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Admissible Inference Rules of Temporal Intransitive Logic with the Operator "tomorrow"

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Abstract. We investigates non-transitive temporal logic with the "tomorrow" operator. In this logic, the operator "necessary" $\Box$ coincides with the operator “possible” $\Diamond$ (or almost coincides in reflexive case). In addition to the basic properties of the reflexive non-transitive logic $\mathcal{L}^r$ (decidability, finite approximability), admissible rules of this logic are investigated. The main result consists in proving the structural completeness of this logic and its tabular extensions.

Keywords: modal logic, frame and model Kripke, admissible and globally admissible inference rule.

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Introduction

The concept of a (structural) admissible inference rule was first introduced by Lorenzen [1] in 1955. For arbitrary logic, admissible rules of inference are those that do not change the set of provable theorems of a given logic. Any inferred rule is valid in the given logic; the reverse is not true in the general case.

Directly from the definition we can conclude that the set of all inference rules admissible in logic forms *the largest* class of inference rules with which we can expand the axiomatic system of a given logic without changing the set of provable theorems. In addition, admissible rules significantly strengthen the deductive system of a given logic. It is known that the derived inference rules can replace a certain reduce the proof linearly. Admissible rules that are not inferred by this logic can shorten the proof more significantly.

The beginning of the history of studying admissible rules can be dated back to 1975 since the appearance of H. Friedman's problem [2] on the existence of an algorithmic criterion for the admissibility of rules in the intuitionistic logic Int. In classical logic, the question of admissibility was resolved trivially — only deducible, provable rules are admissible. In the case of non-classical logics, the examples of Harrop, Mintz, and Post showed that there are admissible, but not provable rules of inference. In the mid-70s G. Mintz [3] obtained sufficient conditions for the deductibility of rules of a special form. A positive solution to Friedman's problem about the existence of an algorithm that recognizes the admissibility of inference rules in the intuitionistic logic Int was obtained by V. Rybakov in 1984 [4]. For a wide class of modal and superintuitionistic logics, the criterion for the admissibility of inference rules was later formulated in [5].

Another way of describing all admissible rules of logic goes back to the problem of A. Kuznetsov (1973) about the existence of a finite basis for admissible rules of inference of

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logic Int. Having a basis for admissible rules, all others are derived from it as consequences. The first positive result in the study of bases for admissible inference rules was obtained by A. Tsitkin [6], who found a basis for all quasi-characteristic inference rules admissible in Int.

In general, Kuznetsov's problem on the existence of a finite basis for admissible inference rules was solved negatively not only for Int (Rybakov, [7]), but also for most other basic logics. V. Rybakov (Chapter 4, [5]) showed that the logics $Int, KC, K4, S4, Grz$ and many other basic, individual logics do not have a finite basis for admissible rules from a finite number variables. Therefore, the problem of an explicit description of an easily observable basis for all admissible inference rules, at least for the main basic logics, becomes relevant.

One of the first results in this direction was obtained in 2000: in the paper [8] a recursive basis was constructed for admissible rules of intuitionistic logic Int, consisting of rules in semi-reduced form. Later, R. Iemhoff [9] obtained an explicit basis for the admissible rules of Int logic. In the article [10] V. Rybakov constructed an exact basis for all admissible rules of logic $S4$. This approach was further developed, for example, in [11, 12].

In the case of temporal (multimodal) or intransitive modal logics, relatively few results are known regarding admissible rules and their bases. The previously developed technique makes significant use of the transitivity of the reachability relation. In this work we make an attempt to fill the gap and explore the admissible rules of intransitive temporal logic $\mathcal{L}_0$ with the "tomorrow" operators and its extensions.

1. Definitions, preliminary facts

It is assumed that the reader is familiar with algebraic and Kripke semantics for modal logics, as well as some initial basic information about the rules of inference and their admissibility (although we briefly recall all the necessary facts below).

As a source on the subject as a whole, we can recommend Rybakov [5] among modern literature for a more developed technique for studying modal logics and rules of inference. In accordance with the modern interpretation, by *logic* we understand the set of all theorems that can be proven in a given axiomatic system.

In the definition, by propositional logic we mean algebraic propositional logic (see [5]), although the reader may consider λ to be modal logic, which is sufficient for our purposes. Initial information and all necessary statements used further, can be found for example in [5, ch. 2.2-2.5; 4.1].

Frame $\mathcal{F} := \langle F, R\rangle$ is a pair, where F is a non-empty set and R is a binary relation on F. The basic set and the frame itself will be further denoted by the same letter. A non-empty set $C \subseteq F$ is called a *cluster* if: 1) for any x, y from C, xRy holds; 2) for any $x \in C$ and $y\ inW$, $((xRy \& yRx) \Longrightarrow y \in C)$ is true. A cluster is called *proper* if $|C| > 1$; otherwise *singleton or degenerate*. For an element $a \in F$, let $C(a)$ denote the cluster (i.e., the set of elements mutually comparable with respect to R with a given element a) generated by the element a.

A sequence of elements $\{a_0; a_1; \ldots; a_n\}$ of an intransitive frame is called a chain of length $n+1$ if, for all $i < n$, element a_{i+1} is R-achievable from element a_i and there are no other frame elements between them.

The depth of element x of the model (frame) F is the maximum number of clusters in chains of clusters starting with the cluster $C(x)$ containing x. The set of all elements in the frame (models) F of depth no more than n will be denoted by $S_{\leqslant n}(F)$, and the set of elements of depth n will be denoted by $S_n(F)$.

Inference rule

$$\frac{\alpha_1(x_1, \ldots, x_n), \ldots, \alpha_k(x_1, \ldots, x_n)}{\beta(x_1, \ldots, x_n)}$$

is called admissible in logic λ if for any formulas $\delta_1, \ldots, \delta_n$ from $(\forall j \;\; \alpha_j(\delta_1, \ldots, \delta_n) \in \lambda)$ it follows $\beta(\delta_1, \ldots, \delta_n) \in \lambda)$.

The rule r is called a consequence of the rules $R := \{r_1; \ldots ; r_k\}$ in logic λ (notation $\mathcal{R} \vdash_\lambda r$), if the conclusion r is deducible from the premises r using theorems, rules $\{r_1; \ldots ; r_k\}$ and postulated rules of inference of λ. Inference rule $r = \{\alpha_1, \ldots, \alpha_k/\beta\}$ is true on the algebra $\mathfrak{A} \in Var(\lambda)$ if and only if for any value of the variables r on $\mathfrak{A}$ as soon as $\forall\; j\; \mathfrak{A} \models_V \alpha_j$, then $\mathfrak{A} \models_V \beta$. Rule r is called a semantic consequence of the system of rules $\mathcal{R}$ in logic λ (notation $\mathcal{R} \models r$), if for any algebra $\mathfrak{A} \in Var(\lambda)$, as soon as all rules from $\mathcal{R}$ and all postulated rules of logic are true on the algebra $\mathfrak{A}$, then the rule r is also true on $\mathfrak{A}$. We say that modal logic is structurally complete if any admissible in the inference rule is deducible in .

Theorem 1.1 (Th. 1.4.11 [5]). *Let a set of inference rules $\mathcal{R} \cup \{r\}$ be given in the language algebraic logic λ. Then $\mathcal{R} \vdash r \iff \mathcal{R} \models r$. In particular, if $\mathcal{R} \not\models r$, then there is an algebra $\mathfrak{A} \in Var(\lambda)$, on which all the rules from $\mathcal{R}$ and all postulated rules of logic λ are true, but $\mathfrak{A} \not\models r$.*

For further presentation we will need n-characteristic Kripke models, with the help of which we will describe free algebras of finite ranks from the variety $Var(\lambda)$. Kripke model $\langle F, R, V\rangle$, where $V : \{p_1, p_2, \ldots, p_n\} \to 2^F$, is called n-characteristic for logic λ if and only if for any formula in variables $p_1, \ldots, p_n$, $\alpha \in \lambda \iff \langle F, R, V\rangle \models \alpha$.

Theorem 1.2 ([5]). *For any finitely approximable modal logic λ, the inference rule r is admissible in λ if and only if r is true on the frame $C_n(\lambda)$ for any n and for any formulaic valuation of the variables r.*

2. Логики $\mathcal{L}_0$ и $\mathcal{L}^r$

In the article [13] a temporal nontransitive logic $\mathcal{L}_0$ with the "tomorrow" operator was introduced. Let $\mathcal{L}_0 = L(F_\infty)$, where frame $F_\infty = \langle N, R\rangle$, N – set of natural numbers; and the relation R is defined as follows: $mRn \iff n = m+1$. This logic is convenient in that from each element of the $\mathcal{L}_0$-frame only one element is reachable with respect to the relation R, i.e. in this logic the operator $\Box$ coincides with the operator $\Diamond$. We also introduce the frame class F_n. Let's define $F_n = \langle\{1, \ldots, n\}, R\rangle$, $n \in N, n > 0$, где $\forall i \in \{1, \ldots, n-1\}(iRj \iff j = i + 1) \wedge (nRn)$

We also define a nontransitive reflexive temporal logic $\mathcal{L}^r$. Let $\mathcal{L}^r = L(F^r_\infty)$, where $F^r_\infty = \langle N, R\rangle$, N is the set of natural numbers, and the relation R is defined as follows: $mRn \iff n = m+1 \vee n = m$. In this logic, from each element of the $\mathcal{L}^r$-frame, only one element different from the given one is reachable. We also introduce the class of frames $F^r_n = \langle\{1, \ldots, n\}, R\rangle$, $n \in N, n > 0$, where $\forall i, j \in \{1, \ldots, n-1\}(iRj \iff j = i+1 \vee i = j) \wedge (nRn)$. Let us define tabular extensions of logics $\mathcal{L}_0$ and $\mathcal{L}^r$. Let $\mathcal{L}_0$ и $\mathcal{L}^r$. Let $\mathcal{L}_n = L(F_n)$, where $F_n = \langle\{1, \ldots, n\}, R\rangle$. Analogically, $\mathcal{L}^r_n = L(F^r_n)$, where $F^r_n = \langle\{1, \ldots, n\}, R\rangle$.

Let α be a modal formula. The modal degree $deg(\alpha)$ of the formula α is determined as follows: $deg(p) = deg(\top) = deg(\bot) = 0$, $deg(\alpha \wedge \beta) = deg(\alpha \to \beta) = deg(\alpha \vee \beta) = \max\{deg(\alpha), deg(\beta)\}$, $deg(\neg\alpha) = deg(\alpha)$, $deg(\Box\alpha) = deg(\Diamond\alpha) = deg(\alpha) + 1$.

It is easy to prove the following statement by induction on the length of the formula:

Theorem 2.1 ([13]). *Let $deg(\alpha) = n$. Then the truth of the formula α on element x of frame F_∞ (F^r_∞) is uniquely determined by the values of all propositional variables included in the formula on elements x; $x + 1$; $x + n$ of frame F_∞ (F^r_∞).*

This implies :

Theorem 2.2 ([13]). *The frame class $\{F_n | n \in N\}[\{F^r_n | n \in N\}]$ is characteristic of logic $\mathcal{L}_0$ [$\mathcal{L}^r$]. In particular, if $deg(\alpha) = n$, $n > 0$, and $F_\infty \not\models \alpha$, then $F_{n+1} \not\models \alpha$ [similarly if $deg(\alpha) = n$, $n > 0$, and $F^r_\infty \not\models \alpha$, then $F^r_{n+1} \not\models \alpha$].*

From the received statement 2.2 should also

Theorem 2.3 ([13]). *Logics $\mathcal{L}_0$ and $\mathcal{L}^r$ are finitely approximable and decidable.*

Let us construct an n-characteristic model $Ch_{\mathcal{L}_0}(n)$ by slices as follows. The first slice of the model consists of 2^n reflective elements, on which all possible valuations of propositional variables $p_1, p_2, \ldots, p_n$. To construct the second slice of this model, on each element of the first slice c_1 we hang from below $2^n - 1$ irreflexive elements with all sorts of different valuations of the propositional variables $p_1, p_2, \ldots, p_n$, different from the valuation of element c_1. We will construct the third slice of this model as follows. To each element of the second slice we assign from below 2^n irreflexive elements with all possible different valuations of the variables $p_1, p_2, \ldots, p_n$. We build all subsequent slices similarly to the third layer. Continuing the described process, as a result of construction we obtain the model $Ch_{\mathcal{L}_0}(n)$.

The n-characteristic model $Ch_{\mathcal{L}_m}(n)$ for tabular logic $\mathcal{L}_m$ is constructed in a similar way, with the only difference that the construction process continues until step (depth) m and ends at this step m. The model $Ch_{\mathcal{L}^r}(n)$ of reflexive logic is constructed in a similar way, with the only difference that at each construction step for each element c_1 we hang from below $2^n - 1$ reflexive elements with all possible different values of propositional variables $p_1, p_2, \ldots, p_n$, different from the valuation of element c_1.

Note that the frame generated by an arbitrary element of a given n-characteristic model is isomorphic to the frame F_k for some k. In the tabular case, the n-characteristic model $Ch_{\mathcal{L}_m}(n)$ is the p-morphic image of the direct union of a sufficient number of frames F_m.

Theorem 2.4. *The model $Ch_{\mathcal{L}_0}(n)$ ($Ch_{\mathcal{L}_m}(n)$, $Ch_{\mathcal{L}^r}(n)$) is n-characteristic for the logic $\mathcal{L}_0$ ($\mathcal{L}_m$, $\mathcal{L}^r$) respectively.*

Proof. In all three cases, the statement is proved in a similar way, so we will prove it only for logic $\mathcal{L}_0$. Let the formula α depend on n propositional variables. By construction, the frame of the model $\mathcal{L}_0$ is an $\mathcal{L}_0$-frame. This means that if $\alpha \in \mathcal{L}_0$, then it is true for all elements of this model.

If formula $\alpha \notin \mathcal{L}_0$, then due to the finite approximability of logic, there is a finite $\mathcal{L}_0$-frame F_m such that $F_m \not\models_V \alpha$ for some valuation V of the variables of the formula. Let's consider all possible cases:

1) All elements of F_m have different valuations of variables, i.e. an arbitrary element j and its predecessor $j-1$ are designated differently. In this case, the model is $\langle F_m, V\rangle$is an open submodel of the model $Ch_{\mathcal{L}_0}(n)$ (by construction of the latter). Therefore, $Ch_{\mathcal{L}_0}(n) \not\models_V \alpha$.

2) The reflexive element m and its R-predecessor $(m-1)$ have the same valuation for the variables of the formula α. In this case, if the elements $m,\ (m-1),\ (m-2),\ \ldots, (m-k),\ k \leqslant m$ have the same variable valuation, then we glue them slice by slice with the element m. In the resulting p-morphic image of the model, the reflexive element of the first slice and its predecessor are designated differently, and therefore is an open submodel of the n-characteristic model, i.e. $Ch_{\mathcal{L}_0}(n) \not\models_V \alpha$ is true.

For the model $Ch_{\mathcal{L}_m}(n)$ or $Ch_{\mathcal{L}^r}(n)$ the proof is similar. The statement has been proven. □

Since in all cases the various elements of the first slice of the n-characteristic model do not have a common R-predecessor, this model is a direct union of component $\mathcal{M}_i, i \leqslant 2^n$, i.e. $Ch_{\mathcal{L}}(n) = \sqcup \mathcal{M}_i,\ \mathcal{L} \in \{\mathcal{L}_0,\ \mathcal{L}_m,\ \mathcal{L}^r\}$. Each component $\mathcal{M}_i$ has the following structure: the first slice consists of a single reflective element. The second slice consists of $2^n - 1$ irreflexive (reflexive in the case of logic $Ch_{\mathcal{L}^r}(n)$) elements, the valuation of which is different from the valuation of the variables on the element of the first slice. Each element of the second and all subsequent slices has 2^n irreflexive ($2^n - 1$ reflexive in the case of logic $Ch_{\mathcal{L}^r}(n)$) immediate R-predecessor, etc. It is easy to show that in the tabular case the model $Ch_{\mathcal{L}_m}(n)$ is a p-morphic image of a finite direct union combining F_m frames. Accordingly, any $\mathcal{L}$-frame, where $\mathcal{L} \in \{\mathcal{L}_0,\ \mathcal{L}_m,\ \mathcal{L}^r\}$, is also a direct union of the components $\mathcal{M}_i$.

Lemma 2.5. *Each element of the n-characteristic model $Ch_{\mathcal{L}^r}(n)$ is formulaic.*

Proof. Let the model $Ch_{\mathcal{L}^r}(n)$ have valuation V variables $p_1, \dots, p_n$. According to the construction of this model, any two different elements $i, j \in Ch_{\mathcal{L}^r}(n) \ : \ iRj$ have different valuations of the variables. For each $x \in Ch_{\mathcal{L}^r}(n)$ of arbitrary depth m, we define the formulas:

$$\alpha(x) := \bigwedge\{p_j \mid x \models_V p_j\} \wedge \bigwedge\{\neg p_i \mid x \not\models_V p_i\};$$

$$f(x) := \alpha(x) \wedge \Diamond\alpha(x) \wedge \Diamond\alpha(x+1) \wedge \Diamond^2\alpha(x+2) \wedge \dots \wedge \Diamond^{m-1}\alpha(m) \wedge \Diamond^{m-1}\Box\alpha(m).$$

It is easy to see that $x \models_V f(x)$. Let us assume that the formula $f(x)$ is true on an element $i \in Ch_{\mathcal{L}^r}(n)$ under valuation V, other than x, and consider all possible cases of the location of this element.

1) If $x < i$ (i is located above x), then after $m - i < m - 1$ steps in relation R, stabilization occurs: $\alpha(m-i) = \alpha(m-i+1) = \dots = \alpha(m-1) = \alpha(m)$, which impossible, because according to the construction of the model, $\alpha(m-1) \neq \alpha(m)$ should be fulfilled.

2) Let now $x > i$ (i is located below x). Then, after $m - 1$ steps in relation R, stabilization should occur. Due to $i \models_V \Diamond^{m-1}\alpha(m) \wedge \Diamond^{m-1}\Box\alpha(m)$, elements reachable from element i in $m-1$ and m steps in relation R must have the same valuation. But this is not possible according to the construction of the model $Ch_{\mathcal{L}^r}(n)$.

3) If $x = i$, then after m steps with respect to R from both elements the same final element m is reachable by R. Consequently, elements x and i belong to the same component $\mathcal{M}_i$ of the n-characteristic model. Reasoning in a similar way, we find that all elements that are reachable from x and i are designated identically, i.e. according to the construction of the model, these elements $Ch_{\mathcal{L}^r}(n)$ elements coincide. □

This implies :

Lemma 2.6. *Each element of the n-characteristic model $Ch_{\mathcal{L}_m}(n)(Ch_{\mathcal{L}^r_m}(n))$ is formulaic.*

3. About structural completeness

Theorem 3.1. *Any finitely generated algebra generated by some $\mathcal{L}^r$-frame belongs to the quasivariety $\mathcal{F}^Q_w(\mathcal{L}^r)$. In particular, the variety $Var(\mathcal{L}^r)$ and the quasivariety $\mathcal{F}^Q_w(\mathcal{L}^r)$ coincide.*

Proof. Let $\mathcal{A} = G^+$ be a finitely generated $\mathcal{L}^r$-algebra. Hence, $\mathcal{A} \in HS\prod \mathcal{F}^Q_w(\mathcal{L}^r)$, i.e. this algebra is a homomorphic image of a subalgebra of the direct product of a certain number of free algebras of countable rank from the variety $Var(\mathcal{L}^r)$. Due to the local finiteness of the logic $\mathcal{L}^r$ (which is easy to verify), the algebra $\mathcal{A}$ is finite and generated by a certain finite $\mathcal{L}^r$-frame G. Consequently, this frame is an open subframe of the p-morphic image of the direct union of frames of the w-characteristic model $Ch_{\mathcal{L}^r}(w) = \sqcup\mathcal{M}_i$. Since the frame G is finite, we can take a direct union of a finite number of frames of the k-characteristic model $Ch_{\mathcal{L}^r}(k) = \sqcup\mathcal{M}_i$ for some suitable k.

As previously noted, any finite $\mathcal{L}^r$-frame is a direct union of the components $\mathcal{G}_j$. Therefore, the frame $G = \sqcup\,\mathcal{G}_j$ is an open subframe of the n-characteristic model frame $Ch_{\mathcal{L}^r}(n) = \sqcup\mathcal{M}_i$ for some suitable n. In particular, for all j we can assume without loss of generality that $\mathcal{G}_j \sqsubseteq \mathcal{M}_j$. Let us define a p-morphism g of a component $\mathcal{M}_j$ onto $\mathcal{G}_j$ for an arbitrary j as follows.

(1) for all elements of components $\mathcal{G}_j$ ($\mathcal{G}_j \sqsubset \mathcal{M}_j$), we define a p-morphism g as identical, i.e. $\forall\ x \in \mathcal{G}_j\ g(x) := x$. In particular, for the element $x_0 \in S_1(\mathcal{G}_j)$ we define $g(x_0) := x_0$.

(2) Let us now define g by slices on the entire component $\mathcal{M}_i \sqsubset Ch_{\mathcal{L}^r}(n)$ as follows. Let the p-morphism not yet be defined on the elements $y_1, \dots, y_k \in S_2(\mathcal{M}_j)$. Let's choose an arbitrary element $x_1 \in S_2(\mathcal{G}_j) \sqsubseteq S_2(\mathcal{M}_j)$. By (1) on such an element the p-morphism is already defined as

identical. Then we define $g(y_i) := x_1,\ 1 \leqslant i \leqslant k$. Thus, we define a p-morphism on the entire second slice $S_2(\mathcal{M}_j)$, preserving the depth of the elements.

Now let the p-morphism on the element $y \in S_3(\mathcal{M}_j)$ not yet be defined and let $yRz \& y \neq z$, where $z \in S_2(\mathcal{M}_j))$. The image $g(z) = e \in S_2(\mathcal{G}_j)$ is already defined. If e is R-maximal in $\mathcal{G}_j$, i.e. is not reachable from elements of a strictly greater depth in $\mathcal{G}_j$, then for all elements $\{t | \exists m \in N\ tR^m y\}$ we define $g(t) := e$ (i.e. the entire lower cone of the element y is p-morphically compressed into the element e).

If a given element e has immediate R-predecessors $\{e_1, \ldots, e_k\}$ in the component $\mathcal{G}_j$, then element y is compressible with one of the elements $e_i, i \leqslant k$. For definiteness, we put $g(y) := e_1$. With this additional definition of p-morphism, the depth of the element is preserved.

By force of the arbitrariness of the choice of element y, we extend the p-morphism on the entire third layer of the component $S_3(\mathcal{M}_j)$. For elements of depth 4 and all subsequent layers of the component $\mathcal{M}_j$ we define a p-morphism in exactly the same way as above. Thus, as a result, the p-morphism g will be defined on the entire component $\mathcal{M}_j$ and $g(\mathcal{M}_j) = \mathcal{G}_j$.

(3) For all $\mathcal{M}_j \sqsubset Ch_{\mathcal{L}^r}(n) \setminus G$ we define $g(\mathcal{M}_j) := x_0$, where x_0 some fixed element of the first slice of an arbitrary component $\mathcal{G}_j \sqsubset G$.

Again, from the arbitrariness of the choice of j, we conclude that the required p-morphism g is defined on the entire frame of the n-characteristic model $Ch_{\mathcal{L}^r}(n) = \sqcup \mathcal{M}_i$. By Theorem 3.3.8 [5], the algebra generated by an arbitrary $\mathcal{L}^r$-frame G is a subalgebra of the free algebra $\mathcal{F}_q$ for some q, and therefore belongs to the quasivariety $\mathcal{F}_w^Q(\mathcal{L}^r)$.

A similar statement is also true for tabular irreflexive logic. It is easy to show in a similar way that for an arbitrary $\mathcal{L}_m$-frame there is a p-morphism from the frame of the n-characteristic model $Ch_{\mathcal{L}_m}(n)$ for a given frame. Taking into account the formulaicity and finiteness of this model, the following theorem is valid:

Theorem 3.2. *The algebra generated by an arbitrary finite $\mathcal{L}_m$-frame belongs to the quasivariety $\mathfrak{F}_w^Q(\mathcal{L}_m)$. In particular, the variety $Var(\mathcal{L}_m)$ and the quasivariety $\mathfrak{F}_w^Q(\mathcal{L}_m)$ coincide.*

Next, suppose that the rule $r := \alpha_1, \ldots, \alpha_n/\beta$ not derivable in logic $\mathcal{L}_m$. Then, by Theorem 1.4.11 [5], this rule will be refuted on some finite $\mathcal{L}_m$-algebra A. It follows from the theorem that the rule will also be refuted on a free algebra of countable rank $\mathfrak{F}_w(\mathcal{L}_m)$. Therefore, rule r is not admissible in logic $\mathcal{L}_m$. Because any derived rule is also admissible, then the statement is proven:

Theorem 3.3. *The inference rule r is admissible in the logic $\mathcal{L}_m \iff$ this rule is derivable in the logic $\mathcal{L}_m$. In particular, the logic $\mathcal{L}_m$ is structurally complete.*

By virtue of Theorem 3.1, we can prove in exactly the same way

Theorem 3.4. *The inference rule r is admissible in the logic $\mathcal{L}_0^r \iff$ this rule is derivable in the logic $\mathcal{L}_0^r$. In particular, the logic $\mathcal{L}_0^r$ is structurally complete.*

Note that due to $\mathcal{L}_0 \subseteq \mathcal{L}_m$, for all natural numbers $n; m$ is executed $Ch_{\mathcal{L}_m}(n) \sqsubseteq Ch_{\mathcal{L}_0}(n)$. Directly from the definition of logics we conclude $\mathcal{L}_0 = \bigcap \mathcal{L}_m$. This implies:

Theorem 3.5. *If for all natural numbers m the inference rule r is admissible in the logic $\mathcal{L}_m$, then the rule r is admissible in the logic $\mathcal{L}_0$.*

Proof. Let the inference rule $r := \{\alpha_1, \ldots, \alpha_n/\beta\}$ is not admissible in logic $\mathcal{L}_0$. Let us show that this rule r is not admissible in some tabular logic $\mathcal{L}_k$. In this case, the rule r is refuted at some formulaic valuation V on the frame of the n-characteristic model $Ch_{\mathcal{L}_0}(n)$ for some n: the premise of the rule is true on $Ch_{\mathcal{L}_0}(n)$, but there is an element $b \in Ch_{\mathcal{L}_0}(n)$, on which the conclusion of the rule is refuted with a given valuation:

$$Ch_{\mathcal{L}_0}(n) \models_V \alpha_i,\ 1 \leqslant i \leqslant n; \quad b \not\models_V \beta.$$

Without loss of generality, we can assume that a given element b is R-maximal among all such elements on which the conclusion of the rule is refuted and has depth k. Then frame

$$b^{\leqslant R} := \{x \mid \exists\, l\ : bRx_l Rx_{l-1} \dots Rx\},$$

generated by element b is isomorphic to frame F_k, where k is the depth of element b. Therefore it is fulfilled

$$\langle F_k,\ V\rangle \models_V \alpha_i,\ 1 \leqslant i \leqslant n; \quad b \not\models_V \beta.$$

Let us show that the rule r will be inadmissible in the logic $\mathcal{L}_k = L(F_k)$. Indeed, as noted earlier, the frame of the k-characteristic model $Ch_{\mathcal{L}_k}(n)$ is the p-morphic image of the direct union of a sufficient (finite!) number of isomorphic copies of the frame F_k. Consequently, transferring the valuation V from the model $\langle F_k,\ V\rangle$ with the help of this p-morphism to the model $Ch_{\mathcal{L}_k}(n)$ we obtain $Ch_{\mathcal{L}_k}(n) \models_V \alpha_i,\ 1 \leqslant i \leqslant n; \quad \exists\, b \in Ch_{\mathcal{L}_k}(n)\ :\ b \not\models_V \beta$. Therefore, the inference rule $r := \{\alpha_1,\ \dots,\ \alpha_n/\beta\}$ is not admissible in the logic $\mathcal{L}_k$. □

Theorem 3.6 ([14]). *Logic $\mathcal{L}_0$ is not structurally complete.*

Proof. Let's define the inference rule

$$\mathcal{R} = \frac{(p \wedge \Box\neg p) \vee (\neg p \wedge \Box p)}{\perp}.$$

Recall that the first skice of the n-characteristic model contains only reflexive elements. On these elements, for any value of the variable p, the premise of the rule R is not satisfied, which entails its admissibility in logic $\mathcal{L}_0$.

Let us now assume that this rule is derivable in $\mathcal{L}_0$. Then, by the deduction theorem, the following is true for modal logics:

$$\exists n_1, \dots, n_k\ \vdash_{\mathcal{L}_0} \Box^{n_1}((p \wedge \Box\neg p) \vee (\neg p \wedge \Box p)) \wedge \dots \wedge \Box^{n_k}((p \wedge \Box\neg p) \vee (\neg p \wedge \Box p)) \to \perp.$$

At the same time, with the valuation $V(p) = \{2k \mid k \in N\}$, the premise of the rule is true on frame $\mathcal{F}_\infty$, and the conclusion is false, which entails the falsity of this formula, and therefore the rule R on frame $\mathcal{F}_\infty$. Therefore, rule R is not derivable in logic $\mathcal{L}_0$. □

Thus, the question arises about the existence of a finite basis for the admissible rules of logic $\mathcal{L}_0$ and its description, if it exists. Note that if an arbitrary $\mathcal{L}_0$-frame does not contain infinitely increasing chains of elements and R-maximal elements of finite depth (i.e. elements of finite depth that are not reachable with respect to R from other elements), then similarly to how it was done earlier, we can prove that the associated algebra belongs to the quasivariety $\mathfrak{F}^Q_w(\mathcal{L}_0)$. Consequently, as soon as on some $\mathcal{L}_0$-frame a rule admissible in the logic $\mathcal{L}_0$ is refuted, then a given frame contains an infinitely increasing chain of elements, or a chain of irreflexive elements of finite length. It is clear that in the first case on this frame Rule R is refuted.

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Допустимые правила временной нетранзитивной логики с оператором "завтра"

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Аннотация. В статье исследуется нетранзитивная временная логика с оператором "завтра". В этой логике оператор "необходимо" $\Box$ совпадает с оператором "возможно" $\Diamond$ (или почти совпадает в рефлексивном случае). Помимо базовых свойств рефлексивной нетранзитивной логики $\mathcal{L}^r$ (разрешимость, финитная аппроксимируемость) исследуются допустимые правила этой логики. Основной результат состоит в доказательстве структурной полноты данной логики и ее табличных расширений.

Ключевые слова: модальная логика, фрейм и модель Крипке, допустимое правило вывода, глобально допустимые правила вывода.