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On the Bipolar Classification of Endomorphisms of a Groupoid

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Abstract. In this paper, a method is obtained for calculating the bipolar type of endomorphism of an arbitrary groupoid. For groupoids with pairwise distinct left translations of elements, the described method for calculating the bipolar type of an endomorphism leads to a criterion for the fixed point of a given endomorphism. In particular, such groupoids include groupoids with a right neutral element, monoids, loops and groups. It turned out that the bipolar type of endomorphisms of a groupoid with pairwise distinct left translations ones contains all the information about the fixed points of endomorphisms of this type. A basic set of endomorphisms of a group is established, containing all regular automorphisms. A method is found for calculating the bipolar type of an inner automorphism of a monoid. We obtain upper bounds for the order of the monoid of all endomorphisms (and the group of all automorphisms) of an algebraic system with finite support that has a binary algebraic operation.

Keywords: groupoid, groupoid endomorphism, groupoid automorphism, bipolar type of endomorphism of groupoid, bipolar type of regular automorphism, bipolar type of inner automorphism, conservative estimates.

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Introduction

The work is devoted to the study of the properties of a groupoid that follow directly from the bipolar classification of endomorphisms of an arbitrary groupoid, introduced in [1] (the bipolar classification of antiendomorphisms is considered in [2]). The introduction of this classification of endomorphisms was due to the interest of various researchers in the following general problems.

Problem 1. *For a fixed groupoid G, list the elements of the monoid of all endomorphisms of this groupoid.*

Problem 2. *For a fixed groupoid G, list the elements describing the group of all automorphisms of this groupoid.*

Problem 3. *For a fixed groupoid G, give qualitative properties of all endomorphisms and all automorphisms.*

An *element-by-element description of endomorphisms* is a description of the elements of the monoid of all endomorphisms (similarly to the group of automorphisms) as transformations of the support set of the groupoid. The papers aimed at solving problems 1 and 2 include the papers [3–6], in which endomorphisms of matrix semigroups of a special form are studied; papers [7,8], which describe the group of all automorphisms of unipotent subgroups of Chevalley groups (see also works [9–13]). In [14], automorphisms of finitely presented quasigroups are studied.

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Endomorphisms of commutative (but generally not associative) finite groupoids associated with a multilayer neural network of forward signal propagation were studied in [15, 16]. Problem 1 remains open for some groupoids from [15] (see Problem 2 from [15]). Thus, Problems 1 and 2 are studied for various groupoids, semigroups, quasigroups, and groups.

By a *qualitative description of endomorphisms* we mean finding the properties that endomorphisms (or automorphisms) of various groupoids have. The results of such studies may be useful for solving Problems 1 and 2, or be of independent value. Examples of such studies can be found in [17, 18]. There are many examples of studies closely related to endomorphisms (in particular, automorphisms) of various groupoids. For example, the investigations [19, 20] in which finite groupoids are classified that have automorphism groups of a special form. Automorphism groups of finite groupoids are studied in [21]. Endomorphism semigroups for some semigroups of a special kind are studied in [22]. The concept of endomorphism (the ring of endomorphisms of an Abelian group) is intensively used in the study of Abelian groups. This extensive direction is presented, for instance, in [23].

Main results of the paper include the Theorem 2.1. According to it, for any groupoid G, any element $g \in G$ and any endomorphism ϕ of the groupoid G the following equivalences hold:

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) \in M_g, \quad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \in G \setminus M_g,$$

where $M_g := \{m \in G \mid h_m = h_g\}$. This theorem leads to a method for calculating the bipolar type of a fixed endomorphism of an arbitrary groupoid.

An important consequence of Theorem 2.1 is Theorem 2.2. This theorem for a groupoid G with different left translations gives a criterion that $g \in G$ is a fixed point of an endomorphism ϕ of the groupoid G. For any $g \in G$ and $\phi \in \mathrm{End}(G)$ the equality $\Gamma_\phi(g) = 1$ holds iff the equality $\phi(g) = g$ holds. The results of this theorem extend to all groups, monoids, loops and groupoids with a right neutral element. In particular, see Corollary 2.1 for monoids of all endomorphisms of an arbitrary groupoid. On the other hand, the Theorem 2.2 gives a practical way to calculate the bipolar type of a endomorphism of groupoid with pairwise distinct left translations. This method extends to loops, monoids, and groups. The bipolar type of endomorphism of a groupoid with different left translations is completely characterized by all fixed points of this endomorphism.

Theorem 2.3 describes the basic sets of endomorphisms of a monoid, consisting of endomorphisms of a monoid. An endomorphism of a monoid is an endomorphism of a groupoid such that the neutral element of the monoid is a fixed point of this endomorphism. The Theorem 2.4 gives a description of the basic set of endomorphisms of the group, which contains all the regular automorphisms of the group. For groups, such a base set is unique. The Theorem 2.5 gives a way to calculate the bipolar type of a inner automorphism of a monoid.

Theorem 3.1 establishes upper bounds for the order of the group of all automorphisms and the monoid of all endomorphisms of an algebraic system with finite support that has a binary algebraic operation. The resulting estimates are called *conservative estimates* of the order of the monoid of all endomorphisms and the order of the group of all automorphisms. This theorem can be used to study endomorphisms and automorphisms of such algebraic objects as rings and semifields (see [24]). The Theorem 3.1 can be used to investigate the question (**D**) from [25].

1. Basic concepts and definitions

The symmetric semigroup of all transformations of the set G will be denoted by $\mathcal{I}(G)$. The composition of transformations α_1, α_2 from $\mathcal{I}(G)$ will be denoted by $(\cdot)$ and defined by the equality

$$(\alpha_1 \cdot \alpha_2)(g) = \alpha_1(\alpha_2(g)) \quad (g \in G).$$

Endomorphisms and their compositions will be considered in the notation of a symmetric semigroup. For an arbitrary groupoid $G = (G, *)$ the *inner left translation* of an element $x \in G$ will be denoted by h_x (for any $x, y \in G$ the equality $h_x(y) = x * y$ holds).

Below is Definition 1 from [1].

Definition 1.1. By $\mathrm{Bte}(G)$ we denote the set of all possible mappings of the set G into the set $\{1, 2\}$. Mappings from this set will be called *bipolar types of endomorphism* of the groupoid G (or simply *types*). If $\gamma \in \mathrm{Bte}(G)$ and $\gamma(g) = 1$ for any $g \in G$ (similarly, $\gamma(g) = 2$), then the mapping γ will be called *first type* (similarly, *second type*). In this paper, the first type will be denoted by A, and the second type by Ω. If the mapping $\gamma \in \mathrm{Bte}(G)$ is not constant on elements of G, then γ will be called *mixed type*.

The centralizer of the transformation α in the symmetric semigroup $\mathcal{I}(G)$ will be denoted by the symbols

$$C(\alpha) := \{\beta \in I(X) \mid \alpha \cdot \beta = \beta \cdot \alpha\}.$$

Definition 2 of [1] for each $g \in G$ introduces the sets $L^{(1)}(g)$ and $L^{(2)}(g)$ (further, *type-generating sets*). Definition 3 of [1] for any type $\gamma \in \mathrm{Bte}(G)$ introduces the set $D(\gamma)$, which is called the *base set of endomorphisms of type* γ of the groupoid G. We present the sets $L^{(1)}(g)$, $L^{(2)}(g)$ and $D(\gamma)$ below:

$$L^{(1)}(g) := \{\alpha \in C(h_g) \mid h_{\alpha(g)} = h_g\}; \quad L^{(2)}(g) := \{\alpha \in \mathcal{I}(G) \mid h_{\alpha(g)} \neq h_g, \quad \alpha \cdot h_g = h_{\alpha(g)} \cdot \alpha\};$$

$$D(\gamma) := \bigcap_{s \in G} L^{(\gamma(s))}(s); \quad D(A) := \bigcap_{s \in G} L^{(1)}(s), \quad D(\Omega) := \bigcap_{s \in G} L^{(2)}(s).$$

We present Theorem 1 from [1].

Theorem 1.1. *For any groupoid G the equality holds*

$$\mathrm{End}(G) = \bigcup_{\gamma \in \mathrm{Bte}(G)} D(\gamma). \tag{1}$$

Moreover, if τ and ω are two different types from $\mathrm{Bte}(G)$*, then the intersection of $D(\tau)$ and $D(\omega)$ is empty.*

An endomorphism ϕ has bipolar type γ from $\mathrm{Bte}(G)$ if $\phi \in D(\gamma)$ (see Definition 5 from [1]). In particular, the following terminology applies:
1) an endomorphism ϕ has the *first type* if $\phi \in D(A)$;
2) an endomorphism ϕ has *second type* if $\phi \in D(\Omega)$;
3) an endomorphism ϕ is of *mixed type* in other cases.

The above assignment of types to endomorphisms of a groupoid leads to a bipolar classification of endomorphisms.

2. Bipolar types of groupoid endomorphisms

For any element g of the groupoid G we define the set $M_g := \{m \in G \mid h_m = h_g\}$. The bipolar type of the endomorphism ϕ will be denoted by Γ_ϕ. For any endomorphism ϕ of the groupoid G the following alternative holds: either $\Gamma_\phi(g) = 1$ or $\Gamma_\phi(g) = 2$. This is deduced from the definition of bipolar type and Theorem 1.1. There is an equivalence

$$\Gamma_\phi(g) = i \Leftrightarrow \phi \in L^{(i)}(g) \quad (g \in G,\ \phi \in \mathrm{End}(G),\ i \in \{1, 2\}), \tag{2}$$

which establishes a connection between $\Gamma_\phi(g)$ and type-generating sets $L^{(1)}(g)$, $L^{(2)}(g)$. This equivalence follows from Theorem 1.1, the definition of the basic set of endomorphisms, and the definition of the type of endomorphism. Note that for any $g \in G$ the intersection of the sets $L^{(1)}(g)$ and $L^{(2)}(g)$ is empty. That's why for any endomorphism ϕ of the groupoid G the following alternative holds: either $\phi \in L^{(1)}(g)$ or $\phi \in L^{(2)}(g)$.

Theorem 2.1. *Let G be an arbitrary groupoid. Then for an arbitrary $g \in G$ and every endomorphism ϕ of the groupoid G the following equivalences hold:*

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) \in M_g, \quad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \in G \setminus M_g. \tag{3}$$

Proof. Let us show that the first equivalence holds. Let the first condition of the first equivalence be satisfied for some $g \in G$. Then the endomorphism ϕ belongs to the type-generating set $L^{(1)}(g)$. Hence $h_{\phi(g)} = h_g$. Then, by the definition of the set M_g, we obtain that the element $\phi(g)$ belongs to the set M_g.

On the other hand, suppose that the condition $\phi(g) \in M_g$ is satisfied. In this case, we have the equality $h_{\phi(g)} = h_g$. Therefore, ϕ cannot belong to $L^{(2)}(g)$ (due to the definition of type-generating sets). Since ϕ is an endomorphism, then for any $g \in G$ there is an alternative: either $\phi \in L^{(1)}(g)$ or $\phi \in L^{(2)}(g)$. Consequently, the set $L^{(1)}(g)$ contains the endomorphism ϕ, hence we have the equality $\Gamma_\phi(g) = 1$. The first equivalence is shown.

Since for any element $g \in G$ and endomorphism ϕ the alternative hold either $\phi \in L^{(1)}(g)$ or $\phi \in L^{(2)}(g)$ and the first equivalence from (3) is proved, then we get the truth of the second equivalent from (3). The theorem is proved. □

We say that G *is a groupoid with pairwise distinct left translations* if for any $x, y \in G$ the following equivalence holds: $h_x = h_y \Leftrightarrow x = y$.

Theorem 2.2. *Let G be a groupoid with pairwise distinct left translations. Then for an arbitrary $g \in G$ and every endomorphism ϕ of the groupoid G the following equivalences hold:*

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) = g, \quad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \neq g. \tag{4}$$

Proof. If all left translations of the elements of the groupoid G are pairwise distinct, then for any $g \in G$ the equality of the sets $M_g = \{g\}$ holds. Therefore, equivalency (4) follows from (3). The theorem is proved. □

The theorem above covers all groupoids satisfying the monoid, loop, and group axioms. In addition, groupoids with pairwise distinct left translations include groupoids with a right neutral element.

The conjunction will be denoted by the symbol $(\wedge)$. Because the $\mathrm{End}(G)$ is a monoid for any groupoid G, then

Corollary 2.1. *Let G be an arbitrary groupoid and Ψ an arbitrary endomorphism from* $\mathrm{End}(\mathrm{End}(G))$ *then for any ϕ from* $\mathrm{End}(G)$ *the following equivalences hold:*

$$\Gamma_\Psi(\phi) = 1 \Leftrightarrow \Psi(\phi) = \phi, \quad \Gamma_\Psi(\phi) = 2 \Leftrightarrow \Psi(\phi) \neq \phi.$$

Moreover, if G is a groupoid with pairwise distinct left translations, then for any $g \in G$ the following implication holds:

$$(\Gamma_\Psi(\phi) = 1) \wedge (\Gamma_\phi(g) = 1) \Rightarrow [\Psi(\phi)](g) = g.$$

In the last implication, $[\Psi(\phi)](g)$ denotes the image of the element g under the endomorphism $\Psi(\phi)$, which is the image of ϕ under the action of Ψ.

Next, we formulate results on endomorphisms of a monoid, the bipolar type of a regular automorphism of a group, and the inner automorphism of a monoid with invertible elements.

Monoid endomorphisms. Let M be a monoid with a neutral element e (at the same time M is a groupoid satisfying the monoid axioms). Then, in terms of universal algebra, M is an algebra with one binary operation and one nullary operation (the distinguished element is the neutral element). Therefore, an endomorphism of the monoid M is any endomorphism ϕ of the groupoid M such that the identity $\phi(e) = e$ holds. As usual, $\mathrm{End}(M)$ is the monoid of all endomorphisms of the groupoid M and $\mathrm{End}_M(M)$ is the monoid of all endomorphisms of the monoid M.

In the set of all bipolar types $\mathrm{Bte}(M)$ of an arbitrary monoid M, select the set $\mathrm{MBte}(M)$ of bipolar types γ such that $\gamma(e) = 1$.

Theorem 2.3. *If M is a monoid, then the sets are equal:*

$$\mathrm{End}_M(M) = \bigcup_{\gamma \in \mathrm{MBte}(M)} D(\gamma).$$

Proof. Indeed, by the theorem 2.2 the endomorphism ϕ of the groupoid M satisfies the condition $\phi(e) = e$ iff $\Gamma_\phi(e) = 1$. Therefore, the endomorphism ϕ of the monoid M belongs to the base set $D(\gamma)$, where γ belongs to set $\mathrm{MBte}(M)$. The theorem is proved. □

Regular automorphisms of a group. An automorphism ϕ of a group G with a neutral element e will be called a *regular automorphism* if for any $g \in G$ different from e the condition $\phi(g) \neq g$ holds. The set of all regular automorphisms is denoted by $\mathrm{RAut}(G)$ (the identity automorphism is not contained in the constructed set). The automorphism group H of the group G will be called the *group of regular automorphisms* if H consists of regular automorphisms (that is, automorphisms occurring in $\mathrm{RAut}(G)$) and the identity automorphism. Identity automorphism we denote by ε.

In the set of all bipolar types $\mathrm{Bte}(G)$ of an arbitrary group G with neutral element e, we fix a bipolar type Λ such that

$$\Lambda(g) := \begin{cases} 1, & g = e \\ 2, & g \neq e. \end{cases}$$

Theorem 2.4. *Let G be a group. Then the set $\mathrm{RAut}(G)$ is a subset of the base set of endomorphisms $D(\Lambda)$. Moreover, if H is the group of regular automorphisms of the group G, then the inclusion $H \subseteq D(\Lambda) \cup D(A)$ holds.*

Proof. By Theorem 2.2 and the definition of the set $\mathrm{RAut}(G)$, every regular automorphism has a bipolar type γ such that $\gamma(e) = 1$ and $\gamma(g) = 2$ for any $g \in G$ different from e. Therefore $\gamma = \Lambda$, hence the inclusion $\mathrm{RAut}(G) \subseteq D(\Lambda)$ hold. The group of regular automorphisms contains the identity automorphism ε, which by virtue of the Theorem 2.2 belongs to $D(A)$, hence the inclusion $H \subseteq D(\Lambda) \cup D(A)$ holds. The theorem is proved. □

The inclusion of the element ε in the set $D(A)$ was given in Remark 1 of [1].

Inner automorphisms of a monoid. Let $G = (G, *)$ be a monoid with the set of all invertible elements G^*. For each $g \in G^*$, the permutation

$$\phi_g(m) = g^{-1} * m * g \quad (m \in G)$$

we will call the *inner automorphism of the monoid* G. For any element $g \in G^*$, the permutation ϕ_g is an automorphism of the monoid G (the proof of automorphism is similar to the proof for the case of groups). If $G^* = G$, then G is a group and the above definition coincides with the definition of an inner automorphism of a group.

Theorem 2.5. *Let G be a monoid. Then for any $g \in G^*$ and any $d \in G$ the following equivalences hold:*

$$\Gamma_{\phi_g}(d) = 1 \Leftrightarrow g * d = d * g, \quad \Gamma_{\phi_g}(d) = 2 \Leftrightarrow g * d \neq d * g.$$

Proof. Consider the first equivalence. Let $\Gamma_{\phi_g}(d) = 1$. Since G is a monoid, by virtue of the Theorem 2.2 we have the relation $\Gamma_{\phi_g}(d) = 1 \Leftrightarrow \phi_g(d) = d$. In this case, $\phi_g(d) = g^{-1} * d * g$. Since g is an invertible element of the monoid, we obtain that the equality $\Gamma_{\phi_g}(d) = 1$ is equivalent to the condition $g * d = d * g$. The first equivalence is proved. The second equivalence is derived from the truth of the first equivalence and the alternative: either $\Gamma_{\phi_g}(d) = 1$ or $\Gamma_{\phi_g}(d) = 2$. The theorem is proved. □

3. Conservative estimates for the monoid order of all endomorphisms

As usual, $|X|$ is the cardinality of the set X and $S(X)$ is the symmetric permutation group of X. An algebraic system will be denoted by $V = (V, F, P)$, where F is the set of operations of the system and P is the set of relations. The definition of a homomorphism of an algebraic system into an algebraic system of the same type can be found in [26] (see p. 49). The concept of endomorphism of an algebraic system V can be formulated as a homomorphism of an algebraic system V into itself. From the definition of an endomorphism it follows

Proposition 3.1. *Let $V = (V, F, P)$ be an algebraic system and the set F contains the binary algebraic operation $(*)$. Then the inclusions hold*

$$\mathrm{End}(V) \subseteq \mathrm{End}(V_{(*)}), \quad \mathrm{Aut}(V) \subseteq \mathrm{Aut}(V_{(*)}),$$

where $\mathrm{End}(V_{(*)})$ and $\mathrm{Aut}(V_{(*)})$ are the set of all endomorphisms and the set of all automorphisms of the groupoid $V_{(*)} := (V, *)$, respectively.

Theorem 3.1. *Let $V = (V, F, P)$ be an algebraic system with finite support V. If the system of operations F contains a binary algebraic operation $(*)$, then the inequalities hold*

$$|\mathrm{End}(V)| \leqslant \min_{g \in V}(|L^{(1)}_{(*)}(g)| + |L^{(2)}_{(*)}(g)|), \tag{5}$$

$$|\mathrm{Aut}(V)| \leqslant \min_{g \in V}(|L^{(1)}_{(*)}(g) \cap S(V)| + |L^{(2)}_{(*)}(g) \cap S(V)|), \tag{6}$$

where $L^{(1)}_{(*)}(g)$ and $L^{(2)}_{(*)}(g)$ are type-generating sets of the groupoid $V_{(*)} := (V, *)$.

Proof. For each fixed $g \in V$ we introduce the notation

$$J_1(g) := \{\gamma \in \mathrm{Bte}(V) \mid \gamma(g) = 1\}; \quad J_2(g) := \{\gamma \in \mathrm{Bte}(V) \mid \gamma(g) = 2\}.$$

For any $g \in V$, the equality $J_1(g) \cup J_2(g) = \mathrm{Bte}(V)$ and the condition $J_1(g) \cap J_2(g) = \varnothing$ holds.

By virtue of Theorem 1.1, we have the equalities

$$\mathrm{End}(V_{(*)}) = \bigcup_{\gamma \in \mathrm{Bte}(V)} D(\gamma) = \left[\bigcup_{\gamma \in J_1(g)} D(\gamma)\right] \bigcup \left[\bigcup_{\gamma \in J_2(g)} D(\gamma)\right]. \tag{7}$$

Conditions are met

$$L^{(1)}_{(*)}(g) \in \{L^{(\gamma(s))}_{(*)}(s) \mid s \in V,\ \gamma \in J_1(g)\},\quad L^{(2)}_{(*)}(g) \in \{L^{(\gamma(s))}_{(*)}(s) \mid s \in V,\ \gamma \in J_2(g)\}.$$

In this case, the relations hold

$$\bigcup_{\gamma \in J_1(g)} D(\gamma) = \bigcup_{\gamma \in J_1(g)} \bigcap_{s \in V} L^{(\gamma(s))}_{(*)}(s) \subseteq L^{(1)}_{(*)}(g),\quad \bigcup_{\gamma \in J_2(g)} D(\gamma) = \bigcup_{\gamma \in J_2(g)} \bigcap_{s \in V} L^{(\gamma(s))}_{(*)}(s) \subseteq L^{(2)}_{(*)}(g),$$

$$\mathrm{Aut}(V_{(*)}) = \mathrm{End}(V_{(*)}) \cap S(V).$$

These relations, together with the equalities (7), give inclusions

$$\mathrm{End}(V_{(*)}) \subseteq L^{(1)}_{(*)}(g) \cup L^{(2)}_{(*)}(g),\quad \mathrm{Aut}(V_{(*)}) \subseteq (L^{(1)}_{(*)}(g) \cap S(V)) \cup (L^{(2)}_{(*)}(g) \cap S(V)). \tag{8}$$

The type-generating sets $L^{(1)}_{(*)}(s)$ and $L^{(2)}_{(*)}(s)$ have an empty intersection for any $s \in V$ (follows trivially from the definition of these sets). By virtue of the Proposition 3.1, the relations (8) and the arbitrariness of g in the reasoning above, we obtain that for any $g \in G$ the inequalities are satisfied

$$|\mathrm{End}(V)| \leqslant |L^{(1)}_{(*)}(g)| + |L^{(2)}_{(*)}(g)|,\quad |\mathrm{Aut}(V)| \leqslant |L^{(1)}_{(*)}(g) \cap S(V)| + |L^{(2)}_{(*)}(g) \cap S(V)|.$$

Therefore, the inequalities (5) and (6) are satisfied. □

The estimates (5) and (6) will be called *conservative estimates* of the order of the monoid of all endomorphisms and the group of all automorphisms of the algebraic system V. Particular cases of the algebraic system V from the 3.1 theorem are such algebras as *groupoids* ($|F| = 1$, $|P| = 0$), *rings*, *quasifields*, *semifields*, etc. Note that inclusions (8) hold for non-finitary algebraic systems with a binary algebraic operation.

Proposition 3.2. *There are finite groupoids for which the estimates (5) and (6) are achievable.*

Indeed, such groupoids include finite groupoids in which all left translations are pairwise equal (as transformations). Let $G = (G, *)$ be a finite groupoid with pairwise equal left translations. Then, by definition, the type-generating set $L^{(2)}(g)$ is an empty set for any $g \in G$ and the type generating sets $L^{(1)}(g)$ pairwise coincide. Therefore, by virtue of the Theorem 1.1, we obtain the equalities of the sets

$$\mathrm{End}(G) = D(A) = \bigcap_{g \in G} L^{(1)}(g) = L^{(1)}(g^*),\quad \mathrm{Aut}(G) = D(A) \cap S(G) = L^{(1)}(g^*) \cap S(G),$$

which hold for any $g^* \in G$.

Consider examples of constructing estimates (5) for specific finite groupoids. These examples will show that there exist finite groupoids G for which the estimates (5) are better than the *natural upper bound* which is expressed the order of the symmetric semigroup $\mathcal{I}(G)$.

Example 3.1. Let $G = (G, *)$ be a groupoid with support $G = \{1, 2, 3, 4\}$ and multiplication $(*)$ defined by the Cayley table:

$*$	1	2	3	4
1	1	2	3	4
2	1	2	3	4
3	2	2	3	4
4	2	2	3	4

In this case $|\mathcal{I}(G)| = 4^4 = 256$. With any transformation $\alpha \in \mathcal{I}(G)$ we will associate the notation:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix} = (a_1, a_2, a_3, a_4) = (\alpha(1), \alpha(2), \alpha(3), \alpha(4)).$$

Left translations have the form: $h_1 = (1,2,3,4)$, $h_2 = (1,2,3,4)$, $h_3 = (2,2,3,4)$, $h_4 = (2,2,3.4)$. Let us introduce the notation $Est_g =: |L^{(1)}(g)| + |L^{(2)}(g)|$ associated with estimates (5) for the groupoid G. Computer calculations based on the enumeration principle show that the following relations hold: $|\mathrm{End}(G)| = 38$, $Est_1 = 182$, $Est_2 = 182$, $Est_3 = 56$, $Est_4 = 56$.

These relations show that for any $g \in G$ the estimates (5) are better than the natural estimate $|\mathcal{I}(G)|$.

Example 3.2. Consider the cyclic group C_5 of order 5 given by the system of left translations:

$$h_1 = (1,2,3,4,5),\ h_2 = (2,3,4,5,1),\ h_3 = (3,4,5,1,2),\ h_4 = (4,5,1,2,3),\ h_5 = (5,1,2,3,4).$$

In this case $|\mathcal{I}(G)| = 3125$. And computer calculations give the ratios:

$$|\mathrm{End}(G)| = 5,\ Est_1 = 625,\ Est_2 = Est_3 = Est_4 = Est_5 = 5.$$

In this case, the results of conservative estimates (5) are better than in the previous example, except for the estimate $Est_1 = 5^4$.

Example 3.3. Consider the Klein four-group given by the set of left translations:

$$h_1 = (1,2,3,4),\ h_2 = (2,1,4,3),\ h_3 = (3,4,1,2),\ h_4 = (4,3,2,1).$$

In this case $|\mathcal{I}(G)| = 256$, and computer calculations give the ratios:

$$|\mathrm{End}(G)| = 16,\ Est_1 = 64,\ Est_2 = Est_3 = Est_4 = Est_5 = 16.$$

The situation is similar to the previous case.

In the context of the examples above, the following is interesting question.

Question 1. When the inequality (5) turns into equality? What the conditions must a finite groupoid G satisfy for exactness of conservative estimates?

The existence of groupoids from question 1 follows from examples 3.2 and 3.3.

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О биполярной классификации эндоморфизмов группоида

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Аннотация. В работе получен способ вычисления биполярного типа эндоморфизма произвольного группоида. Для группоидов с попарно различными левыми сдвигами элементов (в частности, группоидов с правым нейтральным элементом, моноидов, луп и групп) описанный способ вычисления биполярного типа эндоморфизма приводит к критерию неподвижной точки данного эндоморфизма. Выяснилось, что биполярный тип эндоморфизмов группоида с попарно различными левыми сдвигами содержит всю информацию о неподвижных точках эндоморфизмов этого типа. Установлено базовое множество эндоморфизмов группы, содержащее все регулярные автоморфизмы. Найден способ вычисления биполярного типа внутреннего автоморфизма моноида. Получены верхние оценки порядка моноида всех эндоморфизмов (и группы всех автоморфизмов) алгебраической системы с конечным носителем, которая обладает бинарной алгебраической операцией.

Ключевые слова: группоид, эндоморфизм группоида, автоморфизм группоида, биполярный тип эндоморфизма группоида, биполярный тип регулярного автоморфизма, биполярный тип внутреннего автоморфизма, консервативные оценки.