EDN: GZNSAL
УДК 621

The Weighted Hardy Operators and Quasi-monotone Functions

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Received 21.04.2022, received in revised form 30.09.2023, accepted 14.02.2024

Abstract. Some Hardy-type inequalities are established by W. T. Sulaiman. The aim of this work is to extend these inequalities for weighted Hardy operators with quasi-monotone functions. Moreover some new integral weighted inequalities were obtained.

Keywords: inequalities, Hardy's operator, quasi-monotones functions.

Citation: A. Senouci, A. Zanou, The Weighted Hardy Operators and Quasi-monotone Functions, J. Sib. Fed. Univ. Math. Phys., 2024, 17(3), 318–325. EDN: GZNSAL.

1. Introduction and preliminaries

The classical Hardy inequality (See [2]) has been proved for $f(x) \geqslant 0$, $p > 1$

$$\int_0^{+\infty} \left(\frac{F(x)}{x}\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx, \tag{1}$$

where

$$F(x) = \int_0^x f(t)dt. \tag{2}$$

The constant $\left(\frac{p}{p-1}\right)^p$ is sharp (the best possible).

This inequality have many applications in the theory of differential equations (Ordinary and Partial) and led to many interesting questions and connections between different areas of mathematical analysis.

The following inequalities were proved in [4].

Let $f \geqslant 0$, $g > 0$.

1. If $\frac{x}{g(x)}$ is a non-increasing function, $p > 1$ and $0 < a < 1$, then

$$\int_0^{\infty} \left(\frac{F(x)}{g(x)}\right)^p dx \leqslant \frac{1}{a(p-1)(1-a)^{p-1}} \int_0^{\infty} \left(\frac{tf(t)}{g(t)}\right)^p dt. \tag{3}$$

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2. If $\dfrac{x}{g(x)}$ is a non-decreasing function, $0 < p < 1$ and $a > 0$, then

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \geqslant \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^\infty \left(\frac{tf(t)}{g(t)}\right)^p dt. \tag{4}$$

3. If $p \geqslant 2$, then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leqslant \int_0^\infty t^{-1} f^{p-1}(t) F(t) dt. \tag{5}$$

4. If $p > 1$, $h \geqslant 0$, h convex and non-decreasing function, then

$$\int_0^\infty h^p \left(\frac{F(x)}{x}\right) dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty h^p \left(f(t)\right) dt. \tag{6}$$

The Hardy inequalities for quasi-monotone functions are discussed for example in [1] and [3].

The objective of this work is to generalize the inequalities (3)–(5) and (6) for weighted Hardy operator and its dual with quasi-monotone functions. Moreover, other integral inequalities were obtained for quasi-monotone functions.

Throughtout this paper, we will assume that functions are non-negative integrable and the integrals are supposed to exist and are finite.

2. Main results

Consider the weighted Hardy operator and its dual

$$F_w(x) = \int_0^x f_w(t)dt, \; F_w^*(x) = \int_x^\infty f_w(t)dt,$$

where $f_w(t) = f(t)w(t)$ and $g > 0$, $f \geqslant 0$ are Lebesgue measurable functions on $(0, \infty)$. Let ω and $v > 0$ be weight functions on $(0, \infty)$, $V(x) = \int\limits_0^x v(t)dt$, $V^*(x) = \int\limits_x^\infty v(t)dt$, and $G_V(x) = g(x)V(x)$, $G_{V^*}(x) = g(x)V^*(x)$.

The following definition was introduced in [1].

Definition 1. *We say that a non-negative function f is quasi-monotone on $]0, \infty[$, if for some real number α, $x^\alpha f(x)$ is a decreasing or an increasing function of x. More precisely, given $\beta \in \mathbb{R}$, we say that $f \in Q_\beta$ if $x^{-\beta} f(x)$ is non-increasing and $f \in Q^\beta$ if $x^{-\beta} f(x)$ is non-decreasing.*

Theorem 1. *Let $p > 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)} \in Q_\beta$, $0 < a < 1$ and $\beta < a(\dfrac{p-1}{p})$, then*

$$\int_0^\infty \left(\frac{F_w(x)}{G_V(x)}\right)^p dx \leqslant \frac{1}{\left(a(p-1) - p\beta\right)(1-a)^{p-1}} \int_0^\infty \left(\frac{tf_w(t)}{G_V(t)}\right)^p dt. \tag{7}$$

Proof. Since $\dfrac{x}{g(x)} \in Q_\beta$ and $V(x)$ is non-decreasing, then $\dfrac{x}{G_V(x)} \in Q_\beta$.

Let $K = \int\limits_0^\infty \left(\dfrac{F_w(x)}{G_V(x)}\right)^p dx$, then

$$\begin{aligned} K &= \int_0^\infty G_V^{-p}(x) \left(\int_0^x f_w(t)dt\right)^p dx = \\ &= \int_0^\infty G_V^{-p}(x) \left(\int_0^x t^{a(1-\frac{1}{p})} f_w(t) t^{-a(1-\frac{1}{p})} dt\right)^p dx. \end{aligned}$$

$\frac{x}{G_V(x)} \in Q_\beta$, implies that $\left(\frac{x}{G_V(x)}\right)^p \in Q_\beta$. By Hölder's inequality and Fubini's theorem, we get

$$\begin{aligned} K &\leqslant \int_0^\infty G_V^{-p}(x)\left(\int_0^x t^{a(p-1)} f_w^p(t)dt\right)\left(\int_0^x t^{-a}dt\right)^{p-1} dx = \\ &= \frac{1}{(1-a)^{p-1}}\int_0^\infty x^{(1-a)(p-1)}G_V^{-p}(x)\int_0^x t^{a(p-1)} f_w^p(t)dtdx = \\ &= \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)} f_w^p(t)\int_t^\infty x^{(1-a)(p-1)}G_V^{-p}(x)dxdt \leqslant \\ &\leqslant \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)} f_w^p(t)\left(\frac{t^{1-\beta}}{G_V(t)}\right)^p\int_t^\infty x^{p\beta-a(p-1)-1}dxdt = \\ &= \frac{1}{(a(p-1)-p\beta)(1-a)^{p-1}}\int_0^\infty \left(\frac{tf_w(t)}{G_V(t)}\right)^p dt. \end{aligned}$$

□

Remark 1. *If in* (7)*, we set* $w(x) = 1$*,* $V(x) = 1$ *and* $\beta = 0$*, we get the inequality* (3).

Now we consider the converse inequality of (2.1).

Theorem 2. *Let* $0 < p < 1$*,* $f \geqslant 0$*,* $g > 0$*,* $\frac{x}{g(x)} \in Q^\beta$*,* $a > 0$*,* $\beta < a\left(\frac{1-p}{p}\right)$ *and* $\beta \in \mathbb{R}$*, then*

$$\int_0^\infty \left(\frac{F_w(x)}{G_{V^*}(x)}\right)^p dx \geqslant \frac{1}{(a(1-p)-p\beta)(1+a)^{p-1}}\int_0^\infty \left(\frac{tf_w(t)}{G_{V^*}(t)}\right)^p dt. \tag{8}$$

Proof. Since $\frac{x}{g(x)} \in Q^\beta$ and $V^*(x)$ is non-increasing, then $\left(\frac{x}{G_{V^*}(x)}\right)^p \in Q^\beta$.

Let $I = \int\limits_0^\infty \left(\frac{F_w(x)}{G_{V^*}(x)}\right)^p dx$, then

$$\begin{aligned} I &= \int_0^\infty G_{V^*}^{-p}(x)\left(\int_0^x f_w(t)dt\right)^p dx = \\ &= \int_0^\infty G_{V^*}^{-p}(x)\left(\int_0^x t^{a(\frac{1}{p}-1)} f_w(t) t^{-a(\frac{1}{p}-1)}dt\right)^p dx. \end{aligned}$$

By converse Hölder inequality and Fubini's theorem, we get

$$\begin{aligned} I &\geqslant \int_0^\infty G_{V^*}^{-p}(x)\left(\int_0^x t^{a(1-p)} f_w^p(t)dt\right)\left(\int_0^x t^{+a}dt\right)^{p-1} dx = \\ &= \frac{1}{(1+a)^{p-1}}\int_0^\infty x^{(1+a)(p-1)}G_{V^*}^{-p}(x)\int_0^x t^{a(1-p)} f_w^p(t)dtdx = \\ &= \frac{1}{(1+a)^{p-1}}\int_0^\infty t^{a(1-p)} f_w^p(t)\int_t^\infty x^{(1+a)(p-1)}G_{V^*}^{-p}(x)dxdt \geqslant \\ &\geqslant \frac{1}{(1+a)^{p-1}}\int_0^\infty t^{a(1-p)} f_w^p(t)\left(\frac{t^{1-\beta}}{G_{V^*}(t)}\right)^p\int_t^\infty x^{p\beta-a(1-p)-1}dxdt = \\ &= \frac{1}{((a(1-p)-p\beta)(1+a)^{p-1}}\int_0^\infty \left(\frac{tf_w(t)}{G_{V^*}(t)}\right)^p dt. \end{aligned}$$

□

Remark 2. *By setting* $w(x) = 1$*,* $V^*(x) = 1$ *and* $\beta = 0$*, in* (8)*, we obtain the inequality* (4).

Theorem 3. *Let $p > 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)} \in Q^{\beta}$, $a > 1$ and $\beta > a(\dfrac{p-1}{p})$, then*

$$\int_0^{\infty} \left(\frac{F_w^*(x)}{G_{V^*}(x)} \right)^p dx \leqslant \frac{1}{(a(p-1) - p\beta)\,(a-1)^{p-1}} \int_0^{\infty} \left(\frac{t f_w(t)}{G_{V^*}(t)} \right)^p dt. \tag{9}$$

Proof. Let $K_1 = \int\limits_0^{\infty} \left(\dfrac{F_w^*(x)}{G_{V^*}(x)} \right)^p dx$, thus

$$\begin{aligned} K_1 &= \int_0^{\infty} G_{V^*}^{-p}(x) \left(\int_x^{\infty} f_w(t)dt \right)^p dx = \\ &= \int_0^{\infty} G_{V^*}^{-p}(x) \left(\int_x^{\infty} t^{a(1-\frac{1}{p})} f_w(t) t^{-a(1-\frac{1}{p})} dt \right)^p dx. \end{aligned}$$

By applying Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned} K_1 &\leqslant \int_0^{\infty} G_{V^*}^{-p}(x) \left(\int_x^{\infty} t^{a(p-1)} f_w^p(t)dt \right) \left(\int_x^{\infty} t^{-a}dt \right)^{p-1} dx = \\ &= \frac{1}{(a-1)^{p-1}} \int_0^{\infty} x^{(a-1)(p-1)} G_{V^*}^{-p}(x) \int_x^{\infty} t^{a(p-1)} f_w^p(t)dtdx = \\ &= \frac{1}{(a-1)^{p-1}} \int_0^{\infty} t^{a(p-1)} f_w^p(t) \int_0^t x^{(1-a)(p-1)} G_{V^*}^{-p}(x)dxdt \leqslant \\ &\leqslant \frac{1}{(a-1)^{p-1}} \int_0^{\infty} t^{a(p-1)} f_w^p(t) \left(\frac{t^{1-\beta}}{G_{V^*}(t)} \right)^p \int_0^t x^{p\beta - a(p-1)-1}dxdt = \\ &= \frac{1}{(a(p-1) - p\beta)\,(a-1)^{p-1}} \int_0^{\infty} \left(\frac{t f_w(t)}{G_{V^*}(t)} \right)^p dt. \end{aligned}$$

□

If in (9), we set $w(x) = 1$, $V(x)^* = 1$ and $\beta = 0$, we obtain the following corollary.

Corollary 1. *Let $p > 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)}$ non-decreasing function, $a > 1$, then*

$$\int_0^{\infty} \left(\frac{F^*(x)}{g(x)} \right)^p dx \leqslant \frac{1}{(a(p-1)(a-1)^{p-1}} \int_0^{\infty} \left(\frac{t f(t)}{g(t)} \right)^p dt. \tag{10}$$

Theorem 4. *Let $0 < p < 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)} \in Q^{\beta}$, $a < -1$, $\beta > a\left(\dfrac{1-p}{p}\right)$ and $\beta \in \mathbb{R}$, then*

$$\int_0^{\infty} \left(\frac{F_w^*(x)}{G_V(x)} \right)^p dx \geqslant \frac{1}{(p\beta - a(p-1))\,((-1)(1+a))^{p-1}} \int_0^{\infty} \left(\frac{t f_w(t)}{G_V(t)} \right)^p dt. \tag{11}$$

The proof is similar to that of Theorem 3.

If in (11), $w(x) = 1$, $V(x) = 1$ and $\beta = 0$, we get the following corollary.

Corollary 2. *Let $0 < p < 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)}$ non-decreasing function, $a < -1$, then*

$$\int_0^{\infty} \left(\frac{F_w^*(x)}{g(x)} \right)^p dx \geqslant \frac{1}{(a(1-p) - p\beta)\,(1+a)^{p-1}} \int_0^{\infty} \left(\frac{t f_w(t)}{g(t)} \right)^p dt. \tag{12}$$

Remark 3. *The inequalities (10) and (12) are the analogs of inequalities (3) and (4) respectively.*

Theorem 5. *Let $p \geqslant 2$, $f \geqslant 0$, $g > 0$, $\frac{x}{g(x)} \in Q_\beta$ and $\beta < \frac{1}{p}$, then*

$$\int_0^\infty \left(\frac{F_w(x)}{G_V(x)}\right)^p dx \leqslant \frac{1}{1-p\beta}\int_0^\infty G_V^{-1}(t)\left(\frac{tf_w(t)}{G_V(t)}\right)^{p-1} F_w(t)dt. \tag{13}$$

Proof. By applying Hölder's inequality with parameters $p-1$ and its conjugate $\frac{p-1}{p-2}$ and the assumption $\frac{x}{g(x)} \in Q_\beta$, it follows that

$$\begin{aligned}
\int_0^\infty \left(\frac{F_w(x)}{G_V(x)}\right)^p dx &= \int_0^\infty G_V^{-p}(x)F_w^{p-1}(x)F_w(x)dx = \\
&= \int_0^\infty G_V^{-p}(x)F_w^{p-1}(x)\int_0^x f_w(t)dtdx = \\
&= \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)F_w^{p-1}(x)dxdt = \\
&= \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)\left(\int_0^x f_w(u)du\right)^{p-1} dxdt \leqslant \\
&\leqslant \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)\int_0^x f_w^{p-1}(u)du\left(\int_0^x du\right)^{p-2} dxdt = \\
&= \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)x^{p-2}\left(\int_0^x f_w^{p-1}(u)du\right) dxdt \leqslant \\
&\leqslant \int_0^\infty f_w(t)\int_t^\infty f_w^{p-1}(u)\left(\frac{u^{1-\beta}}{G_V(u)}\right)^p\int_u^\infty x^{p\beta-2}dxdudt = \\
&= \frac{1}{1-p\beta}\int_0^\infty f_w(t)\int_t^\infty f_w^{p-1}(u)\left(\frac{u^{1-\beta}}{G_V(u)}\right)^p u^{p\beta-1}dudt = \\
&= \frac{1}{1-p\beta}\int_0^\infty f_w^{p-1}(u)u^{p\beta-1}\left(\frac{u^{1-\beta}}{G_V(u)}\right)^p\int_0^u f_w(t)dtdu = \\
&= \frac{1}{1-p\beta}\int_0^\infty G_V^{-1}(u)\left(\frac{uf_w(u)}{G_V(u)}\right)^{p-1} F_w(u)du.
\end{aligned}$$

□

Further, setting $V(x) = 1$ and $g(x) = x$ in Theorem 5, yields the following corollary.

Corollary 3. *Let $p \geqslant 2$, $f \geqslant 0$, $\beta < \frac{1}{p}$, then*

$$\int_0^\infty \left(\frac{F_w(x)}{x}\right)^p dx \leqslant \frac{1}{1-p\beta}\int_0^\infty t^{-1}f_w^{p-1}(t)F_w(t)dt. \tag{14}$$

Remark 4. *If we take $w = 1$ and $\beta = 0$ in (14), we obtain inequality (5).*

Theorem 6. *Let $p > 1$, $\beta < \frac{1}{p}(1-\frac{1}{p})$, $h > 0$ be a convex function, $\frac{x}{g(x)} \in Q_\beta$, then*

$$\int_0^\infty x^{p\beta}h^p\left(x^{-\beta}\frac{F_w(x)}{G_V(x)}\right)dx \leqslant$$

$$\leqslant \left(\frac{p}{p-p^2\beta-1}\right)\left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{p\beta}h^p\left(\frac{(t^{1-\beta}f_w(t))}{G_V(t)}\right)dt. \tag{15}$$

Proof. By using the convexity of h and Hölder's inequality, we get

$$\int_0^\infty x^{p\beta} h^p \left(x^{-\beta}\frac{F_w(x)}{G_V(x)}\right)^p dx = \int_0^\infty \left(x^\beta h\left(x^{-\beta}\frac{F_w(x)}{G_V(x)}\right)\right)^p dx =$$

$$= \int_0^\infty \left(x^\beta h\left(\frac{1}{x}\frac{x^{1-\beta}}{G_V(x)}\int_0^x f_w(t)dt\right)\right)^p dx \leqslant$$

$$\leqslant \int_0^\infty \left(\frac{x^\beta}{x}\int_0^x h\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dt\right)^p dx =$$

$$= \int_0^\infty x^{p(\beta-1)}\left(\int_0^x h\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dt\right)^p dx =$$

$$= \int_0^\infty x^{p(\beta-1)}\left(\int_0^x t^{\frac{1}{p}(1-\frac{1}{p})} h\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right) t^{-\frac{1}{p}(1-\frac{1}{p})}dt\right)^p dx \leqslant$$

$$\leqslant \int_0^\infty x^{p(\beta-1)}\int_0^x t^{1-\frac{1}{p}} h^p\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dt\left(\int_0^x t^{-\frac{1}{p}}dt\right)^{p-1} dx =$$

$$= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty x^{p\beta+\frac{1}{p}-2}\int_0^x t^{1-\frac{1}{p}} h^p\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dtdx =$$

$$= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}} h^p\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)\int_t^\infty x^{p\beta+\frac{1}{p}-2}dxdt =$$

$$= \left(\frac{p}{p-p2\beta-1}\right)\left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{p\beta} h^p\left(\frac{(t^{1-\beta}f_w(t))}{G_V(t)}\right)dt.$$

□

If in (15), we set $V(x) = 1$ and $g(x) = x$, we get the following corollary.

Corollary 4. *Let* $p > 1$, $\beta < \frac{1}{p}\Big(1 - \frac{1}{p}\Big)$, $f \geqslant 0$, $h > 0$, *h be a convex and non-decreasing function, then*

$$\int_0^\infty x^{p\beta} h^p\left(x^{-\beta}\frac{F_w(x)}{x}\right)dx \leqslant \left(\frac{p}{p-p^2\beta-1}\right)\left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{p\beta}h^p\left(t^{-\beta}f_w(t)\right)dt. \quad (16)$$

Remark 5. *If in* (16), *we put* $w = 1$ *and* $\beta = 0$, *we get* (6).

The following lemma was proved in [4].

Lemma 1. *Let* $h \geqslant 0$ *be convex, and* $h(0) = 0$, *then* $h(x)/x$ *is non-decreasing.*

Theorem 7. *Let* $p > 1$, $h > 0$ *be a convex, non-decreasing function* $h(0) = 0$, $g > 0$, $\frac{x}{g(x)} \in Q_\beta$ *and* $\beta < p - 1$, *then*

$$\int_0^\infty \frac{x^{2-p+\beta}h\left(\frac{x^{-\beta}F_w(x)}{G_V(x)}\right)}{h(x)}dx \leqslant \frac{1}{p-\beta-1}\int_0^\infty \frac{t^{2-p+\beta}h\left(\frac{t^{-\beta}f_w(t)}{G_V(t)}\right)}{h(t)}dt. \quad (17)$$

Proof. Let $I_1 = \int\limits_0^\infty \frac{x^{2-p+\beta}h\left(\frac{x^{-\beta}F_w(x)}{G_V(x)}\right)}{h(x)}dx,$

then

$$\begin{aligned} I_1 &= \int_0^\infty \frac{x^{2-p+\beta} h\left(\frac{1}{x}\frac{x^{1-\beta}}{G_V(x)}\int_0^x f_w(t)dt\right)}{h(x)}dx = \\ &= \int_0^\infty \frac{x^{2-p+\beta}}{h(x)} h\left(\frac{1}{x}\int_0^x \frac{x^{1-\beta}}{G_V(x)} f_w(t)dt\right) dx. \end{aligned}$$

Since $\frac{x}{g(x)} \in Q_\beta$, h is non-decreasing convex and $\frac{x}{G_V(x)} \in Q_\beta$, we have

$$\begin{aligned} \int_0^\infty \frac{x^{2-p+\beta}}{h(x)} h\left(\frac{1}{x}\int_0^x \frac{x^{1-\beta}}{G_V(x)} f_w(t)dt\right) dx &\leqslant \int_0^\infty \frac{x^{2-p+\beta}}{h(x)} h\left(\frac{1}{x}\int_0^x \frac{t^{1-\beta} f_w(t)}{G_V(t)}dt\right) dx \leqslant \\ &\leqslant \int_0^\infty \frac{x^{1-p+\beta}}{h(x)}\int_0^x h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dtdx. \end{aligned}$$

Applying Fubini's theorem and Lemma 1, we get

$$\begin{aligned} \int_0^\infty \frac{x^{1-p+\beta}}{h(x)}\int_0^x h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dtdx &= \int_0^\infty h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right)\int_t^\infty \frac{x^{1-p+\beta}}{h(x)} dxdt \leqslant \\ &\leqslant \int_0^\infty h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right)\left(\frac{t}{h(t)}\right)\int_t^\infty x^{\beta-p} dxdt = \\ &= \frac{1}{p-\beta-1}\int_0^\infty \frac{t^{2-p+\beta}}{h(t)} h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dt. \end{aligned}$$

Thus

$$\int_0^\infty \frac{x^{2-p+\beta} h\left(\frac{x^{-\beta}F_w(x)}{G_V(x)}\right)}{h(x)}dx \leqslant \frac{1}{p-\beta-1}\int_0^\infty \frac{t^{2-p+\beta}}{h(t)} h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dt.$$

□

If in (17), we set $V(x) = 1$, $v = \omega = 1$ and $\beta = 0$, we have the following corollary.

Corollary 5. *Let $p > 1$, $h > 0$ be a convex, non-decreasing function $h(0) = 0$, $g > 0$, $\frac{x}{g(x)}$ non-increasing function, then*

$$\int_0^\infty \frac{x^{2-p} h\left(\frac{x^{-\beta}F(x)}{g(x)}\right)}{h(x)}dx \leqslant \frac{1}{p-1}\int_0^\infty \frac{t^{2-p} h\left(\frac{t^1 f_w(t)}{g(t)}\right)}{h(t)}dt. \tag{18}$$

Definition 2. *A function h is said to be submultiplicative if $h(xy) \leqslant h(x)h(y)$.*

Theorem 8. *Let $p > 1$, $f \geqslant 0$, $\beta \in \mathbb{R}$, $\beta < p$. If $h \geqslant 0$ is a convex function and submultiplicative, $h(0) = 0$ such that $\frac{x}{h(x)} \in Q_\beta$, then*

$$\int_0^\infty \frac{x^{1-p}}{h^2(x)} h(F(x))dx \leqslant \frac{1}{p-\beta}\int_0^\infty \frac{t^{1-p}}{h(t)} h(f(t))dt. \tag{19}$$

Proof. By using the assumption of convexity and submultiplicativity of h, $\frac{x}{h(x)} \in Q_\beta$ and the Fubini theorem, we get

$$\begin{aligned}
\int_0^\infty \frac{x^{1-p}}{h^2(x)} h(F(x))dx &= \int_0^\infty \frac{x^{1-p}}{h^2(x)} h\left(x\frac{1}{x}\int_0^x f(t)dt\right) dx \leqslant \\
&\leqslant \int_0^\infty \frac{x^{-p}}{h(x)} \int_0^x h(f(t))dtdx = \\
&= \int_0^\infty h(f(t)) \int_t^\infty \frac{x^{-p}}{h(x)} dxdt = \\
&= \int_0^\infty h(f(t)) \int_t^\infty \frac{x^{1-\beta}}{h(x)} x^{\beta-p-1} dxdt \leqslant \\
&\leqslant \int_0^\infty h(f(t)) \frac{t^{1-\beta}}{h(t)} \int_t^\infty x^{\beta-p-1} dxdt = \\
&= \frac{1}{p-\beta} \int_0^\infty \frac{t^{1-p}}{h(t)} h(f(t))dt.
\end{aligned}$$

□

If we set $\beta = 0$ in (19), we obtain the following corollary.

Corollary 6. *Let $p > 1$, $f \geqslant 0$. If $h \geqslant 0$ be a convex submultiplicative function and $h(0) = 0$, then*

$$\int_0^\infty \frac{x^{1-p}}{h^2(x)} h(F(x))dx \leqslant \frac{1}{p} \int_0^\infty \frac{t^{1-p}}{h(t)} h(f(t))dt. \tag{20}$$

This work is supported by University of Tiaret, PRFU projet, COOL03UN140120220001.

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Весовые операторы Харди и квазимонотонные функции

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Аннотация. Некоторые неравенства типа Харди установлены У. Т. Сулейманом. Целью данной работы является распространение этих неравенств на весовые операторы Харди с квазимонотонными функциями. Кроме того, были получены новые интегральные весовые неравенства.

Ключевые слова: неравенства, оператор Харди, квазимонотонные функции.