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An Application of the Plane Curve's Standard Basis to a Certain Class of Problems from Classical Mechanics

Sophie B. Bogdanova*
Sergey O. Gladkov†
Moscow Aviation Institute
(National Research University) MAI
Moscow, Russian Federation

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Abstract. It is shown that the moving basis of a curve in polar coordinates can always be considered as a right-handed reference frame moving with acceleration. A system of differential equations is obtained that describes the trajectory of a freely falling body in a non-inertial reference frame coinciding with the standard basis of the curve. Finally, they were solved numerically using the Archimedean spiral, the three-petal rose and the cardioid as examples.

Keywords: relative motion, line curvature, mechanics of curvilinear motion, computer simulation.

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Introduction

In this paper we will consider one of the aspects of using the plane curve's moving basis [1–3] and, with its help, we will provide solutions to a number of curvilinear motion problems. The effectiveness of the moving basis method has been sufficiently demonstrated in a series of original papers [4–6]. However, attention should be paid to the disadvantage of the standard basis of the $\boldsymbol{\tau} - \mathbf{n}$ curve, where the unit tangent vector $\boldsymbol{\tau}$ and the unit normal vector $\mathbf{n}$ are related by simple linear relationships:

$$\begin{cases} \dfrac{d\boldsymbol{\tau}}{ds} = K\mathbf{n}, \\ \dfrac{d\mathbf{n}}{ds} = -K\boldsymbol{\tau}, \end{cases} \tag{1}$$

where curvature $K = \dfrac{|y''|}{\left(1+y'^2\right)^{3/2}} = \dfrac{|\ddot{y}\dot{x} - \ddot{x}\dot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} > 0$ (see [1–3]). As shown in Fig. 1, the moving basis with this definition changes its orientation from "right" to "left", moving from the concave region to the convex region.

It should be noted that it was possible to avoid the influence of this circumstance on the re-sults obtained in papers [4–6]. Although it is clear that a reference frame that constantly changes its orientation is extremely inconvenient.

*sonjaf@list.ru https://orcid.org/0000-0001-8503-1794
†sglad51@mail.ru https://orcid.org/0000-0002-2755-9133

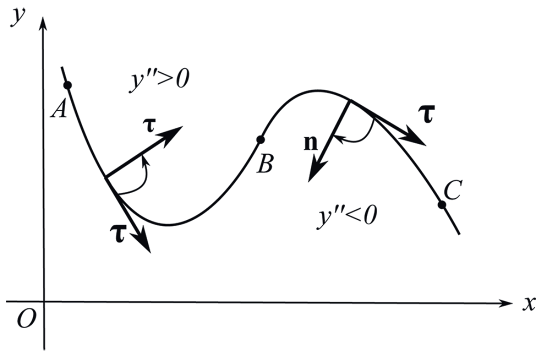

Fig. 1. In the concave section AB, the $\boldsymbol{\tau} - \mathbf{n}$ basis is right-handed, and in the convex section BC, it al-ready has a left orientation

1. Physical moving basis

Here we will consider two fundamentally different cases.

1. Let the curve be a graph of the function $y = y(x)$. Then the situation can be improved by transforming the standard basis to the $\mathbf{T} - \mathbf{N}$ basis (Fig. 2). Its unit vectors are quite

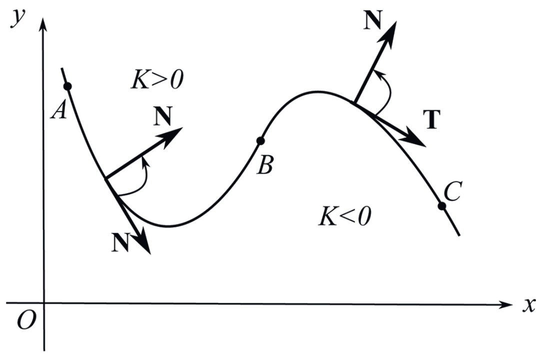

Fig. 2. The moving $\mathbf{T} - \mathbf{N}$ basis is right-handed on any section of the curve

similarly related (1):

$$\begin{cases} \dfrac{d\mathbf{T}}{ds} = K \cdot \mathbf{N}, \\ \dfrac{d\mathbf{N}}{ds} = -K \cdot \mathbf{T}, \end{cases} \tag{2}$$

Here the curvature of the curve is defined as

$$K = \frac{y''}{\left(1 + y'^2\right)^{3/2}} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}. \tag{3}$$

As a result of transformations (2)–(3), the $\mathbf{T} - \mathbf{N}$ basis can be considered as a convenient physical reference frame.

2. Let the curve be given in polar coordinates as a function $\mathbf{R}_0(\varphi) = \mathbf{i} \cdot r(\varphi) \cos\varphi + \mathbf{j} \cdot r(\varphi) \sin\varphi$ where the polar angle increases in counterclock-wise direction. A careful examination of all curves known in polar coordinates [7] shows that the moving basis of such curves never

changes its orientation (and always remains "right-handed"), which happens due to the specific direction of change in the parameter φ (see Fig. 3) Thus, the invariance of the

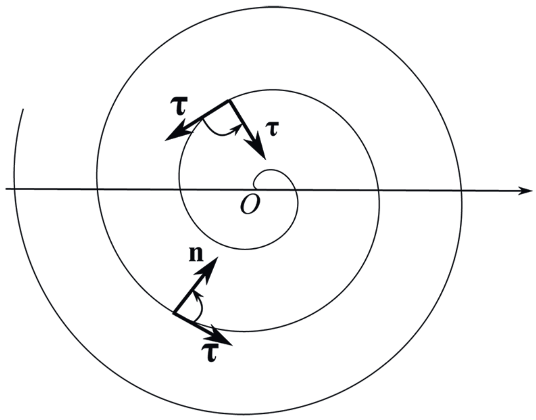

Fig. 3. The moving basis on the Archimedean spiral is always right-handed

moving basis of curves in polar coordinates with respect to changes in the convex and concave sections allows it to be used as a physical reference frame for solving curvilinear motion problems.

2. Trajectory of a freely falling body in a moving basis

Any curvilinear motion is accelerated, and therefore a moving basis moving along its trajectory can be considered as a non-inertial reference frame [8–10]. As we know [8–10], the trajectories of moving bodies are different in different reference frames. For example, from the perspective of a stationary observer, a freely falling body moves along a vertical straight line, but the same trajectory from the center of a moving basis of a curve moving at a given speed will differ significantly from a straight vertical fall. Thus, the task is to de-scribe the trajectory of a body freely falling from point from the perspective of an observer lo-cated in the center of the moving basis of the curve $\mathbf{R_0}(\varphi)$ (see Fig. 4). The binormal vector $\mathbf{b}$ is determined by the well-known rule $\mathbf{b} = \boldsymbol{\tau} \times \mathbf{n}$ (see [1]). For a plane curve it is constant, i.e., $\dot{\mathbf{b}} = 0$. According to [4] we have

$$\ddot{\mathbf{r}}(\varphi) = \ddot{\mathbf{R}}(\varphi) - \ddot{\mathbf{R}}_\mathbf{0}(\varphi), \tag{4}$$

where

$$\begin{cases} \ddot{\mathbf{R}}_0(\varphi) = \dot{v} \cdot \boldsymbol{\tau} + v^2 K \cdot \mathbf{n}, \\ \ddot{\mathbf{R}}(\varphi) = -g \cdot \mathbf{b}, \\ \ddot{\mathbf{r}}(\varphi) = \dfrac{d}{d\varphi} \left(X(\varphi) \cdot \boldsymbol{\tau} + Y(\varphi) \cdot \mathbf{n} + Z(\varphi) \cdot \mathbf{b} \right), \end{cases} \tag{5}$$

where g is the acceleration of gravity. Below, we will use uppercase letters $X(\varphi), Y(\varphi), Z(\varphi)$ to denote the coordinates of the falling body in the non-inertial $\boldsymbol{\tau} - \mathbf{n} - \mathbf{b}$ frame, and lowercase letters $x(\varphi), y(\varphi), z(\varphi)$ to denote its coordi-nates in the stationary basis $\mathbf{i} - \mathbf{j} - \mathbf{k}$. As a result of double differentiation of the radius vector $\mathbf{r}(\varphi)$, taking into account relations (1), we get:

$$\begin{aligned} \ddot{\mathbf{r}}(t) \;=\; & \ddot{X} \cdot \boldsymbol{\tau} + \ddot{Y} \cdot \mathbf{n} + \ddot{Z} \cdot \mathbf{b} + \\ & + \boldsymbol{\tau} \cdot \left(-2\dot{Y} v K - X v^2 K^2 - Y \dot{v} K - Y v \dot{K} \right) + \\ & + \mathbf{n} \cdot \left(2\dot{X} v K + X \dot{v} K + X v \dot{K} - Y v^2 K^2 \right). \end{aligned} \tag{6}$$

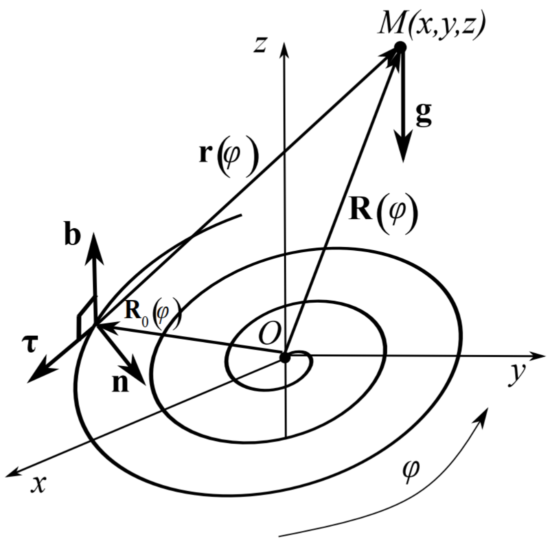

Fig. 4. Problem geometry. The moving basis $\boldsymbol{\tau} - \mathbf{n} - \mathbf{b}$ moves along the curve $\mathbf{R_0}\,(\varphi)$ in the direction of in-creasing parameter φ

Substituting further (5) and (6) into equality (4) and equating the projections onto the corresponding moving unit vectors, we obtain the following system of differential equations:

$$\begin{cases} \ddot{X} = 2\dot{Y}vK + Xv^2K^2 + Y\dot{v}K + Yv\dot{K} - \dot{v}, \\ \ddot{Y} = Yv^2K^2 - v^2K^2 - 2\dot{X}vK - X\dot{v}K - Xv\dot{K}, \\ \ddot{Z} = -g. \end{cases} \tag{7}$$

Their solution determines the trajectory of a freely falling body in the moving basis.

3. Analysis of the results obtained

The results of numerical simulation of system (5) are analyzed in three specific cases.

1. Archimedean spiral $r = a\varphi$. Its curvature according to [7] is as follows

$$K = \frac{1}{a} \cdot \frac{\varphi^2 + 2}{(\varphi^2 + 1)^{3/2}}.$$

2. Three-petal rose $r = a \sin 3\varphi$ with curvature [7].

$$K = \frac{2}{a} \cdot \frac{9\cos^2 3\varphi + 4\sin^2 3\varphi}{\left(9\cos^2 3\varphi + \sin^2 3\varphi\right)^{3/2}},\ 0 < \varphi < 2\pi.$$

3. A cardioid represented by the equation $r = a\,(1 - \cos\varphi)$, whose curvature according to [7] is as follows

$$K = \frac{3}{4a\sqrt{2}\sin\dfrac{\varphi}{2}},\ 0 < \varphi < 2\pi$$

The solution results are illustrated in Fig. 5–7.

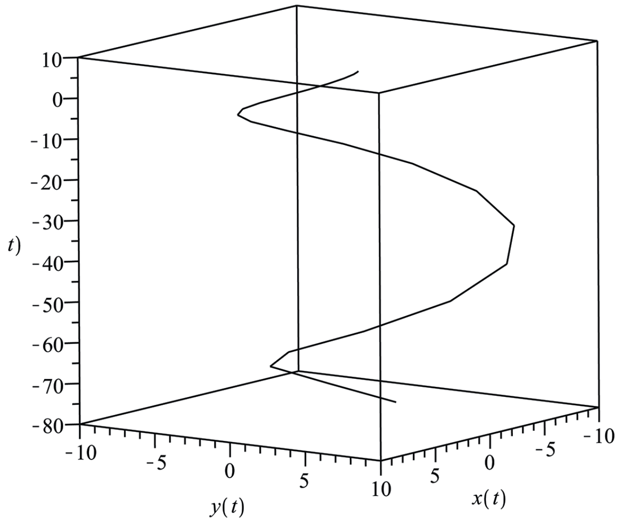

Fig. 5. The trajectory of a freely falling body from the center of the moving basis of the Archimedean spiral $r = \varphi$, moving with the following speed $v = \varphi^2$ for $0 \leqslant \varphi \leqslant 3\pi$. Initial conditions $X(0) = Y(0) = 5$, $Z(0) = 10$, $X'(0) = Y'(0) = Z'(0) = 0$

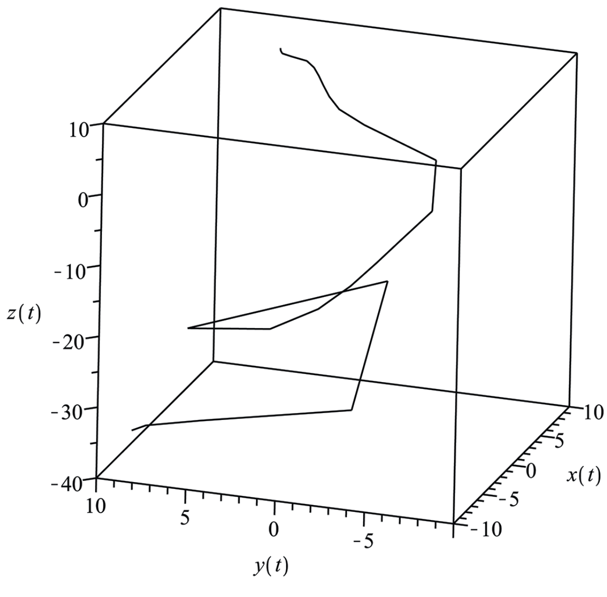

Fig. 6. The trajectory of a freely falling body from the center of the moving basis of the three-petal rose $r = 2\sin 3\varphi$, moving with the following speed $v = \varphi^2$ for $0 \leqslant \varphi \leqslant 2\pi$. Initial conditions $X(0) = Y(0) = 5$, $Z(0) = 10$, $X'(0) = Y'(0) = Z'(0) = 0$

Conclusion

1. The fundamental possibility of using a moving basis chosen along a given curve in solving a number of physical problems in polar coordinates is shown.

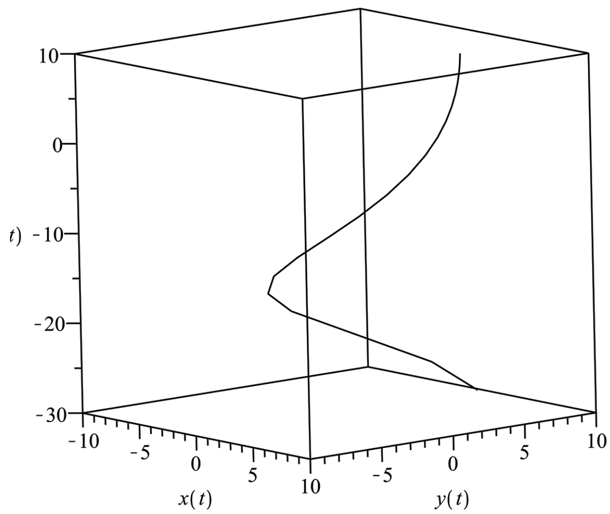

Fig. 7. The trajectory of a freely falling body from the center of the moving basis of the cardioid $r = 2(1 - \cos\varphi)$, moving with the following speed $v = \varphi^3$ for $0 \leqslant \varphi \leqslant 2\pi$. Initial conditions $X(0) = Y(0) = 5$, $Z(0) = 10$, $X'(0) = Y'(0) = Z'(0) = 0$

2. A system of differential equations is presented that describes the free fall of a body in the non-inertial reference frame moving along a plane curve specified in polar coordinates.

3. The results of the numerical solution of the resulting system of equations are graphically illustrated using the Archimedean spiral, the rose and the cardioid as examples.

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Об одном применении естественного базиса плоской кривой к решению задач механики

Софья Б. Богданова
Сергей О. Гладков
Московский авиационный институт (Национальный исследовательский университет)
Москва, Российская Федерация

Аннотация. Показано, что подвижный базис кривой в полярной системе координат можно рассматривать как правую систему отсчета, движущуюся с ускорением. Построена система дифференциальных уравнений, описывающая траекторию движения свободно падающего тела в неинерциальной системе координат, совпадающей с естественным базисом кривой. Приведены результаты моделирования этой системы на примере спирали Архимеда, трехлепестковой розы и кардиоиды.

Ключевые слова: относительное движение, кривизна линии, механика криволинейного движения, компьютерное моделирование.