natural sciences. Often these equations are difficult to solve by integrating them directly. To solve complex systems of differential equations, their group properties are studied, that is, the properties of leaving the differential manifold of the equation under consideration invariant when the independent and differential variables undergo transformations of a certain group of transformations. If this property exists, they say that the system of equations admits a group, and when transforming from this group, any solution of the system goes back into the solution of this system, which makes it possible to obtain various classes of partial solutions of the system by integrating simpler systems of equations.

In this paper, the equations of a viscous inhomogeneous fluid in a Hele-Shaw cell are researched by group analysis method. The movement of a viscous fluid is described by the Navier-Stokes equations. Article [3] describes two-layer flows in a hele-show cell. The geometry of the flow of a two-layer fluid in a cell is shown in Figure 1. The dimensions of the cell in the direction of the x and y axes significantly exceed the width of the gap between the cell plates.

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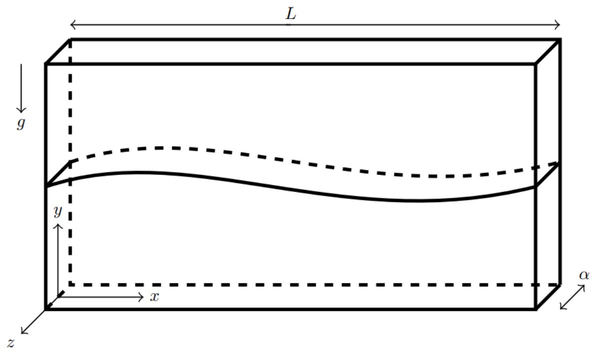

Fig. 1. Geometry of Hele-Shaw cell [3]

Article [3] presents a system of differential equations of the form

$$\begin{aligned} \rho(u_t + \beta(uu_x + vu_y)) + p_x &= -\mu u, \\ p_y &= -\rho g, \\ u_x + v_y &= 0, \\ \rho_t + u\rho_x + v\rho_y &= 0, \\ \mu &= \mu(\rho). \end{aligned} \tag{1}$$

Here the independent variables — x and y are spatial coordinates in the Hele–Shaw cell and t — time. The differential variables are u — horizontal and v — vertical components of velocity, p — pressure, ρ — density. In addition, there is an arbitrary element μ — viscosity, which is some unknown function of density. From the physical meaning we assume $\mu > 0$. The equations also include two constants: g — acceleration of gravity and $\beta = 1.2$.

1. Infinitesimal operator and its extension

Let's study the equations (1) using group analysis. To do this, we will follow the algorithm for searching for group transformations allowed by a differential equation given in [1, 2].

We introduce notation for independent variables $x^1 = x,\ x^2 = y,\ x^3 = t$, and for differential variables $u^1 = u,\ u^2 = v,\ u^3 = p,\ u^4 = \rho$. Let us denote partial derivatives with respect to independent variables

$$\begin{array}{llll} u_x = u^1_1, & v_x = u^2_1, & p_x = u^3_1, & \rho_x = u^4_1, \\ u_y = u^1_2, & v_y = u^2_2, & p_y = u^3_2, & \rho_y = u^4_2, \\ u_t = u^1_3, & v_t = u^2_3, & p_t = u^3_3, & \rho_t = u^4_3. \end{array}$$

In the new notation equations (1) take the form

$$\begin{aligned} u^4u^1_3 + \beta u^1u^4u^1_1 + \beta u^2u^4u^1_2 + u^3_1 + \mu(u^4)u^1 &= 0, \\ u^3_2 + gu^4 &= 0, \\ u^1_1 + u^2_2 &= 0, \\ u^4_3 + u^1u^4_1 + u^2u^4_2 &= 0. \end{aligned} \tag{2}$$

The equations (1) define the manifold E in the space of variables, the following substitution is required to move to the manifold E

$$\begin{aligned} u_1^3 &= -u^4 u_3^1 - \beta u^4 (u^1 u_1^1 + u^2 u_2^1) - \mu(u^4) u^1, \\ u_2^3 &= -g u^4, \\ u_2^2 &= -u_1^1, \\ u_3^4 &= -u^1 u_1^4 - u^2 u_2^4. \end{aligned} \tag{3}$$

We look for the infinitesimal operator in the form

$$X = \xi^1 \frac{\partial}{\partial x_1} + \xi^2 \frac{\partial}{\partial x_2} + \xi^3 \frac{\partial}{\partial x_3} + \eta^1 \frac{\partial}{\partial u_1} + \eta^2 \frac{\partial}{\partial u_2} + \eta^3 \frac{\partial}{\partial u_3} + \eta^4 \frac{\partial}{\partial u_4},$$
$$\xi^i = \xi^i(x^1, x^2, x^3, u^1, u^2, u^3, u^4), \quad \eta^j = \eta^j(x^1, x^2, x^3, u^1, u^2, u^3, u^4).$$

To obtain the continuation of the operator to the first derivatives its necessary to use the formula $\underset{1}{X} = X + \zeta_i^\alpha \dfrac{\partial}{\partial u_i^\alpha}$, where $\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j)$. Summation is carried out according to the indices of independent $(i,\, j)$ and differential (α) variables; $D_i = \dfrac{\partial}{\partial x^i} + u_i^\alpha \dfrac{\partial}{\partial u^\alpha} + \ldots$ [1].

2. Defining equations

Let us apply the invariance criterion [2] by acting on the second equation of system (2) by the operator $\underset{1}{X}$

$$\underset{1}{X}(u_2^3 + g u^4) = \underset{1}{X}(u_2^3) + g \underset{1}{X}(u^4) = \zeta_2^3 + g\eta^4 \Big|_{[E]} = 0.$$

Let's move to the manifold $[E]$ using substitutions (3)

$$\begin{aligned} &\eta_2^3 + u_2^1 \eta_{u^1}^3 + (-u_1^1)\eta_{u^2}^3 + (-gu^4)\eta_{u^3}^3 + u_2^4 \eta_{u^4}^3 - \\ &- (-u^4 u_3^1 - \beta u^4 u^1 u_1^1 - \beta u^4 u^2 u_2^1 - \mu u^1)(\xi_2^1 + u_2^1 \xi_{u^1}^1 + (-u_1^1)\xi_{u^2}^1 + \\ &+ (-gu^4)\xi_{u^3}^1 + u_2^4 \xi_{u^4}^1) - (-gu^4)(\xi_2^2 + u_2^1 \xi_{u^1}^2 + (-u_1^1)\xi_{u^2}^2 + (-gu^4)\xi_{u^3}^2 + u_2^4 \xi_{u^4}^2) - \\ &- u_3^3(\xi_2^3 + u_2^1 \xi_{u^1}^3 + (-u_1^1)\xi_{u^2}^3 + (-gu^4)\xi_{u^3}^3 + u_2^4 \xi_{u^4}^3) + g\eta^4 = 0. \end{aligned}$$

Here the quantities u_i^α are independent variables. We split the equation into independent variables $u_1^1, u_2^1, u_3^1, u_3^3, u_2^4$ and get

$$\begin{aligned} \xi_{u^1}^1 = \xi_{u^2}^1 = \xi_{u^4}^1 = \xi_{u^1}^3 = \xi_{u^2}^3 = \xi_{u^4}^3 &= 0, \\ \eta_2^3 - \eta_{u^3}^3 g u^4 + \eta^4 g + \mu \xi_2^1 u^1 - \mu \xi_{u^3}^1 g u^1 u^4 + \xi_2^2 g u^4 - \xi_{u^3}^2 g g u^4 u^4 &= 0, \\ \beta \xi_2^1 u^1 u^4 - \beta \xi_{u^3}^1 g u^1 u^4 u^4 - \eta_{u^2}^3 - \mu \xi_{u^2}^1 u^1 - \xi_{u^2}^2 g u^4 &= 0, \\ \beta \xi_{u^1}^1 u^1 u^4 - \beta \xi_{u^2}^1 u^2 u^4 &= 0, \\ \beta \xi_2^1 u^2 u^4 - \beta \xi_{u^3}^1 g u^2 u^4 u^4 + \eta_{u^1}^3 + \mu \xi_{u^1}^1 u^1 + \xi_{u^1}^2 g u^4 &= 0, \\ \xi_2^1 u^4 - \xi_{u^3}^1 g u^4 u^4 &= 0, \\ -\xi_2^3 + \xi_{u^3}^3 g u^4 &= 0, \\ \eta_{u^4}^3 + \mu \xi_{u^4}^1 u^1 + \xi_{u^4}^2 g u^4 &= 0. \end{aligned}$$

Subjecting the first, third and fourth equations of the system to similar processing, we get

$$
\begin{aligned}
\xi^1_2 = \xi^1_{u^1} = \xi^1_{u^2} = \xi^1_{u^3} = \xi^1_{u^4} &= 0,\\
\xi^2_{u^1} = \xi^2_{u^2} = \xi^2_{u^3} = \xi^2_{u^4} &= 0,\\
\xi^3_1 = \xi^3_2 = \xi^3_{u^1} = \xi^3_{u^2} = \xi^3_{u^3} = \xi^3_{u^4} &= 0,\\
\eta^1_{u^2} = \eta^1_{u^3} = \eta^1_{u^4} &= 0,\\
\eta^2_{u^4} &= 0,\\
\eta^3_{u^1} = \eta^3_{u^2} = \eta^3_{u^4} &= 0,\\
\eta^4_{u^1} = \eta^4_{u^2} = \eta^4_{u^3} &= 0,
\end{aligned}
$$

as well as a set of defining equations
DE1.1: $\eta^1_3 u^4 + \beta\eta^1_1 u^1 u^4 + \beta\eta^1_2 u^2 u^4 + \eta^3_1 - \mu\eta^3_{u^3} u^1 + \mu\xi^1_1 u^1 + \xi^2_1 g u^4 + \eta^4 \mu_{u^4} u^1 + \eta^1 \mu = 0,$
DE1.2: $-\beta^{-1}\xi^1_3 u^4 + \eta^4 u^1 + \eta^1 u^4 + u^1 u^4(\eta^1_{u^1} - \eta^3_{u^3}) = 0,$
DE1.3: $u^4\left[u^2(\eta^1_{u^1} - \eta^3_{u^3} + \xi^1_1 - \xi^2_2) + \eta^2 - \beta^{-1}\xi^2_3 - \xi^2_1 u^1\right] + \eta^4 u^2 = 0,$
DE1.4: $\eta^4 + u^4(\eta^1_{u^1} - \eta^3_{u^3} + \xi^1_1 - \xi^3_3) = 0,$
DE2.1: $\eta^3_2 + g(\eta^4 + u^4(\xi^2_2 - \eta^3_{u^3})) = 0,$
DE3.1: $\eta^1_1 + \eta^2_2 - \eta^2_{u^3} g u^4 = 0,$
DE3.2: $\eta^1_{u^1} - \eta^2_{u^2} - \xi^1_1 + \xi^2_2 = 0,$
DE3.3: $\eta^2_{u^1} - \xi^2_1 = 0,$
DE4.1: $\eta^4_1 u^1 + \eta^4_2 u^2 + \eta^4_3 = 0,$
DE4.2: $\eta^1 - \xi^1_3 + u^1(\xi^3_3 - \xi^1_1) = 0,$
DE4.3: $\eta^2 - \xi^2_3 - u^1\xi^2_1 + u^2(\xi^3_3 - \xi^2_2) = 0.$

3. Operators allowed by the system of equations

Considering differential and algebraic consequences from the defining equations, we obtain the coordinates of the tangent vector field of the infinitesimal operator

$$
\begin{aligned}
\xi^1 &= C_2 x^1 + C_3,\\
\xi^2 &= (2C_2 - 2C_4)x^2 + C_6,\\
\xi^3 &= C_4 x^3 + C_5,\\
\eta^1 &= (C_2 - C_4)u^1,\\
\eta^2 &= (2C_2 - 3C_4)u^2,\\
\eta^3 &= (C_1 + 2C_2 - 2C_4)u^3 + A(x^3),\\
\eta^4 &= C_1 u^4.
\end{aligned} \tag{4}
$$

The solution (4) of the defining equations depends on six arbitrary constants $C_1 \dots C_6$ and on an arbitrary function $A(x^3)$. Since there are infinitely many options for choosing function $A(x^3)$, then the solution space L is infinite-dimensional. The space L can be represented as a direct sum $L = L^6 \bigoplus L^\infty$, where L^6 is a six-dimensional space of solutions for which $A(x^3) = 0$, and the subspace L^∞ is infinite-dimensional and consists of solutions such that $C_1 = \dots = C_6 = 0$, $A(x^3) \neq 0$ with the operator $X_A = A(x^3)\dfrac{\partial}{\partial u^3}$.

We alternately set one of the constants $C_{i=1\ldots6}$ equal to one, and the rest equal to zero

$$\begin{aligned}
\zeta_1 &= (0,0,0,0,u^3,u^4),\\
\zeta_2 &= (x^1,2x^2,0,u^1,2u^2,2u^3,0),\\
\zeta_3 &= (1,0,0,0,0,0),\\
\zeta_4 &= (0,-2x^2,x^3,-u^1,-3u^2,-2u^3,0),\\
\zeta_5 &= (0,0,1,0,0,0),\\
\zeta_6 &= (0,1,0,0,0,0).
\end{aligned}$$

We scalarly multiply the resulting vectors by $\partial = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3}, \partial_{u^1}, \partial_{u^2}, \partial_{u^3}, \partial_{u^4})$ and get the operators

$$\begin{aligned}
X_1 &= \zeta_1 \cdot \partial = u^3\partial_{u^3} + u^4\partial_{u^4},\\
X_2 &= \zeta_2 \cdot \partial = x^1\partial_{x^1} + 2x^2\partial_{x^2} + u^1\partial_{u^1} + 2u^2\partial_{u^2} + 2u^3\partial_{u^3},\\
X_3 &= \zeta_3 \cdot \partial = \partial_{x^1},\\
X_4 &= \zeta_4 \cdot \partial = -2x^2\partial_{x^2} + x^3\partial_{x^3} - u^1\partial_{u^1} - 3u^2\partial_{u^2} - 2u^3\partial_{u^3},\\
X_5 &= \zeta_5 \cdot \partial = \partial_{x^3},\\
X_6 &= \zeta_6 \cdot \partial = \partial_{x^2}.
\end{aligned}$$

Operators X_3, X_6 correspond to shifts along spatial coordinates, while X_5 is time shift. Operator X_1 sets uniform stretching, operators X_2 и X_4 are heterogeneous stretch.

4. Classification equation

The equation $(-C_1 + C_4)\,\mu + C_1u^4\mu_{u^4} = 0$ is a classification equation with solution $\mu = C\left(u^4\right)^{\dfrac{C_1 - C_4}{C_1}}$, $C = const$.

This equation does not include the constants C_2, C_3, C_5, C_6, therefore, the transformations corresponding to the operators X_2, X_3, X_5, X_6 retain the form of equations for any type of dependence of the liquid viscosity on density.

Let us consider different types of $\mu(u^4)$.

1) μ — arbitrary function. This means that the classification equation is satisfied when $-C_1 + C_4 = 0$ end $C_1u^4 = 0$. From here, $C_1 = C_4 = 0$. We get the kernel of transformations with operators $\{X_2, X_3, X_5, X_6, X_A\}$.

2) $\mu = \mathbf{0}$. In the case of a non-viscous liquid, we substitute $\mu = 0$, $\mu_{u^4} = 0$ in the classifying equation and we obtain the identity. In this case we can choose arbitrary C_1, C_4, and the basis of operators is $\{X_1, X_2, X_3, X_4, X_5, X_6, X_A\}$.

3) $\mu = \mathbf{C} \neq \mathbf{0}$. In this case $\mu_{u^4} = 0$ end $C_1 = C_4$. Then the coordinates of the vector tangent field $\xi^1 = C_2x^1 + C_3$, $\xi^2 = (2C_2 - 2C_1)x^2 + C^6$, $\xi^3 = C^1x^3 + C^5$, $\eta^1 = (C_2 - C_1)u^1$, $\eta^2 = (2C^2 - 3C^1)u^2$, $\eta^3 = (2C_2 - C_1)u^3 + A(x^3)$, $\eta^4 = C_1u^4$. basis of operators is $\{X_1 + X_4, X_2, X_3, X_5, X_6, X_A\}$.

4) $\mu = \mathbf{C} \cdot (\mathbf{u^4})^{\mathbf{k}}$**.** Substituting μ end $\mu_{u^4} = Ck(u^4)^{k-1}$ in the classifying equation, we get $C_4 = C_1(1-k)$. Basis of operators is $\{X_1 + (1-k)X_4, X_2, X_3, X_5, X_6, X_A\}$.

5. Operator invariants

Operator invariants are found from invariance criterion $X_i J = 0$ [2].

$$\begin{aligned}
&J_{X_1} = \left\{x^1, x^2, x^3, u^1, u^2, \frac{u^3}{u^4}\right\} = \left\{x, y, t, u, v, \frac{p}{\rho}\right\},\\
&J_{X_2} = \left\{x^3, u^4, \frac{x^2}{(x^1)^2}, \frac{u^1}{x^1}, \frac{u^2}{(x^1)^2}, \frac{u^3}{(x^1)^2}\right\} = \left\{\frac{y}{x^2}, t, \frac{u}{x}, \frac{v}{x^2}, \frac{p}{x^2}, \rho\right\},\\
&J_{X_3} = \left\{x^2, x^3, u^1, u^2, u^3, u^4\right\} = \{y, t, u, v, p, \rho\},\\
&J_{X_4} = \left\{x^1, x^2(x^3)^2, \frac{(u^1)^2}{x^2}, \frac{(u^2)^2}{(x^2)^3}, \frac{u^3}{x^2}, u^4\right\} = \left\{x, yt^2, \frac{u^2}{y}, \frac{v^2}{y^3}, \frac{p}{y}, \rho\right\},\\
&J_{X_5} = \left\{x^1, x^2, u^1, u^2, u^3, u^4\right\} = \{x, y, u, v, p, \rho\},\\
&J_{X_6} = \left\{x^1, x^3, u^1, u^2, u^3, u^4\right\} = \{x, t, u, v, p, \rho\}.
\end{aligned}$$

6. Invariant solutions for the operator $\langle X_2, X_5\rangle$

Let's take two operators from the core of the main group of transformations and let's create a two-parameter group $H = \langle X_2, X_5\rangle$. We transform the basis of invariants of the operator X_5

$$J_{X_5} = \{x, y, u, v, p, \rho\} \to \left\{x, \frac{y}{x^2}, \frac{u}{x}, \frac{v}{x^2}, \frac{p}{x^2}, \rho\right\}.$$

Then for the group H the basis of invariants $J_{X_2} \cap J_{X_5} = \left\{\frac{y}{x^2}, \frac{u}{x}, \frac{v}{x^2}, \frac{p}{x^2}, \rho\right\}$.

We take the invariant $\lambda = \dfrac{y}{x^2}$ as an independent variable. Let's calculate its partial derivatives $\lambda_t = 0$, $\lambda_x = -\dfrac{2y}{x^3} = -\dfrac{2\lambda}{x}$, $\lambda_y = \dfrac{1}{x^2} = \dfrac{\lambda}{y}$. We take the remaining four invariants as new required functions

$$\begin{gathered}
U(\lambda) = \frac{u}{x} \implies u = Ux, \qquad V(\lambda) = \frac{v}{x^2} \implies v = Vx^2,\\
P(\lambda) = \frac{p}{x^2} \implies p = Px^2, \qquad R(\lambda) = \rho.
\end{gathered} \tag{5}$$

After transforming the original system of equations to new variables and the required functions, we obtain a factor system $E|H$ which contains ordinary differential equations for λ

$$\begin{cases}
\beta R\,(UV' + VU') + 2P - 2\lambda P' = -\mu U,\\
P' = -Rg,\\
U - 2\lambda U' + V' = 0,\\
R'\,(V - 2\lambda U) = 0.
\end{cases}$$

From the last equation of the factor system we have two cases.

Case 1. $V = 2\lambda U$. Substitute $V' = 2U + 2\lambda U'$ in third equation of $E|H$ and get U=0. Then what remains from the first equation is an equation with separable variables $2P - 2\lambda P' = 0$, end $P = C_1\lambda$, $P' = C_1 = const$, from third equation $V' = 0$, $V = C_2 = const$, from second equation $R = -\dfrac{C_1}{g}$.

We return to the "physical" variables and write down the invariant solution

$$u = Ux = 0, \quad v = Vx^2 = C_2x^2, \quad p = Px^2 = C_1y, \quad \rho = R = -\frac{C_1}{g}.$$

Case 2. $R' = 0$, $R = C_1 = const$. From second equation we get $P' = -C_1g$ end $P = -C_1g\lambda + C_2$. Note that in this case $\mu(\rho) = \mu(R) = \mu(C_1) = const$. Let $k = -\dfrac{\beta C_1}{\mu}$. Then first equation of $E|H$ takes the form

$$k(VU' - UV') + 2C_2 = U.$$

Let's consider the case when $C_2 = 0$. Let us express V from the first equation $E|H$

$$V = \frac{U}{kU'} + 2\lambda U - \frac{U^2}{U'}. \tag{6}$$

$$V' = \frac{1}{k}\frac{U'U' - UU''}{U'U'} + 2U + 2\lambda U' - \frac{2UU'U' - U^2U''}{U'U'}.$$

On the other hand, from the third equation $V' = 2\lambda U' - U$, and we get an ordinary differential equation with respect to U

$$U'U'(1 + kU) = UU''(1 - kU).$$

This equation does not include an independent variable λ, therefore, we can lower the order of the equation by taking U as the independent variable and the unknown function is $U' = f(U)$, and we get

$$U'' = \frac{dU'}{d\lambda} = \frac{df(U)}{d\lambda} = \frac{df}{dU}\frac{dU}{d\lambda} = f'f.$$

Then

$$f\,[f[1 + kU] - Uf'[1 - kU]] = 0.$$

Case 2.1. Если $f = 0$, $U' = 0$, $U = C_3 = const$.

Then $V = \dfrac{U}{U'}\left(\dfrac{1}{k} + 2\lambda U' - U\right) = \infty, \quad P = -C_1g\lambda, \quad R = C_1.$

Case 2.2. Solving differential equation $\dfrac{df}{f} = \dfrac{1 + kU}{U(1 - KU)}dU$, we get dependence $\lambda(U) = \dfrac{1}{A}\left[\ln BU - 2kU + \dfrac{k^2U^2}{2}\right]$, $A = const$, $B = const$.

The function $\lambda(U)$ is continuous and increases strictly monotonically by $(0; +\infty)$, therefore it is a bijection $\lambda : (0; +\infty) \rightleftarrows \mathbb{R}$. This means there is an inverse function $U(\lambda)$.

To find the dependence $U(\lambda)$, it's need to solve the equation for U

$$f(U) = \frac{1}{A}\left[\ln BU - 2kU + \frac{k^2U^2}{2}\right] - \lambda = 0.$$

We use Newton's iterative scheme

$$U_{k+1} = U_k - \frac{f(U_k)f'(U_k)}{(f'(U_k))^2 - 0.5f''(U_k)f(U_k)},$$

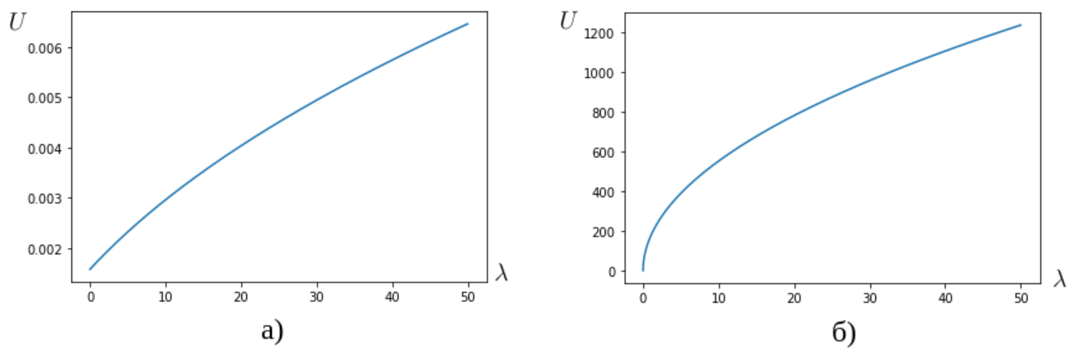

Fig. 2. Dependence $U(\lambda)$ for a) water, $\mu = 8.9410^{-4}$ Pa·s, б) glycerin, $\mu = 1.49$ Pa·s

$$f' = \frac{1}{A}\left[\frac{B}{U} - 2k + k^2 U\right], \quad f'' = \frac{1}{A}\left[-\frac{B}{U^2} + k^2\right].$$

The derivative U' is approximated to the second order of accuracy by the difference relation

$$U'(\lambda) = \frac{U(\lambda + h) - U(\lambda - h)}{2h} + o(h^2).$$

Now using (6), we can calculate $V(\lambda)$. Let's go through the formulas (5) to the original differential variables and obtain a numerical solution for a fixed $\lambda = \dfrac{y}{x^2}$. We see that the resulting solution is stationary.

We impose a rectangular grid with a step $(\Delta x, \Delta y)$ on the Hele-Shaw cell and calculate the vector field of the fluid flow velocity (Fig. 3).

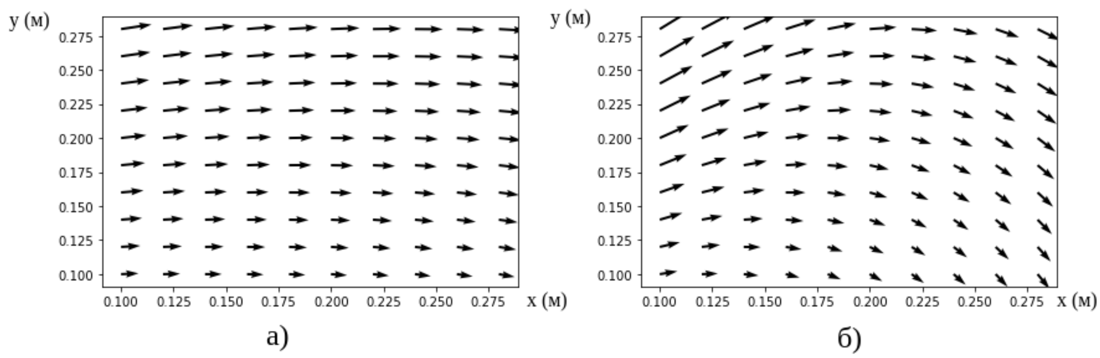

Fig. 3. Vector field of current velocities for a) water at $A = 10$, б) glycerin at $A = 10^3$

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Исследование уравнений вязкой неоднородной жидкости в ячейке Хеле-Шоу методом группового анализа

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Аннотация. В работе построена основная группа преобразований, допускаемых системой дифференциальных уравнений течения вязкой неоднородной жидкости в ячейке Хеле–Шоу, получено классифицирующее уравнение относительно функции вязкости, построен базис операторов, сохраняющих вид исходных уравнений. Описан базис пространства решений определяющих уравнений. Найдены инварианты операторов и получены инвариантные решения уравнений.

Ключевые слова: групповой анализ, уравнения жидкости, инфинитезимальный оператор, инвариант, инвариантное решение, ячейка Хеле–Шоу.