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Integral Operator of Potential Type for Infinitely Differentiable Functions

Simona G. Myslivets*
Siberian Federal University
Krasnoyarsk, Russian Federation

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Abstract. In this paper, we prove the infinite differentiability of an integral operator of the potential type for an infinitely differentiable function defined on the boundary of a bounded domain with the boundary of the class $\mathcal{C}^\infty$ up to the boundary of the domain on both sides.

Keywords: the differentiability of an integral operator of the potential type.

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We consider n-dimensional complex space $\mathbb{C}^n$, $n > 1$ with variables $z = (z_1, \ldots, z_n)$. Let's introduce the vector module $|z| = \sqrt{z_1^2 + \ldots + z_n^2}$ and the differential forms $dz = dz_1 \wedge \ldots \wedge dz_n$ and $d\bar{z} = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$ and also $dz[k] = dz_1 \wedge \ldots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \ldots \wedge dz_n$.

A bounded domain $D \subset \mathbb{C}^n$ has boundary of class $\partial D \in \mathcal{C}^\infty$ if $D = \{z \in \mathbb{C}^n : \ \rho(z) < 0\}$, where ρ is real-valued function of class $\mathcal{C}^\infty$ on some neighborhood of the closure of domain D, and the differential $d\rho \neq 0$ on ∂D. Let's denote the "complex" guiding cosines

$$\rho_k = \frac{1}{|\operatorname{grad}\rho|}\frac{\partial \rho}{\partial z_k}, \qquad \rho_{\bar{k}} = \frac{1}{|\operatorname{grad}\rho|}\frac{\partial \rho}{\partial \bar{z}_k}.$$

We will also consider infinitely differentiable functions $f \in \mathcal{C}^\infty(\partial D)$ on the boundary of the domain D.

Consider the Bochner–Martinelli kernel, which is an exterior differential form $U(\zeta, z)$ of type $(n, n-1)$ (see, for example, [1, Ch. 1]), given by

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n}\sum_{k=1}^{n}(-1)^{k-1}\frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}}\, d\bar{\zeta}[k] \wedge d\zeta.$$

This kernel plays an important role in multidimensional complex analysis (see, for example, [1,2]).

Let $g(\zeta, z)$ be the fundamental solution to the Laplace equation:

$$g(\zeta, z) = -\frac{(n-2)!}{(2\pi i)^n}\frac{1}{|\zeta - z|^{2n-2}}, \quad n > 1,$$

then

$$U(\zeta, z) = \sum_{k=1}^{n}(-1)^{k-1}\frac{\partial g}{\partial \zeta_k}\, d\bar{\zeta}[k] \wedge d\zeta.$$

For the function $f \in \mathcal{C}^\infty(\partial D)$, we introduce the Bochner–Martinelli integral (integral operator)

$$M(f) = \int_{\partial D} f(\zeta)U(\zeta, z), \quad z \notin \partial D,$$

*asmyslivets@sfu-kras.ru
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and also the single-layer potential (integral operator)

$$\Phi(f) = i^n 2^{n-1} \int\limits_{\partial D} f(\zeta) g(\zeta, z)\, d\sigma(\zeta), \quad z \notin \partial D,$$

where $d\sigma$ is the Lebesgue surface measure on ∂D.

We formulate theorems on the derivatives of integrals $M(f)$ and $\Phi(f)$, proved in [1, Ch. 1]. These statements are derived from the classical formulas of the potential theory [3].

Theorem 1. *If $\partial D \in \mathcal{C}^2$ and $f \in \mathcal{C}^2(\partial D)$, then the integral $M(f)$ extends to $\overline{D}$ and $\mathbb{C}^n \setminus D$ as a function of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$. At the same time, the formulas are valid*

$$\frac{\partial M(f)}{\partial z_m} = \int\limits_{\partial D} \Big(\frac{\partial f}{\partial \zeta_m} - \rho_m \sum_{k=1}^{n} \rho_k \frac{\partial f}{\partial \bar{\zeta}_k} \Big) U(\zeta, z) +$$
$$+ i^n 2^{n-1} \int\limits_{\partial D} \sum_{s,k=1}^{n} \Big[\rho_k \frac{\partial}{\partial \zeta_s} \Big(\rho_m \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \Big) - \rho_m \frac{\partial}{\partial \zeta_k} \Big(\rho_m \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \Big) \Big] g(\zeta, z) d\sigma(\zeta)$$

and

$$\frac{\partial M(f)}{\partial \bar{z}_m} = \int\limits_{\partial D} \Big(\frac{\partial f}{\partial \bar{\zeta}_m} - \rho_{\bar{m}} \sum_{k=1}^{n} \rho_k \frac{\partial f}{\partial \bar{\zeta}_k} \Big) U(\zeta, z) +$$
$$+ i^n 2^{n-1} \int\limits_{\partial D} \sum_{s,k=1}^{n} \Big[\rho_k \frac{\partial}{\partial \bar{\zeta}_s} \Big(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \Big) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k} \Big(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \Big) \Big] g(\zeta, z) d\sigma(\zeta).$$

Theorem 2. *If $\partial D \in \mathcal{C}^2$ and $f \in \mathcal{C}^2(\partial D)$, then for the integral $\Phi(f)$ the formulas are valid*

$$\frac{\partial \Phi(f)}{\partial z_m} = - \int\limits_{\partial D} f \rho_m U(\zeta, z) + i^n 2^{n-1} \int\limits_{\partial D} \sum_{k=1}^{n} \Big[\rho_k \frac{\partial}{\partial \zeta_m} (f \rho_{\bar{k}}) - \rho_m \frac{\partial}{\partial \zeta_k} (f \rho_{\bar{k}}) \Big] g(\zeta, z) d\sigma(\zeta)$$

and

$$\frac{\partial \Phi(f)}{\partial \bar{z}_m} = - \int\limits_{\partial D} f \rho_{\bar{m}} U(\zeta, z) + \ + i^n 2^{n-1} \int\limits_{\partial D} \sum_{k=1}^{n} \Big[\rho_k \frac{\partial}{\partial \bar{\zeta}_m} (f \rho_{\bar{k}}) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k} (f \rho_{\bar{k}}) \Big] g(\zeta, z) d\sigma(\zeta).$$

It follows from the theorems 1 and 2 that the partial derivatives of the integrals $M(f)$ and $\Phi(f)$ are the application of the integral operators M and Φ to some differential operators of the function f.

Thus, if we denote the integral operator

$$I(f^1, f^2) = M(f^1) + \Phi(f^2), \quad z \notin \partial D,$$

for some functions $f^1(z)$, $f^2(z)$ of class $\mathcal{C}^\infty$ on the boundary of the domain D, then the statement is true

Corollary 1. *These equalities are valid*

$$\frac{\partial I(f^1, f^2)}{\partial z_m} = I(L_m(f^1, f^2), K_m(f^1, f^2)) = M(L_m(f^1, f^2)) + \Phi(K_m(f^1, f^2)),$$

where

$$L_m(f^1,f^2) = \frac{\partial f^1}{\partial \zeta_m} - \rho_m \sum_{k=1}^{n} \rho_k \frac{\partial f^1}{\partial \bar{\zeta}_k} - f^2 \rho_m,$$

$$K_m(f^1,f^2) = \sum_{s,k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_s} \left(\rho_m \rho_{\bar{k}} \frac{\partial f^1}{\partial \bar{\zeta}_s} \right) - \rho_m \frac{\partial}{\partial \zeta_k} \left(\rho_m \rho_{\bar{k}} \frac{\partial f^1}{\partial \bar{\zeta}_s} \right) \right] +$$
$$+ i^n 2^{n-1} \sum_{k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_m} (f^2 \rho_{\bar{k}}) - \rho_m \frac{\partial}{\partial \zeta_k} (f^2 \rho_{\bar{k}}) \right].$$

correspondingly

$$\frac{\partial I(f^1,f^2)}{\partial \bar{z}_m} = I(L_{\bar{m}}(f^1,f^2), K_{\bar{m}}(f^1,f^2)) = M(L_{\bar{m}}(f^1,f^2)) + \Phi(K_{\bar{m}}(f^1,f^2)),$$

where

$$L_{\bar{m}}(f^1,f^2) = \frac{\partial f^1}{\partial \bar{\zeta}_m} - \rho_{\bar{m}} \sum_{k=1}^{n} \rho_k \frac{\partial f^1}{\partial \bar{\zeta}_k} - f^2 \rho_{\bar{m}},$$

$$K_{\bar{m}}(f^1,f^2) = \sum_{s,k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_s} \left(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f^1}{\partial \bar{\zeta}_s} \right) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k} \left(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f^1}{\partial \bar{\zeta}_s} \right) \right] +$$
$$+ i^n 2^{n-1} \sum_{k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_{\bar{m}}} (f^2 \rho_{\bar{k}}) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k} (f^2 \rho_{\bar{k}}) \right].$$

Thus, the derivatives of the operator $I(f^1,f^2)$ are again the operator I from some derivatives of the functions f^1, f^2.

From corollary 1 we get the statement

Theorem 3. *If $\partial D \in \mathcal{C}^\infty$ and $f^1, f^2 \in \mathcal{C}^\infty(\partial D)$, then both integrals $I(f^1,f^2)$ ($z \in D$, $z \in \mathbb{C}^n \setminus \overline{D}$) continue by $\overline{D}$ and on $\mathbb{C}^n \setminus D$ as infinitely differentiable functions.*

Proof. Let's first find the second derivatives of this integral using the corollary

$$\begin{aligned} \frac{\partial^2 I(f^1,f^2)}{\partial z_l \partial z_m} &= \frac{\partial}{\partial z_l} I\big(L_m(f^1,f^2), K_m(f^1,f^2)\big) = \\ &= I\big(L_l(L_m(f^1,f^2), K_m(f^1,f^2)), K_l(L_m(f^1,f^2), K_m(f^1,f^2))\big) = \\ &= M\big(L_l(L_m(f^1,f^2), K_m(f^1,f^2))\big) + \Phi\big(K_l(L_m(f^1,f^2), K_m(f^1,f^2))\big). \quad (1) \end{aligned}$$

Derivatives are also written out

$$\frac{\partial^2 I(f^1,f^2)}{\partial \bar{z}_l \partial z_m}, \frac{\partial^2 I(f^1,f^2)}{\partial z_l \partial \bar{z}_m}, \frac{\partial^2 I(f^1,f^2)}{\partial \bar{z}_l \partial \bar{z}_m}.$$

Denote by $\alpha = (\alpha_1, \ldots, \alpha_t)$, $t = 1, 2, \ldots$ a set of indexes of size t that take any values from the set of indexes $1, \ldots, n$ and $\bar{1}, \ldots, \bar{n}$. Therefore, we have

$$\frac{\partial^t I(f^1,f^2)}{\partial z_\alpha} = \frac{\partial^t I(f^1,f^2)}{\partial z_{\alpha_1} \cdots \partial z_{\alpha_t}}$$

where $\dfrac{\partial}{\partial z_{\alpha_j}} = \dfrac{\partial}{\partial z_m}$, if $\alpha_j = m$ и $\dfrac{\partial}{\partial z_{\alpha_j}} = \dfrac{\partial}{\partial \bar{z}_m}$, if $\alpha_j = \bar{m}$.

Then we get that

$$\frac{\partial^t I(f^1, f^2)}{\partial z_\alpha}$$

there is a sum of the Bochner–Martinelli integral of an infinitely differentiable function and the single-layer potential of an infinitely differentiable function. From here and from the properties of the Bochner–Martinelli integral (see, for example, [1, 2]) and the single-layer potential (see, for example, [3]), it follows that the integral $I(f^1, f^2)$ is an infinitely differentiable function up to the boundary. □

Corollary 2. *If $\partial D \in \mathcal{C}^\infty$ and $f \in \mathcal{C}^\infty(\partial D)$, then the Bochner-Martinelli integral $M(f)$ continues on $\overline{D}$ and on $\mathbb{C}^n \setminus D$ as an infinitely differentiable function.*

Consider the case when $f^1 = f$ and $f^2 = 0$. Then

$$L_m(f,0) = L_m(f) = \frac{\partial f}{\partial \zeta_m} - \rho_m \sum_{k=1}^{n} \rho_k \frac{\partial f}{\partial \bar{\zeta}_k},$$

$$K_m(f,0) = K_m(f) = i^n 2^{n-1} \sum_{s,k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_s} \left(\rho_m \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \right) - \rho_m \frac{\partial}{\partial \zeta_k} \left(\rho_m \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \right) \right],$$

correspondingly,

$$L_{\bar{m}}(f,0) = L_{\bar{m}}(f) = \frac{\partial f}{\partial \bar{\zeta}_m} - \rho_{\bar{m}} \sum_{k=1}^{n} \rho_k \frac{\partial f}{\partial \bar{\zeta}_k},$$

$$K_{\bar{m}}(f,0) = K_{\bar{m}}(f) = i^n 2^{n-1} \sum_{s,k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_s} \left(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \right) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k} \left(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s} \right) \right].$$

Then, according to the corollary 1, we get

$$\frac{\partial I(f,0)}{\partial z_m} = \frac{\partial M(f)}{\partial z_m} = I\big(L_m(f,0), K_m(f,0)\big) = M\big(L_m(f)\big) + \Phi\big(K_m(f)\big),$$

$$\frac{\partial I(f,0)}{\partial \bar{z}_m} = \frac{\partial M(f)}{\partial \bar{z}_m} = I\big(L_{\bar{m}}(f,0), K_{\bar{m}}(f,0)\big) = M\big(L_{\bar{m}}(f)\big) + \Phi\big(K_{\bar{m}}(f)\big),$$

Now let's consider the case when $f^1 = 0$ and $f^2 = f$. Then

$$L_m(0,f) = \tilde{L}_m(f) = -f\rho_m,$$

$$K_m(0,f) = \tilde{K}_m(f) = +i^n 2^{n-1} \sum_{k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_m} (f\rho_{\bar{k}}) - \rho_m \frac{\partial}{\partial \zeta_k} (f\rho_{\bar{k}}) \right],$$

correspondingly,

$$L_{\bar{m}}(0,f) = \tilde{L}_{\bar{m}}(f) = -f\rho_{\bar{m}},$$

$$K_{\bar{m}}(0,f) = \tilde{K}_{\bar{m}}(f) = i^n 2^{n-1} \sum_{k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_{\bar{m}}} (f\rho_{\bar{k}}) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k} (f\rho_{\bar{k}}) \right].$$

Then, according to the corollary 1, we get

$$\frac{\partial I(0,f)}{\partial z_m} = \frac{\partial \Phi(f)}{\partial z_m} = I\big(L_m(0,f), K_m(0,f)\big) = M\big(\tilde{L}_m(f)\big) + \Phi\big(\tilde{K}_m(f)\big),$$

$$\frac{\partial I(0,f)}{\partial \bar{z}_m} = \frac{\partial \Phi(f)}{\partial \bar{z}_m} = I\big(L_{\bar{m}}(0,f), K_{\bar{m}}(0,f)\big) = M\big(\tilde{L}_{\bar{m}}(f)\big) + \Phi\big(\tilde{K}_{\bar{m}}(f)\big),$$

We got that

$$\frac{\partial M(f)}{\partial z_m} = M(L_m(f)) + \Phi(K_m(f)) = I(L_m(f), K_m(f))$$
$$\frac{\partial \Phi(f)}{\partial z_m} = M(\tilde{L}_m(f)) + \Phi(\tilde{K}_m(f)) = I(\tilde{L}_m(f), \tilde{K}_m(f)).$$

Consider the second derivative of the Bochner–Martinelli integral

$$\frac{\partial^2 M(f)}{\partial z_m \partial z_l} = \frac{\partial M(L_m(f))}{\partial z_l} + \frac{\partial \Phi(K_m(f))}{\partial z_l} =$$
$$= I\big(L_l \circ L_m(f), K_l \circ L_m(f)\big) + I\big(\tilde{L}_l \circ K_m(f), \tilde{K}_l \circ K_m(f)\big).$$

It follows that

$$\frac{\partial^2 M(f)}{\partial z_m \partial z_l} = \frac{\partial I(L_m(f), K_m(f))}{\partial z_l} = I\big(L_l \circ L_m(f), K_l \circ L_m(f)\big) + I\big(\tilde{L}_l \circ K_m(f), \tilde{K}_l \circ K_m(f)\big).$$

Therefore, the derivative of the integral operator I is the sum of two integral operators I, in which the arguments of the first operator I will be the operators L and K applied to the first argument of this operator, and the arguments of the second operator I will be the operators $\tilde{L}$ and $\tilde{K}$ applied to the second argument of this operator.

It follows that, for example, the following third-order derivative will be equal to

$$\frac{\partial^3 M(f)}{\partial z_m \partial z_l \partial z_t} = I\big(L_t \circ L_l \circ L_m(f), K_t \circ L_l \circ L_m(f)\big) + I\big(\tilde{L}_t \circ K_l \circ L_m(f), \tilde{K}_t \circ K_l \circ L_m(f)\big) +$$
$$+ I\big(L_t \circ \tilde{L}_l \circ K_m(f), K_t \circ \tilde{L}_l \circ K_m(f)\big) + I\big(\tilde{L}_t \circ \tilde{K}_l \circ K_m(f), \tilde{K}_k \circ \tilde{K}_l \circ K_m(f)\big).$$

The derivatives with a different set of variables are calculated in the same way.

We denote, as in the Theorem 3, by $\alpha = (\alpha_1, \ldots, \alpha_t)$, $t = 1, 2, \ldots$ a set of indices of size t that take any values from the set of indices $1, \ldots, n$ and $\bar{1}, \ldots, \bar{n}$. Therefore, we have

$$\frac{\partial^t M(f)}{\partial z_\alpha} = \frac{\partial^t M(f)}{\partial z_{\alpha_1} \cdots \partial z_{\alpha_k}},$$

where $\dfrac{\partial}{\partial z_{\alpha_j}} = \dfrac{\partial}{\partial z_m}$, if $\alpha_j = m$ и $\dfrac{\partial}{\partial z_{\alpha_j}} = \dfrac{\partial}{\partial \bar{z}_m}$, if $\alpha_j = \bar{m}$.

Corollary 3. *The derivative* $\dfrac{\partial^t M(f)}{\partial z_\alpha}$ *of order* t *from the Bochner-Martinelli integral is the sum of* 2^{t-1} *integral operators* I *applied to various compositions of operators* L, K, $\tilde{L}$, $\tilde{K}$ *from the function* f.

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Интегральный оператор типа потенциала для бесконечно дифференцируемых функций

Симона Г. Мысливец
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. В этой статье доказана бесконечная дифференцируемость интегрального оператора типа потенциала для бесконечно дифференцируемых функций, определенных на границе ограниченной области вплоть до границы области с обеих сторон.

Ключевые слова: дифференцируемость интегрального оператора типа потенциала вплоть до границы.