EDN: VEGZRT
УДК 517.9

Equilibrium Problem for a Kirchhoff-Love Plate Contacting by the Side Edge and the Bottom Boundary

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Received 10.03.2023, received in revised form 15.06.2023, accepted 14.02.2024

Abstract. A new model of a Kirchhoff–Love plate which is in contact with a rigid obstacle of a certain given configuration is proposed in the paper. The plate is in contact either on the side edge or on the bottom surface. A corresponding variational problem is formulated as a minimization problem for an energy functional over a non-convex set of admissible displacements subject to a non-penetration condition. The inequality type non-penetration condition is given as a system of inequalities that describe two cases of possible contacts of the plate and the rigid obstacle. Namely, these two cases correspond to different types of contacts by the plate side edge and by the plate bottom. The solvability of the problem is established. In particular case, when contact zone is known equivalent differential statement is obtained under the assumption of additional regularity for the solution of the variational problem.

Keywords: contact problem, non-penetration condition, non-convex set, variational problem.

Citation: N.P. Lazarev, E.M. Rudoy, D.Ya. Nikiforov, Equilibrium Problem for a Kirchhoff-Love Plate Contacting by the Side Edge and the Bottom Boundary, J. Sib. Fed. Univ. Math. Phys., 2024, 17(3), 355–364. EDN: VEGZRT.

Introduction

Contact problems for solids with inequality type constraints have attracted attention of scientists since 1933s [1–3]. Problems of this kind are associated with the use of boundary conditions that describe non-penetration constraints on the contact surfaces or curves. For this Signorini problem it is assumed that some properties of displacements for points where a solid is in contact with a rigid obstacle [4, 5] or with another deformable body [6–9] are know in advance. It was established with the use of the fictitious domain method that a certain class of contact

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problems are connected with crack problems subject to non-penetration conditions on crack faces [10–12]. Point-wise contact problems were considered [13, 14] where minimization problems over non-convex sets were studied.

In contrast to previous studies (see [4, 11]), a certain initial configuration of Kirchhoff–Love plate and an obstacle with a given geometrical shape are considered. The plate is in contact interaction with a rigid obstacle by its side edge or by its given front surface which is located below with respect to the selected coordinate system. In this case two types of restrictions are imposed. Namely, the first type is described by inequality for deflection functions (vertical displacements). The second type is described by inequalities for deflection functions and horizontal displacements. The main problem is formulated as a minimization of an energy functional over a non-convex set of admissible displacements. The solvability of the non-linear equilibrium problem is established. In particular case, when types of contact zones are known in advance an equivalent differential statement is obtained under the assumption of additional regularity for the solution of the variational problem.

1. The variational problem

Let $\Omega \subset \mathbf{R}^2$ be a bounded with a smooth boundary Γ which consists of two continuous curves $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\mathrm{mes}(\Gamma_0) > 0$. For convenience, it is supposed that

$$\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 = \psi(x_2), \quad x_2 \in [a, b]\},$$

where ψ is a given function, $a < b$, $a, b \in \mathbb{R}$. Let us denote the unit normal vector to Γ by $\nu = (\nu_1, \nu_2)$. For simplicity, suppose that plate has uniform thickness $2h$. Let us assign three-dimensional Cartesian space $\{x_1, x_2, z\}$ with the set $\{\Omega\} \times \{0\} \subset \mathbf{R}^3$ corresponding to the middle plane of the plate.

Let us denote the displacement vector of the mid-surface points ($x \in \Omega$) by $\chi = \chi(x) = (W, w)$, displacements in the plane $\{x_1, x_2\}$ by $W = (w_1, w_2)$ and displacements along the axis z (deflections) by w. The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W)$ and $\sigma_{ij} = \sigma_{ij}(W)$, respectively [5]:

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}\right), \quad \sigma_{ij}(W) = a_{ijkl}\varepsilon_{kl}(W), \quad i, j = 1, 2,$$

where $\{a_{ijkl}\}$ is the given elasticity tensor that is assumed to be symmetric and positive definite:

$$a_{ijkl} = a_{klij} = a_{jikl}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega),$$

$$a_{ijkl}\xi_{ij}\xi_{kl} \geqslant c_0|\xi|^2 \quad \forall \xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const > 0.$$

A summation convention over repeated indices is assumed. Bending moments are [5]

$$m_{ij}(w) = -d_{ijkl}w_{,kl}\,, \quad i, j = 1, 2, \quad \left(w_{,kl} = \frac{\partial^2 w}{\partial x_k \partial x_l}\right)$$

where tensor $\{d_{ijkl}\}$ has the same symmetry, boundedness, and positive definiteness characteristics as tensor $\{a_{ijkl}\}$. Let $B(\cdot\,, \cdot)$ be a bilinear form defined by the equality

$$B(\chi, \bar{\chi}) = \int_\Omega \big\{\sigma_{ij}(W)\,\varepsilon_{ij}(\bar{W}) - m_{ij}(w)\bar{w}_{,ij}\,\big\}dx, \tag{1}$$

where $\chi = (W, w)$, $\bar{\chi} = (\bar{W}, \bar{w})$.

Let us introduce Sobolev spaces

$$H^{1,0}_{\Gamma_0}(\Omega) = \left\{ v \in H^1(\Omega) \;\middle|\; v = 0 \text{ on } \Gamma_0 \right\},$$
$$H^{2,0}_{\Gamma_0}(\Omega) = \left\{ v \in H^2(\Omega) \;\middle|\; v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\},$$
$$H(\Omega) = H^{1,0}_{\Gamma_0}(\Omega)^2 \times H^{2,0}_{\Gamma_0}(\Omega).$$

It is well known that standard expression for potential energy functional of a Kirchhoff–Love plate has the following representation

$$\Pi(\chi) = \frac{1}{2} B(\chi, \chi) - \int_\Omega F\,\chi dx, \qquad \chi = (W, w),$$

where vector $F = (f_1, f_2, f_3) \in L_2(\Omega)^3$ describes the body forces [5]. Note that the following inequality providing coercivity of functional $\Pi(\chi)$

$$B(\chi, \chi) \geqslant c\|\chi\|^2 \quad \forall\, \chi \in H(\Omega), \quad (\|\chi\| = \|\chi\|_{H(\Omega)}) \tag{2}$$

with a constant $c > 0$ that is independent of χ holds for bilinear form $B(\cdot, \cdot)$ [5].

An obstacle is described by the following part of the cylindrical surface

$$\{(x_1, x_2, z) \,|\, (x_1, x_2) \in \Gamma_1, \quad z \in (-\infty, -h]\}.$$

It restricts displacements on the side edge of the plate. Deflections are restricted by the following part of the plane

$$\{(x_1, x_2, z) \,|\, x_1 \leqslant \psi(x_2), \quad x_2 \in [a, b], \quad z = -h\,\}.$$

It is assumed that for the initial state the elastic plate touches a rigid obstacle with a given shape by its side edge corresponding to the points of curve Γ_1 as shown in Fig. 1:

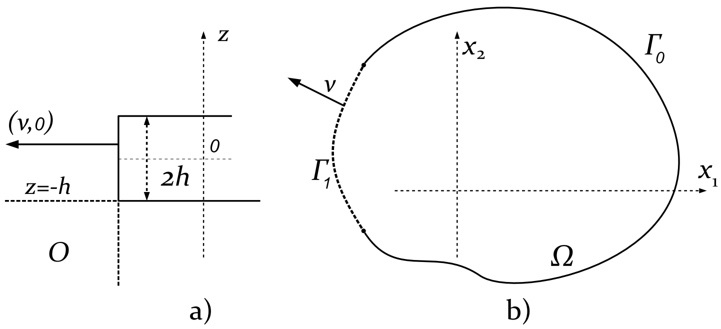

Fig. 1. a) cross section of the plate and the obstacle O; b) midsurface of the plate

At the part Γ_1 of the boundary the condition describing non-penetration of the plate points into the rigid obstacle is considered

$$w \geqslant 0 \quad \text{on} \quad \Gamma_1 \quad \text{or if this is not the case} \quad w \leqslant 0 \quad \text{and} \quad W\nu + h\frac{\partial w}{\partial \nu} \leqslant 0 \quad \text{on} \quad \Gamma_1. \tag{3}$$

It is worth to mention that according to (3) function $\chi = (W, w)$ satisfies either $w \geqslant 0$ on Γ_1 or two last inequalities (3). Now one can introduce the following set of admissible functions

$$K = \{\chi = (W, w) \in H(\Omega) \,|\, \chi \text{ satisfies } (3)\}.$$

Note that K is not convex set since for some $\alpha > 0$ one can construct functions $\tilde{\chi} = (\tilde{W}, \tilde{w}) \in K$ and $\hat{\chi} = (\hat{W}, \hat{w}) \in K$ with properties

$$\tilde{w} < -\alpha, \quad -\frac{\alpha}{2} < \tilde{W}\nu + h\frac{\partial \tilde{w}}{\partial \nu} \leqslant 0, \quad \frac{\alpha}{2} > \hat{w} > 0, \quad \hat{W}\nu + h\frac{\partial \hat{w}}{\partial \nu} > \alpha \quad \text{on} \quad \gamma \subset \Gamma_1, \quad \text{mes}(\gamma) > 0.$$

Obviously, function $\chi_s = \frac{1}{2}(\tilde{\chi} + \hat{\chi})$, $\chi_s = (W_s, w_s)$ does not belong to K because of

$$w_s \leqslant 0 \quad \text{and} \quad W_s\nu + h\frac{\partial w_s}{\partial \nu} > 0 \quad \text{on} \quad \gamma.$$

Let us formulate a variational statement of an equilibrium problem. It is required to find a function $\xi = (U, u) \in K$ such that

$$\Pi(\xi) = \inf_{\chi \in K} \Pi(\chi). \tag{4}$$

Theorem 1.1. *Problem* (4) *has a solution.*

Proof. The existence of a solution of the problem is established in accordance with the Weierstrass theorem [15]. It is well known that energy functional has properties of coercivity and weak lower semicontinuity on $H(\Omega)$ [16]. First, it is proved that set K is weakly closed. Let an arbitrary sequence $\{\chi_n\} \subset K$ be given with the property $\chi_n \to \chi$ in space $H(\Omega)$. By virtue of embedding theorems, this implies that there is a subsequence $\{\chi_n\}$, still denoted in the same way, that converges almost everywhere on Γ to χ. Let us prove that limit function χ also belongs to K. Indeed, there are the following relations for $\chi_n = (W_n, w_n)$

$$w_n \geqslant 0 \quad \text{or} \quad w_n \leqslant 0 \quad \text{and} \quad W_n\nu + h\frac{\partial w_n}{\partial \nu} \leqslant 0 \quad \text{on} \quad \Gamma_1 \backslash B$$

That are satified for each point of $\Gamma_1 \backslash B$, $\text{mes}(B) = 0$ and for all $n \in \mathbb{N}$. Therefore, for every fixed $x \in \Gamma_1 \backslash B$ one can obtain that

$$w_n(x) \geqslant 0 \quad \text{or} \quad w_n(x) \leqslant 0 \quad \text{and} \quad W_n(x)\nu + h\frac{\partial w_n(x)}{\partial \nu} \leqslant 0.$$

There must exists either a subsequence $\{\chi_{n_k}\} \subset \chi_n$ for which

$$w_{n_k}(x) \geqslant 0 \tag{5}$$

or a subsequence $\{\chi_{n_m}\}$ with the following property

$$w_{n_m}(x) \leqslant 0 \quad \text{and} \quad W_{n_m}(x)\nu + h\frac{\partial w_{n_m}(x)}{\partial \nu} \leqslant 0. \tag{6}$$

In both cases one can take the limit in corresponding inequalities, namely, for $\{\chi_{n_k}\}$ in (5), and for $\{\chi_{n_m}\}$ in (6). As a result, the following relations are obtained for limiting function χ

$$w(x) \geqslant 0$$

for the case of subsequence $\{\chi_{n_k}\}$ and

$$w(x) \leqslant 0 \quad \text{and} \quad W(x)\nu + h\frac{\partial w(x)}{\partial \nu} \leqslant 0.$$

for the case of subsequence $\{\chi_{n_m}\}$. Note that if both subsequences $\{\chi_{n_m}\}$ and $\{\chi_{n_k}\}$ with the mentioned properties exist then it means that

$$w(x) = 0 \quad \text{and} \quad W(x)\nu + h\frac{\partial w(x)}{\partial \nu} \leqslant 0.$$

Since point $x \in \Gamma_1 \backslash B$ is arbitrary, condition (3) is fulfilled for limiting function χ. Therefore, set K is weakly closed in $H(\Omega)$. Finally, for problem (4) conditions of the Weierstrass theorem for both functional $\Pi(\chi)$ and set of admissible functions K are satisfied. Then problem (4) has at least one solution. The theorem is proved.

2. Differential statement for the case of known contact zones

In this section the case when types of contact zones are known is considered. Let us assume that curve Γ_1 consists of disjoint curves Γ_1^e and Γ_1^b. Namely, it is supposed that inequalities

$$w \leqslant 0 \quad \text{and} \quad W\nu + h\frac{\partial w}{\partial \nu} \leqslant 0 \quad \text{on} \quad \Gamma_1^e, \tag{7}$$

describing a contact of the plate side edge are fulfilled on Γ_1^e. There is the following condition on the rest part Γ_1^b of curve Γ_1

$$w \geqslant 0 \quad \text{on} \quad \Gamma_1^b \tag{8}$$

which corresponds to a contact of the plate bottom with the rigid obstacle. A new set of admissible functions is introduced as follows

$$K_2 = \{\chi = (W, w) \in H(\Omega) \,|\, \chi \text{ satisfies } (7), (8)\}.$$

One can see that set K_2 is convex and closed. The convexity of set K_2 allow us to represent the following minimization problem

$$\Pi(\xi) = \inf_{\chi \in K_2} \Pi(\chi) \tag{9}$$

as variational inequality [5]

$$\xi \in K_2, \quad B(\xi, \chi - \xi) \geqslant \int_\Omega F(\chi - \xi)dx \quad \forall\, \chi \in K_2. \tag{10}$$

Suppose that solution $\xi = (U, u) \in K$ is sufficiently smooth. Next, let us apply the following Green's formulas for functions $\chi = (W, w) \in K$ [5]

$$\int_\Omega \sigma_{ij}(U)\,\varepsilon_{ij}(W)dx = -\int_\Omega \sigma_{ij,j}(U)w_i\,dx + \int_\Gamma \Big(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau\Big)\,d\Gamma, \tag{11}$$

$$\int_\Omega m_{ij}(u)w_{,ij}\,dx = \int_\Omega m_{ij,ij}(u)\,w\,dx + \int_\Gamma \Big(t^\nu(u)w - m_\nu(u)\frac{\partial w}{\partial \nu}\Big)\,d\Gamma, \tag{12}$$

where

$$\sigma_\nu(U) = \sigma_{ij}(U)\nu_i\nu_j, \quad m_\nu(u) = -m_{ij}\nu_i\nu_j,$$

$$\sigma_\tau(U) = (\sigma_\tau^1(U), \sigma_\tau^2(U)) = (\sigma_{1j}(U)\nu_j, \sigma_{2j}(U)\nu_j) - \sigma_\nu(U)\nu,$$

$$t^\nu(u) = -m_{ij,k}\tau_k\tau_j\nu_i - m_{ij,j}\nu_i, \quad \tau = (-\nu_2, \nu_1),$$

$$W\nu = w_i\nu_i, \quad W\tau = (W_\tau^1, W_\tau^2), \quad w_i = (W\nu)\nu_i + W_\tau^i, \quad i = 1, 2.$$

Along with variational statement (9), one can deal with corresponding differential statement. Namely, the following theorem holds.

Theorem 2.1. *Supposing that solution $\xi = (U, u)$ is sufficiently smooth, variational problem* (9) *is equivalent to the following boundary value problem*

$$-m_{ij,ij}(u) = f_3 \quad in \quad \Omega, \tag{13}$$

$$-\sigma_{ij,j}(U) = f_i \quad in \quad \Omega, \quad i = 1, 2, \tag{14}$$

$$t^\nu(u) \leqslant 0, \quad u \leqslant 0, \quad \sigma_\nu(U) \leqslant 0, \quad \sigma_\nu(U) - \frac{1}{h} m_\nu(u) = 0 \quad on \quad \Gamma_1^e, \tag{15}$$

$$\sigma_\tau(U) = (0, 0), \quad U\nu + h\frac{\partial u}{\partial \nu} \leqslant 0 \quad on \quad \Gamma_1^e, \tag{16}$$

$$\sigma_\tau(U) = (0, 0), \quad \sigma_\nu(U) = m_\nu(u) = 0, \quad t_\nu(u) \geqslant 0, \quad u \geqslant 0 \quad on \quad \Gamma_1^b, \tag{17}$$

$$\sigma_\nu(U)U\nu - t^\nu(u)u + m_\nu(u)\frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_1^e, \quad t^\nu(u)u = 0 \quad on \quad \Gamma_1^b, \tag{18}$$

$$U = u = \frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_0. \tag{19}$$

Proof. Substituting $\bar{\chi} = \xi \pm \tilde{\chi}$, where $\tilde{\chi} \in C_0^\infty(\Omega)^3$, as a test function into (10), one can obtain the following relation

$$\int_\Omega (\sigma_{ij}(U)\, \varepsilon_{ij}(\tilde{W}) - m_{ij}(u)\tilde{w}_{,ij}\,)dx = \int_\Omega F\tilde{\chi}dx$$

, that is, equilibrium equations

$$-m_{ij,ij}(u) = f_3 \quad \text{in} \quad \Omega, \tag{20}$$

$$-\sigma_{ij,j}(U) = f_i \quad \text{in} \quad \Omega, \quad i = 1, 2, \tag{21}$$

hold in terms of distribution.

Applying Green's formulas to (10) and using (20), (21), one can show that

$$\int_\Gamma \Bigg(\sigma_\nu(U)(W - U)\nu + \sigma_\tau(U)(W - U)\tau -$$

$$-t^\nu(u)(w - u) + m_\nu(u)\left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}\right)\Bigg) d\Gamma \geqslant 0 \quad \forall\, \chi = (W, w) \in K. \tag{22}$$

Since K is convex cone in $H(\Omega)$, one can substitute $\chi = \lambda\xi$ in (22) and deduce

$$\int_\Gamma \left(\sigma_\nu(U)U\nu + \sigma_\tau(U)\bar{U}\tau - t^\nu(u)u + m_\nu(u)\frac{\partial u}{\partial \nu} \right) d\Gamma = 0, \tag{23}$$

$$\int_\Gamma \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geqslant 0, \tag{24}$$

for all $\chi = (W, w) \in K$. Let us suppose that $\chi = (W, w) \in K$ and $\chi = 0$ on Γ_1^b. In this case one can rewrite (24) as follows

$$\int_{\Gamma_1^e} \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geqslant 0. \tag{25}$$

Since $W\tau$ is not included in inequalities (7), due to arbitrariness of $W\tau$ on Γ_1^e one can conclude that

$$\sigma_\tau(U) = 0 \quad \text{on} \quad \Gamma_1^e.$$

Therefore, inequality (25) can be reduced to

$$\int_{\Gamma_1^e} \left(\sigma_\nu(U)W\nu - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geqslant 0. \tag{26}$$

By choosing functions $\chi = (W, w)$ such that $W = 0$, $\dfrac{\partial w}{\partial \nu} = 0$ on Γ_1^e, one can obtain in (26) that

$$t^\nu(u) \geqslant 0 \quad \text{on} \quad \Gamma_1^e.$$

Now one can substitute test functions with properties $w = 0$, $W\nu + h\dfrac{\partial w}{\partial \nu} = 0$ and obtain

$$\int_{\Gamma_1^e} \left(\sigma_\nu(U)W\nu - \frac{1}{h} m_\nu(u)W\nu \right) d\Gamma \geqslant 0. \tag{27}$$

Then

$$\sigma_\nu(U) - \frac{1}{h} m_\nu(u) = 0 \quad \text{on} \quad \Gamma_1^e.$$

since the value of $W\nu$ can be arbitrary. The last equality allow us to represent (27) in the form

$$\int_{\Gamma_1^e} \sigma_\nu(U)\Big(W\nu + h\frac{\partial w}{\partial \nu}\Big)\, d\Gamma \geqslant 0,$$

Then, it follows that

$$\sigma_\nu(U) \leqslant 0 \quad \text{on} \quad \Gamma_1^e.$$

Let us assume that $\chi \in K$, $\chi = 0$ on Γ_1^e. Then one can obtain on Γ_1^b that $w \geqslant 0$ and

$$\int_{\Gamma_1^b} \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geqslant 0. \tag{28}$$

Due to arbitrariness of W, $\dfrac{\partial w}{\partial \nu}$ on Γ_1^b there are following equalities

$$\sigma_\tau(U) = (0, 0), \quad \sigma_\nu(U) = m_\nu(u) = 0 \quad \text{on} \quad \Gamma_1^b.$$

Now, it remains to deduce from the reduced inequality

$$-\int_{\Gamma_1^b} t^\nu(u)w\, d\Gamma \geqslant 0 \tag{29}$$

the following inequality

$$t^\nu(u) \leqslant 0 \quad \text{on} \quad \Gamma_1^b.$$

Let us consider relation (23). Let us take into account that $\xi = (U, u) \in K$,

$$\sigma_\nu(U) \leqslant 0, \quad \sigma_\nu(U) - \frac{1}{h} m_\nu(u) = 0, \quad t^\nu(u) \geqslant 0 \quad \text{on} \quad \Gamma_1^e,$$

and

$$\sigma_\tau(U) = (0, 0), \quad \sigma_\nu(U) = m_\nu(u) = 0, \quad t^\nu(u) \leqslant 0 \quad \text{on} \quad \Gamma_1^b.$$

Then corresponding integrand of (23) is non-negative a.e. on Γ. Therefore

$$\sigma_\nu(U)U\nu - t^\nu(u)u + m_\nu(u)\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1^e, \quad t^\nu(u)u = 0 \quad \text{on} \quad \Gamma_1^b.$$

Conversely, in order to obtain variational inequality (10) from (13)–(19) relation (13) is multiplied by $(u-w)$ and relations (14) are multiplied by $(u_i - w_i)$, $i = 1, 2$, where $W = (w_1, w_2)$ and w such that $\chi = (W, w) \in K$. Then after integrating over Ω and summing, one can obtain

$$-\int_\Omega (\sigma_{ij,j}(U)(U-W) + m_{ij,ij}(u)(w-u))dx = \int_\Omega F(\chi - \xi)dx.$$

Now let us use the Green formulas and obtain

$$\begin{aligned}
&\int_\Omega \Big(\sigma_{ij}(U)\,\varepsilon_{ij}(W-U)dx - m_{ij}(u)(w-u)_{,ij}\Big)dx- \\
&-\int_\Gamma \Big(\sigma_\nu(U)(W\nu - U\nu) + \sigma_\tau(U)(W\tau - U\tau)\Big)\,d\Gamma+ \\
&+\int_\Gamma \Big(t^\nu(u)(w-u) - m_\nu(u)(\frac{\partial w}{\partial\nu} - \frac{\partial u}{\partial\nu})\Big)\,d\Gamma = \int_\Omega F(\chi-\xi)dx. \qquad (30)
\end{aligned}$$

Taking into account that $\sigma_\tau(U) = 0$ on Γ_1, $\xi = \chi = 0$ on Γ_0, the sum of integrals over Γ in the left side of (30) can be represented as follows

$$I = \int_{\Gamma_1} \Big(t^\nu(u)(w-u) - m_\nu(u)(\frac{\partial w}{\partial\nu} - \frac{\partial u}{\partial\nu}) - \sigma_\nu(U)(W\nu - U\nu)\Big)\,d\Gamma. \qquad (31)$$

Then bearing in mind the equalities $\sigma_\tau(U) = (0,0)$, $\sigma_\nu(U) = m_\nu(u) = 0$ on Γ_1^b, relation (31) can be represented as the following sum

$$\begin{aligned}
I = &\int_{\Gamma_1^b} t^\nu(u)(w-u)d\Gamma+ \\
&+\int_{\Gamma_1^e} \Big(t^\nu(u)(w-u) - m_\nu(u)(\frac{\partial w}{\partial\nu} - \frac{\partial u}{\partial\nu}) - \sigma_\nu(U)(W\nu - U\nu)\Big)\,d\Gamma. \qquad (32)
\end{aligned}$$

Finally, relation (32) can be transformed into

$$\begin{aligned}
I = &\int_{\Gamma_1^b} t^\nu(u)(w-u)d\Gamma + \int_{\Gamma_1^e} \Big(t^\nu(u)w - \sigma_\nu(U)\big(W\nu + h\frac{\partial w}{\partial\nu})\Big)\,d\Gamma- \\
&-\int_{\Gamma_1^e} \Big(t^\nu(u)u - \sigma_\nu(U)U\nu - m_\nu(u)\frac{\partial u}{\partial\nu}\Big)\,d\Gamma.
\end{aligned}$$

Taking into account relations (15)–(18) and $\chi \in K$, one can see that each term in the last sum is non-positive. It remains to note that since $I \leqslant 0$ equality (30) yields variational inequality (10). □

The results of Section 1 were obtained with the support of the Ministry of Science and Higher Education of the Russian Federation (Grant no. FSRG-2023-0025). The results of Section 2 were obtained by D.Y.N. (the third author) with the support of the Ministry of Science and Higher Education of the Russian Federation (supplementary agreement no. 075-02-2023-947, 16 February 2023.

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Задача о равновесии пластины Кирхгофа-Лява, контактирующей боковой кромкой и лицевой поверхностью

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Аннотация. Предложена новая модель пластины Кирхгофа–Лява, которая может соприкасаться либо по боковой грани, либо по одной из лицевых поверхностей с жестким препятствием определенной заданной конфигурации. Соответствующая вариационная задача формулируется в виде задачи минимизации функционала энергии над невыпуклым множеством допустимых перемещений с условием непроникания. Условие непроникания представлено в виде системы неравенств, описывающей два случая возможного контакта пластины и жесткого препятствия. А именно эти два случая соответствуют разным типам контактов: со стороны боковой кромки пластины и со стороны ее известной лицевой поверхности. Установлена разрешимость задачи. В частном случае, когда зоны контакта заранее известны, найдена эквивалентная дифференциальная постановка в предположении дополнительной регулярности решения вариационной задачи.

Ключевые слова: контактная задача, условие непроникания, невыпуклое множество, вариационная задача.