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Linear Autotopism Subgroups of Semifield Projective Planes

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Abstract. We investigate the well-known hypothesis of D. R. Hughes that the full collineation group of non-Desarguesian semifield projective plane of a finite order is solvable (the question 11.76 in Kourovka notebook was written down by N. D. Podufalov). This hypothesis is reduced to autotopism group that consists of collineations fixing a triangle. We describe the elements of order 4 and dihedral or quaternion subgroups of order 8 in the linear autotopism group when the semifield plane is of rank 2 over its kernel. The main results can be used as technical for the further studies of the subgroups of even order in an autotopism group for a finite non-Desarguesian semifield plane. The results obtained are useful to investigate the semifield planes with the autotopism subgroups from J. G. Thompson's list of minimal simple groups.

Keywords: semifield plane, autotopism, homology, Baer involution, Hughes' problem.

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Introduction

It is well-known that the geometric properties of projective plane are closely connected with the algebraic properties of its coordinatizing set. So, a finite Desarguesian projective plane is coordinatized by the field, a translation plane by the quasifield.

The study of finite semifields and semifield planes started ago with the first examples constructed by L. E. Dickson in 1906. A *semifield* is called a non-associative ring $Q = (Q, +, \cdot)$ with identity where the equations $ax = b$ and $ya = b$ are uniquely solved for any $a, b \in Q$, $a \neq 0$. The abcense of an associative law in a semifield leads to a number of anomalous properties in comparison with a field or a skewfield or even a near-field.

By the mid-1950s, some classes of finite semifield planes had been found. All of them had the common property that the collineation group (automorphism group) is solvable. So D. R. Hughes conjectured in 1959 that any finite projective plane coordinatized by a non-associative semifield has the solvable collineation group. This hypothesis is presented in the monography [1]; it is proved also that the hypothesis is reduced to the solvability of an autotopism group as a group fixing a triangle. Moreover, the hypothesis is reduced to linear autotopism subgroup over the kernel. The Hughes' problem attracted the interest of a wide range of researchers who proved the collineation group solvability for an extensive list of semifield planes with certain restrictions.

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In 1990 the problem was written down by N. D. Podufalov in the Kourovka notebook ([2], the question 11.76).

We represent the approach to study Hughes' problem based on the classification of finite simple groups and theorem of J. G. Thompson on minimal simple groups. The spread set method allows us to identify the conditions when the semifield plane with certain autotopism subgroup exists. This method can be used also to construct examples, including computer calculations. The elimination of some groups from Thompson's list as autotopism subgroups allows us make progress in solving the problem.

We consider, mostly, the case when a semifield plane has a rank 2 over its kernel basing on the theory of M. Biliotti and co-authors [3]. Nevertheless, some results are generalised to N-rank case.

It is shown by the first author in [4, 5], that an autotopism of order two has the matrix representation convenient for calculations and reasoning. Here we use the spread set method to describe the geometric sense of an autotopism of order 4. The matrix representation of this autotopism allows us to prove the criterion of existence for the dihedral and quaternion subgroups of order 8 in an autotopism group. We present the examples of semifield planes of minimal order 625 with this property.

1. Main definitions and preliminary discussion

We use main definitions, according [1, 6], see also [7–9], for notifications.

Consider a linear space Q, n-dimensional over the finite field $GF(p^s)$ (p is prime) and the subset of linear transformations $R \subset GL_n(p^s) \cup \{0\}$ such that:

1) R consists of p^{ns} square $(n \times n)$-matrices over $GF(p^s)$;
2) R contains the zero matrix 0 and the identity matrix E;
3) for any $A, B \in R$, $A \neq B$, the difference $A - B$ is a nonsingular matrix.

The set R is called a *spread set* [1]. Consider a bijective mapping θ from Q onto R and present the spread set as $R = \{\theta(y) \mid y \in Q\}$. Determine the multiplication on Q by the rule $x * y = x \cdot \theta(y)$ $(x, y \in Q)$. Then $\langle Q, +, * \rangle$ is a right quasifield of order p^{ns} [6, 10]. Moreover, if R is closed under addition then $\langle Q, +, * \rangle$ is a semifield.

Note, that if we use a prime field $\mathbb{Z}_p$ as a basic field then the mapping θ is presented using only linear functions; it greatly simplifies reasoning and calculations (also computer).

A semifield Q coordinatizes the projective plane π of order $|\pi| = |Q|$ such that:

1) the affine points are the elements (x, y) of the space $Q \oplus Q$;
2) the affine lines are the cosets to subgroups

$$V(\infty) = \{(0, y) \mid y \in Q\}, \qquad V(m) = \{(x, x\theta(m)) \mid x \in Q\} \quad (m \in Q);$$

3) the set of all cosets to the subgroup is the singular point;
4) the set of all singular points is the singular line;
5) the incidence is set-theoretical.

The solvability of a collineation group $Aut\,\pi$ for a semifield plane is reduced [1] to the solvability of an autotopism group Λ (collineations fixing a triangle). Without loss of generality, we can assume that linear autotopisms are determined by linear transformations of the space $Q \oplus Q$:

$$\lambda : (x, y) \to (x, y) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

here the matrices A and B satisfy the condition (for instance, see [11])

$$A^{-1}\theta(m)B \in R \quad \forall \theta(m) \in R. \tag{1}$$

The collineations fixing a closed configuration have special properties. It is well-known [1], that any involutory collineation is a central collineation or a Baer collineation.

A collineation of a projective plane is called *central*, or *perspectivity*, if it fixes a line pointwise (*the axis*) and a point linewise (*the center*). If the center is incident to the axis then a collineation is called *an elation*, and *a homology* in another case. The order of any elation is a factor of the order $|\pi|$ of a projective plane, and the order of any homology is a factor of $|\pi|-1$. All the perspectivities in an autotopism group are homologies and form the cyclic subgroups [12]:

$$H_1 = \left\{ \begin{pmatrix} M & 0 \\ 0 & E \end{pmatrix} \;\middle|\; M \in R_m^* \right\}, \quad H_2 = \left\{ \begin{pmatrix} E & 0 \\ 0 & M \end{pmatrix} \;\middle|\; M \in R_r^* \right\},$$

$$H_3 = \left\{ \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \;\middle|\; M \in R_l^* \right\}.$$

The matrix subsets R_l, R_m, R_r are defined by a spread set [12]:

$$\begin{aligned} R_l &= \{M \in GL_n(p^s) \cup \{0\} \mid MT = TM \ \forall T \in R\}, \\ R_m &= \{M \in R \mid MT \in R \ \forall T \in R\}, \\ R_r &= \{M \in R \mid TM \in R \ \forall T \in R\}, \end{aligned}$$

they are called *left, middle and right nuclei* of the plane π respectively. These subfields in $GL_n(p^s) \cup \{0\}$ are isomorphic to correspondent nuclei of the coordinatizing semifield Q:

$$\begin{aligned} N_l &= \{x \in Q \mid (x * a) * b = x * (a * b) \ \forall a, b \in Q\}, \\ N_m &= \{x \in Q \mid (a * x) * b = a * (x * b) \ \forall a, b \in Q\}, \\ N_r &= \{x \in Q \mid (a * b) * x = a * (b * x) \ \forall a, b \in Q\}. \end{aligned}$$

The plane π is Desarguesian (classic) iff Q is a field, then $R \simeq Q \simeq GF(p^{ns})$.

An autotopis group of a semifield plane of odd order contains three involutory homologies:

$$h_1 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \in H_1, \quad h_2 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \in H_2, \quad h_3 = h_1 h_2 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix} \in H_3.$$

A collineation of a projective plane π of order m is called *Baer collineation* if it fixes pointwise a subplane of order $\sqrt{|\pi|} = \sqrt{m}$ (*Baer subplane*). We use the results on the matrix representation of a Baer involution $\tau \in \Lambda$ and of a spread set obtained by M. Biliotti with co-authors [3] and by the first author in [4, 5].

2. Linear autotopisms of order 4

We consider now the case when a semifield plane π has a rank 2 over its kernel, $|\pi| = |N_l|^2$. To simplify the notification we use $K = N_l \simeq GF(q)$, $q = p^n$. The point set of the plane is

$$\pi = \{(x_1, x_2, y_1, y_2) \mid x_i, y_i \in GF(q)\},$$

the spread set R consists of (2×2)-matrices determined its second row:

$$R = \left\{ \theta(v,u) = \begin{pmatrix} f(v,u) & g(v,u) \\ v & u \end{pmatrix} \;\middle|\; v,u \in GF(q) \right\}.$$

Here the functions f and g are additive:

$$\begin{aligned} &f(v_1,u_1) + f(v_2,u_2) = f(v_1+v_2, u_1+u_2), \\ &g(v_1,u_1) + g(v_2,u_2) = g(v_1+v_2, u_1+u_2), \qquad v_1,v_2,u_1,u_2 \in GF(q), \end{aligned}$$

so f and g are the additive polynomials:

$$f(v,u) = \sum_{j=0}^{n-1} (f_j u^{p^j} + F_j v^{p^j}), \qquad g(v,u) = \sum_{j=0}^{n-1} (g_j u^{p^j} + G_j v^{p^j}), \qquad f_j, F_j, g_j, G_j \in GF(q).$$

The autotopism group Λ consists of *semi-linear* transformations of the linear space:

$$\lambda : (x,y) \to (x^\sigma, y^\sigma) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where σ is a basic field automorphism:

$$x^\sigma = (x_1, x_2)^\sigma = (x_1^{p^t}, x_2^{p^t}).$$

Evidently, that the subgroup Λ_0 of *linear* autotopisms ($t = 0$) is normal in Λ and the factor Λ/Λ_0 is isomorphic to a subgroup of $Aut\, K$. Therefore, the solvability problem is reduced to the linear autotopism subgroup Λ_0.

G. E. Moorhouse in 1989 proved [13]:

Lemma 1. *Let π be a projective plane of order n^2, $n \equiv 2$ or $3 \pmod 4$, and G is a cyclic collineation group of order 4. Then the involution in G is central.*

We will expand Moorhouse' result for $|\pi| = p^{2n}$ if $p \not\equiv -1 \pmod 4$.

Let π be a non-Desarguesian semifield plane of order q^2 with the kernel $K \simeq GF(q)$ ($q = 2^n$). If $\tau \in \Lambda_0$ is an involution then it is Baer, and we can propose that, in appropriate base, it has a Jordan normal form (see [3]):

$$\tau = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

The spread set $R \in GL_2(q) \cup \{0\}$ consists of matrices

$$\theta(v,u) = \begin{pmatrix} v + u + m(v) & f(v) + m(u) \\ v & u \end{pmatrix}, \quad v,u \in GF(2^n). \tag{3}$$

Lemma 2. *The linear autotopism group Λ_0 of a semifield projective plane π of order 2^{2n} with the kernel $K \simeq GF(2^n)$ does not contain elements of order 4.*

Proof. Let $\alpha \in \Lambda_0$ be an autotopism of order 4, $\alpha^4 = \varepsilon$. Then $\alpha^2 = \tau$ is a Baer involution (2), because $h_1, h_2, h_3 \notin \Lambda$ for $p = 2$. Let

$$\alpha = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

then $A^2 = B^2 = L$, $AL = LA$, $BL = LB$. So we have

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix},$$

where

$$\begin{cases} a_1^2 = 1, \\ a_1a_2 + a_2a_1 = 1; \end{cases} \qquad \begin{cases} b_1^2 = 1, \\ b_1b_2 + b_2b_1 = 1. \end{cases}$$

The systems have no solution in a field of the characteristic 2, the lemma is proved. □

Theorem 2.1. *Let π be a semifield non-Desarguesian plane of order 2^{2n} with the kernel $K \simeq GF(2^n)$. Then the Sylow 2-subgroup of the linear autotopism group Λ_0 has an order at most 2.*

Proof. From the lemma, the Sylow 2-subgroup $S \subset \Lambda_0$ is elementary Abelian. Let $\tau, \alpha \in S$, where τ is (2). Then

$$\alpha = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Consider the condition (1)

$$A^{-1}\theta(v,u)B \in R \quad \forall v, u \in GF(2^n)$$

for the spread set (3). For $\theta(0,1) = E$ we have

$$A^{-1}B = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b+a \\ 0 & 1 \end{pmatrix} \in R, \qquad b = a.$$

Further, for $\theta(v,0)$:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v + m(v) & f(v) \\ v & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} v + m(v) + av & f(v) + av + m(v)a + a^2v \\ v & va \end{pmatrix} = \theta(v, va),$$

and

$$m(va) = av + m(v)a + a^2v, \quad \forall v \in GF(2^n).$$

Consider the polynomial $m(v)$:

$$m(v) = m_0v + m_1v^2 + m_2v^4 + \ldots + m_{n-1}v^{2^{n-1}},$$

$$m_0va + m_1v^2a^2 + m_2v^4a^4 + \ldots = m_0va + m_1v^2a + m_2v^4a + \ldots + va + va^2, \quad a + a^2 = 0.$$

If $a = 0$ then $\alpha = \varepsilon$; if $a = 1$ then $\alpha = \tau$. The theorem is proved. □

Let now $p > 2$, $|\pi| = p^{2n}$, $K \simeq GF(p^n)$. We will not consider the case $p \equiv -1 \pmod 4$: this case is more complicated for a semifield plane of arbitrary rank, see [14]. If $p \equiv 1 \pmod 4$ then the prime field $\mathbb{Z}_p$ of K contains an element i such that $i^2 = -1$. We have $iE = E + \cdots + E$, therefore $iE \in R_r \cap R_m$ and the linear autotopism group Λ_0 contains the homologies of order 4:

$$\alpha_1 = \begin{pmatrix} iE & 0 \\ 0 & E \end{pmatrix} \in H_1, \qquad \alpha_2 = \begin{pmatrix} E & 0 \\ 0 & iE \end{pmatrix} \in H_2, \qquad \alpha_1\alpha_2 \in H_3.$$

As has be proven in [4], a Baer involution $\tau \in \Lambda_0$ can be written as

$$\tau = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \qquad L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

for appropriate Jordan base. The spread set R consists of matrices

$$\theta(v,u) = \begin{pmatrix} m(u) & f(v) \\ v & u \end{pmatrix}, \quad v, u \in GF(p^n), \tag{5}$$

where the functions m and f are injective additive polynomials, $m(1) = 1$, $f(1) \neq \pm 1$.

The following theorem expands Moorhouse' lemma 1.

Theorem 2.2. *Let π be a non-Desarguesian semifield plane of order p^{2n} with the kernel $K \simeq GF(p^n)$, p is prime, $p \equiv 1 \pmod 4$, and $\alpha \in \Lambda_0$ is a linear autotopism of order 4. Then α^2 is a homology and either $\alpha \in \langle \alpha_1, \alpha_2 \rangle$ or π admits a linear Baer involution τ and $\alpha \in \langle \alpha_1, \alpha_2, \tau \rangle$.*

Proof. Let α is not homology,

$$\alpha = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

then α^2 is the homology h_1, h_2, h_3 or a Baer involution τ (4).

If $\alpha^2 = \tau$ then from $A^2 = B^2 = L$, $AL = LA$, $BL = LB$ we have

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},$$

where $a_1, b_1 \in \{i, -i\}$, $a_2, b_2 \in \{1, -1\}$. Thus, the autotopism α can be represented as a product

$$\alpha = \begin{pmatrix} \pm i & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} = \tau\mu h_1$$

or $\alpha = \tau\mu h_2$ or $\alpha = \tau\mu h_3$, where $\mu = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$, $M = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. We consider the condition (1) for the collineation μ for any matrix (5): $M^{-1}\theta(v,u)M \in R$. For any $v \in K$ and $u = 0$ we have:

$$M^{-1}\theta(v,0)M = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & f(v) \\ v & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -if(v) \\ iv & 0 \end{pmatrix} \in R,$$

$f(iv) = -if(v)$. It contradicts with the additivity of $f(x)$, because $i \in \mathbb{Z}_p$ and so $f(iv) = if(v)$. Therefore, the case $\alpha^2 = \tau$ is impossible, and α^2 is a homology.

Let $\alpha^2 = h_1$. Then $A^2 = -E$, $B^2 = E$, and the following Jordan normal form are possible: $\pm iE$, $\pm iL$ for A and $\pm E$, $\pm L$ for B. If $A = \pm iE$ or $B = \pm E$ then $\alpha \in \langle \alpha_1, \alpha_2 \rangle$. The remaining possibility

$$\alpha = \begin{pmatrix} iL & 0 \\ 0 & L \end{pmatrix} \cdot h_1^{k_1} h_2^{k_2}$$

leads to $\alpha \cdot (\alpha_1 h_1^{k_1} h_2^{k_2})^{-1} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} = \tau$ is the Baer involution, $\alpha \in \langle \alpha_1, \alpha_2, \tau \rangle$.

For $\alpha^2 = h_2$ and $\alpha^2 = h_3$ we obtain the analogous result. The theorem is proved. □

Note that the homologies generate the normal subgroup in the autotopism group [1]. Therefore, if $F < \Lambda_0$ is a simple non-Abelian subgroup then it does not contain elements of order 4 for $p \not\equiv -1 \pmod 4$. Further, any elementary Abelian subgroup of Λ_0 is of order at most 8. Maximal its order is for $\langle \tau, h_1, h_2 \rangle$; but $\langle h_1, h_2 \rangle$ is normal in Λ_0, thus $|F|$ is either odd or $2 \cdot (2m+1)$. This contradicts to conjecture that F is simple non-Abelian group.

Corollary 1. *Let π be a non-Desarguesian semifield plane of order p^{2n} with the kernel $K \simeq GF(p^n)$, p is prime, $p \equiv 1 \pmod 4$. Then its autotopism group contains no simple non-Abelian subgroups.*

For $p = 2$ the more significant result has been proven by M. J. Ganley in 1974 ([15], see also [16]).

Theorem 2.3. *Let Q be a finite semifield of order 2^s. If Q has dimension 2 over one of its nuclei then its autotopism group is solvable.*

Extend the results obtained by the information on coordinatizing semifield automorphisms. It has been proven in [11], that the linear transformation $x \to xA$ is the automorphism of a semifield Q iff the matrix

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \tag{6}$$

is the autotopism of the semifield plane π with the condition $A^{-1}\theta(m)A = \theta(mA)$ for any $m \in Q$. Therefore, we have the following result.

Corollary 2. *Let Q be a non-associative semifield of order p^{2n} with the left nucleus $K \simeq GF(p^n)$, p is prime, $p \not\equiv -1 \pmod 4$. Then the automorphism subgroup $Aut_K Q$ of Q that fixes K has a Sylow 2-subgroup of order at most 2.*

Proof. For the even case $p = 2$ the result is a direct consequence of the theorem 2.1. Let $p \equiv 1 \pmod 4$ and the transformation $x \to xA$ be an automorphism from $Aut_K Q$. Then we can assume that $A \in GL_2(p^n)$. According to [11], the involution (6) may be Baer only. Up to base chosen, we can suppose $A = L$. The centralizer of τ in Λ_0 contains involutions τ, $h_1\tau$, $h_2\tau$, $h_3\tau$ only. All these possibilities lead to the contradiction, because the involution $\tau \cdot (h_i\tau)$ is not Baer. Thus, the elementary Abelian 2-subgroup of $Aut_K Q$ is of order at most 2. Finally, if the autotopism α (6) is of order 4 then α^2 is an involutory homology h_3 with $A = -E$; we have the contradiction. □

3. Dihedral and quaternion subgroups

The question on autotopism subgroups isomorphic to D_8 or Q_8 is explained by the fact that such subgroups are contained in the Sylow 2-subgroup of a large number of simple non-Abelian groups. For semifield plane of arbitrary rank over the kernel, the first author proved [8] that a dihedral autotopism subgroup of order 8 must contain the homology if $p \equiv 1 \pmod 4$. Now we describe the matrix representation of subgroup

$$H = \langle \alpha, \beta \mid \alpha^4 = \beta^2 = 1,\ \beta\alpha\beta = \alpha^{-1} \rangle \simeq D_8 \tag{7}$$

and the spread set matrices. The generalization of this result for N-dimensional case see in the next section.

Theorem 3.4. *Let π be a non-Desarguesian semifield plane of order p^{2n} with the krenel $K \simeq GF(p^n)$, p is prime, $p \equiv 1 \pmod 4$, and the linear autotopism group Λ_0 contains a subgroup isomorphic to the dihedral group of order 8 (7). Then the base of 4-dimensional vector space over K can be chosen such that $H = \langle \alpha, \beta \rangle$,*

$$\alpha = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \cdot h_1^{k_1} h_2^{k_2}, \qquad \beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad i^2 = -1, \quad i \in \mathbb{Z}_p,$$

$k_1, k_2 \in \{0, 1\}$. The spread set of π consist of matrices (5), where m and f are injective involutory functions on $GF(p^n)$, $m(m(x)) = x$, $f(f(x)) = x$.

Proof. If $\alpha \in \langle \alpha_1, \alpha_2 \rangle$ then $\alpha \in Z(\Lambda)$ and the condition $\beta\alpha\beta = \alpha^{-1}$ is not satisfied for any autotopism β. Therefore, the plane π admits the Baer involution τ, it has the spread set (5), and $\alpha \in \langle \alpha_1, \alpha_2, \tau \rangle$, from the theorem 2.2. Then α^2 is one of involutory homologies h_1, h_2, h_3. Consider the possible cases.

Let $\alpha^2 = h_1$. Then, up to involutory homologies, $\alpha = \begin{pmatrix} iL & 0 \\ 0 & L \end{pmatrix} = \alpha_1\tau$, and $\beta(\alpha_1\tau)\beta = (\alpha_1\tau)^3\beta$, $\beta\tau\beta = h_1\tau$. Denote $\beta = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$, then $C^2 = D^2 = E$, $CLC = -L$, $DLD = L$, and either $D = \pm E$ or $D = \pm L$. If $D = \pm E$ then $\begin{pmatrix} C & 0 \\ 0 & E \end{pmatrix}$ is the involutory homology, and so $C = -E$, this contradicts to condition. Therefore, up to involutory homology, $D = L$,

$$\beta = \begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} CL & 0 \\ 0 & E \end{pmatrix} \cdot \tau,$$

and $\beta\tau$ is the homology of order 4: $\beta\tau = \alpha_1$ or $\beta\tau = \alpha_1^3$, $CL = \pm iE$, $C = \pm iL$, $C^2 = -E \neq E$. Thus, the case $\alpha^2 = h_1$ is impossible.

By similar reasoning we come to a contradiction in the case $\alpha^2 = h_2$.

Let now $\alpha^2 = h_3$, and $\alpha = \begin{pmatrix} iL & 0 \\ 0 & iL \end{pmatrix} = \alpha_1\alpha_2\tau$, without the involutory homologies. Then $\beta(\alpha_1\alpha_2\tau)\beta = (\alpha_1\alpha_2\tau)^3 = \alpha_1\alpha_2 h_3\tau$, $\beta\tau\beta = h_3\tau$, $CLC = DLD = -L$, and we have

$$C = \begin{pmatrix} 0 & c \\ c^{-1} & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & d \\ d^{-1} & 0 \end{pmatrix}.$$

Consider the transition matrix T:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{8}$$

Then, for the new base,

$$T\alpha T^{-1} = \alpha, \qquad T\beta T^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

To complete the proof, it is enough to test the involutority of the functions m and f. Indeed, β is a collineation, and, for any matrix $\theta(v,u)$ from the spread set R, the product

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m(u) & f(v) \\ v & u \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ f(v) & m(u) \end{pmatrix}$$

must belong to R, see (1). Therefore, $m(m(u)) = u$ and $f(f(v)) = v$ for all $v, u \in GF(p^n)$. The theorem is proved. □

Now let the linear autotopism group Λ_0 contains a quaternion subgroup of order 8:

$$F = \langle \alpha, \gamma \mid \alpha^4 = \gamma^4 = 1,\ \alpha^2 = \gamma^2,\ \alpha\gamma\alpha = \gamma \rangle \simeq Q_8. \tag{9}$$

Theorem 3.5. *Let π be a non-Desarguesian semifield plane of order p^{2n} with the kernel $K \simeq GF(p^n)$, p is prime, $p \equiv 1 \pmod 4$, and the linear autotopism group Λ_0 contains a subgroup isomorphic to the quaternion group of order 8 (9). Then the base of 4-dimensional vector space over K can be chosen such that $F = \langle \alpha, \gamma \rangle$,*

$$\alpha = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \cdot h_1^{k_1} h_2^{k_2}, \qquad \gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad i^2 = -1,\ i \in \mathbb{Z}_p,$$

$k_1, k_2 \in \{0,1\}$. *The spread set of π consist of matrices (5), where m and f are injective involutory functions on $GF(p^n)$, $m(m(x)) = x$, $f(f(x)) = x$.*

Proof. If either α or γ belongs to subgroup $\langle \alpha_1, \alpha_2 \rangle < Z(\Lambda_0)$ then $\alpha\gamma = \gamma\alpha$, it is impossible. Therefore, the plane π admits a Baer involution τ (4) and, for instance, $\alpha \in \langle \alpha_1, \alpha_2, \tau \rangle$. Evident, that we can ignore the involutory homologies factors, because $\langle \alpha, \gamma \rangle \simeq Q_8$ leads to $\langle \alpha h_1^{k_1} h_2^{k_2}, \gamma \rangle \simeq Q_8$. So, we suppose further $\alpha = \alpha_1\tau$, $\alpha = \alpha_2\tau$ or $\alpha = \alpha_1\alpha_2\tau$.

In the first case $\alpha^2 = \gamma^2 = h_1$. Notify

$$\gamma = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \quad C^2 = -E, \quad D^2 = E,$$

and consider the condition $\alpha\gamma\alpha = \gamma$:

$$(\alpha_1\tau)\gamma(\alpha_1\tau) = \gamma, \quad h_1\tau\gamma\tau = \gamma, \qquad -LCL = C, \quad LDL = D.$$

Then $D \in \{E, -E, L, -L\}$, but the case $D = \pm E$ is impossible: $\begin{pmatrix} C & 0 \\ 0 & E \end{pmatrix}$ is a homology of order 4, $C = \pm iE$, $\alpha\gamma = \gamma\alpha$. Therefore, up to involutory homologies, $D = L$,

$$\gamma = \begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} CL & 0 \\ 0 & E \end{pmatrix} \cdot \tau,$$

then $\gamma\tau$ is the involutory homology too (from $CLCL = E$), that is $\gamma\tau = h_1$, $\gamma = h_1\tau$, and $\alpha\gamma = \gamma\alpha$. This contradiction shows that the case $\alpha^2 = h_1$ is impossible; for $\alpha^2 = h_2$, similarly.

Let now $\alpha^2 = \gamma^2 = h_3$, $\alpha = \begin{pmatrix} iL & 0 \\ 0 & iL \end{pmatrix} = \alpha_1\alpha_2\tau$. Then

$$\alpha\gamma\alpha = (\alpha_1\alpha_2\tau)\gamma(\alpha_1\alpha_2\tau) = h_1h_2\tau\gamma\tau = h_3\tau\gamma\tau = \gamma,$$

and we have the conditions $-LCL = C$, $-LDL = D$, $C^2 = D^2 = -E$, leading to

$$C = \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix}.$$

Choose the transition matrix T (8), then for the new base we obtain

$$T\alpha T^{-1} = \alpha, \qquad T\gamma T^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Prove the involutority of m and f form the condition (1):

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \theta(v,u) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} u & -v \\ -f(v) & m(u) \end{pmatrix} \in R \quad \forall v, u \in GF(p^n).$$

Thus, $f(-f(v)) = -v$, $m(m(u)) = u$, and the additivity leads to $f(f(v)) = v$. The theorem is proved. □

Remark 1. *Note that, according to the theorem 2.2 on autotopisms of order 4, the collineation γ must be the product of homologies to a Baer involution. Indeed,*

$$\gamma = \alpha_1\alpha_2\sigma, \qquad \sigma = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},$$

where σ is the Baer involution fixing pointwise the subplane

$$\pi_\sigma = \{(x_1, ix_1, y_1, iy_1) \mid x_1, y_1 \in GF(p^n)\}.$$

Rewrite the autotopism γ of order 4 as

$$\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \beta\tau.$$

We see that $\langle \alpha, \beta \rangle \simeq D_8$ and $\langle \alpha, \beta\tau \rangle \simeq Q_8$, and the following corollary is proved.

Corollary 3. *Let π be a non-Desarguesian semifield plane of order p^{2n} with the krenel $K \simeq GF(p^n)$, p is prime, $p \equiv 1 \pmod 4$. The linear autotopism group Λ_0 contains a subgroup isomorphic to the quaternion group of order 8 iff Λ_0 contains a subgroup isomorphic to the dihedral group of order 8.*

4. Examples

Construct the semifield planes of minimal order satisfying the condition of theorems 3.4 and 3.5. It is well-known [1] that a semifield of order p^2 is a field $GF(p^2)$. Therefore the minimal examples are the planes of order $5^4 = 625$. Let the field $K \simeq GF(25)$ is an algebraic extension of $\mathbb{Z}_5$, $K = \mathbb{Z}_5(a)$, where a is the root of the irreducible polynomial $x^2 + 3x + 3 \in \mathbb{Z}_5[x]$. Then the semifield Q of order 625 is the vector space $Q = \{x = (v, u) \mid v, u \in K\}$ with the multiplication law $y * x = y \cdot \theta(x)$ $(x, y \in Q)$; the spread set R consists of matrices (5). The functions m and f are the polynomials from $K[x]$:

$$m(u) = m_0u + m_1u^5, \quad \varphi(v) = f_0v + f_1v^5,$$

which satisfy the conditions $m(m(u)) = u$, $f(f(v)) = v$ for all $u, v \in K$, $m(1) = 1$, $f(1) \neq \pm 1$. There exist 34 pairs of functions m, f such that $\det \theta(v, u) = 0$ only for $(v, u) = (0, 0)$. Therefore, we obtain 34 semifield planes of order 625 with the kernel of order 25 which admit the linear autotopism subgroup isomorphic to D_8 (or Q_8). At most 11 pairwise non-isomorphic planes are among them. The isomorphism is either multiplication by a suitable matrix (i. e. changing of base) or the automorphism of K:

$$(m_0, m_1, f_0, f_1) \to (m_0^5, m_1^5, f_0^5, f_1^5)$$

The table below represents the coefficients m_i, f_i together with the nuclei of the semifields.

Table 1. Information on the planes of order 625

№	m_0	m_1	f_0	f_1	N_l	$N_m = N_r$
1	0	1	0	a	K	$\{(0, y) \mid y \in K\}$
2	0	1	0	$a + 1$		$\lvert N_m\rvert = \lvert N_r\rvert = 25$
3	0	1	$2a + 1$	2		
4	0	1	$2a + 1$	3		
5	$4a + 2$	$a + 4$	$a + 3$	$a + 2$		$\{(0, y) \mid y \in \mathbb{Z}_5\}$
6	$4a + 2$	$a + 4$	$a + 3$	$3a + 4$	K	$\lvert N_m\rvert = \lvert N_r\rvert = 5$
7	$3a + 4$	$2a + 2$	0	$a + 1$		
8	$3a + 4$	$2a + 2$	$2a + 1$	2		
9	$3a + 4$	$2a + 2$	$2a + 1$	$2a$		
10	$3a + 4$	$2a + 2$	$2a + 1$	$2a + 2$		$\{(x, y) \mid$
11	$a + 3$	$4a + 3$	$a + 3$	$a + 2$	K	$x \in \{0, a + 3, 2a + 1,$
						$3a + 4, 4a + 2\}, y \in \mathbb{Z}_5\}$
						$\lvert N_m\rvert = \lvert N_r\rvert = 25$

5. Generalization for arbitrary dimension

Here we consider the case when a semifield plane π has the order p^N, without restriction to the order of the kernel. In this case we can represent the point set of π as a $2N$-dimensional vector space over $\mathbb{Z}_p$, with the spread set $R \subset GL_N(p) \cup \{0\}$. Some results from the previous sections can be generalized for any N and $p \equiv 1 \pmod 4$.

Let π be a non-Desarguesian semifield plane of order p^N ($p > 2$ be prime). According to the results of [5], if the autotopism group Λ contains the Baer involution τ then $N = 2n$ is even and we can choose the base of $4n$-dimensional linear space over $\mathbb{Z}_p$ such that

$$\tau = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \qquad L = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}. \tag{10}$$

The spread set R in $GL_{2n}(p) \cup \{0\}$ consists of matrices

$$\theta(V, U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix}, \tag{11}$$

where $V \in Q$, $U \in K$, Q, K are the spread sets in $GL_n(p) \cup \{0\}$, K is the spread set of the Baer subplane π_τ, m, f are additive injective functions from K and Q into $GL_n(p) \cup \{0\}$, $m(E) = E$. Note that throughout the section, the blocks-submatrices have the same dimension by default. Instead of linear autotopism group Λ_0 we will consider the autotopism group Λ:

$$\Lambda = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;\middle|\; A, B \in GL_N(p),\ A^{-1}\theta(m)B \in R\ \forall \theta(m) \in R \right\}.$$

Unfortunately, we can not now extend the result of the theorem 2.2 to the general case. The geometric sense of order 4 autotopism α was presented in [14] when $\alpha^2 = \tau$ is a Baer involution. Perhaps, one will construct the examples illustrating the matrix representation of a spread set in this case; there is no evident contradiction.

Theorem 5.6. *Let π be a non-Desarguesian semifield plane of order p^N, p is prime, $p \equiv 1 \pmod 4$, and $\alpha \in \Lambda$ is an autotopism of order 4. If α^2 is a homology then either $\alpha \in \langle \alpha_1, \alpha_2 \rangle$ or π admits a Baer involution τ and $\alpha \in \langle \alpha_1, \alpha_2, \tau \rangle$.*

Proof. Let $\alpha^2 = h_1$ and $\alpha \notin \langle \alpha_1, \alpha_2 \rangle$. Then $\alpha = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where $A^2 = -E$, $B^2 = E$. Therefore $A = \operatorname{diag}(i, -i) \neq \pm iE$, $B = \operatorname{diag}(1, -1) \neq \pm E$. The number of 1 among the diagonal elements of B equals to the number of -1, because else we have the autotopism that fixes more than a Baer subplane, it is impossible. So, we can assume, up to base changing and involution homologies factors, that $B = L$ and $A = iL$, $\alpha = \alpha_1\tau$. For $\alpha^2 = h_2$ and $\alpha^2 = h_3$ the consideration is similar. □

We extend now the main result of [8]:

Theorem 5.7. *Any non-Desarguesian semifield plane π of order p^N, where $p > 2$ is prime and $p \equiv 1 \pmod 4$, does not admit an autotopism subgroup isomorphic to the dihedral group of order 8 without homologies.*

Denote the following autotopisms:

$$\alpha = \begin{pmatrix} -iE & 0 & 0 & 0 \\ 0 & iE & 0 & 0 \\ 0 & 0 & -iE & 0 \\ 0 & 0 & 0 & iE \end{pmatrix} h_1^{k_1} h_2^{k_2},$$
$$\beta = \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & -E & 0 \end{pmatrix}, \tag{12}$$

where $i \in \mathbb{Z}_p$, $i^2 = -1$, $k_1, k_2 \in \{0, 1\}$. Using the result of [18], we prove.

Theorem 5.8. *Let π be a non-Desarguesian semifield plane of order p^N, p is prime, $p \equiv 1 \pmod 4$. The autotopism group Λ contains a subgroup $H \simeq D_8$ (7) iff it contains a subgroup $F \simeq Q_8$ (9). Then $N = 2n \geqslant 4$, the subgroups H and F contains the involutory homology h_3, the plane π admits a Baer involution τ (10). The base of linear space can be chosen such that $H = \langle \alpha, \beta \rangle$, $F = \langle \alpha, \gamma \rangle$, where the autotopisms are (12). The spread set R consists of matrices (11), where $V \in Q$, $U \in K$, the sets $Q, K \subset GL_n(p) \cup \{0\}$ are closed under addition. The additive injections $m : K \to K$ and $f : Q \to Q$ are non-trivial involutions.*

Proof. The result for Q_8 had been proven in [18]. We will repeat now the proof of the theorem 3.4 with generalization for N-dimensional case. Let $\alpha \in H \simeq D_8$, $\alpha^4 = \varepsilon$. Then α^2 is the homology by the theorem 5.7.

1. If $\alpha^2 = h_1$ then $\alpha \notin \langle \alpha_1, \alpha_2 \rangle \subset Z(\Lambda)$. Therefore we can assume, up to base changing, that $\alpha = \alpha_1 \tau$, the spread set R consists of matrices (11). Further, let the Baer involution β be the matrix $\beta = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$. Then, from $\alpha\beta = \beta\alpha^{-1}$ we have

$$C = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad C_1 C_2 = E,\ D_1^2 = D_2^2 = E.$$

We can use the block-diagonal transition matrix T similar (8) and obtain $C_1 = C_2 = E$. Moreover, we can assume that D_1 and D_2 are either diagonal matrices $\operatorname{diag}(1, -1)$ or $\pm E$. From the condition (1) we have

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} 0 & D_2 \\ D_1 & 0 \end{pmatrix} = \theta(D_1, 0) \in R,$$

but either the matrix $E + \theta(D_1, 0) = \theta(D_1, E) \neq 0$ or the matrix $\theta(D_1, iE) \neq 0$ is singular, it is impossible. Thus, the conjecture $\alpha^2 = h_1$ leads to contradiction, similar for h_2.

2. Let $\alpha^2 = h_3$. Then α is the matrix (12), up to base changing. The transition matrix

$$T = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & D_1 \end{pmatrix}$$

preserves α and maps the Baer involution

$$\beta = \begin{pmatrix} 0 & C_1 & 0 & 0 \\ C_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1 \\ 0 & 0 & D_1^{-1} & 0 \end{pmatrix}$$

to the matrix (12). The condition (1) for $\theta(V, U)$ leads to the involutivity $m(m(U)) = U$, $f(f(V)) = V$. The theorem is proved. □

Conclusion

We can see that the properties and the structure of the linear autotopism group for a two-dimensional semifield plane may be considerably generalized to the N-dimensional case. The proof technique can be used with more careful consideration.

In order to study Hughes' problem on the solvability of the full collineation group of a finite non-Desarguesian semifield plane, the authors consider it possible to use the obtained results to further investigations. The method applied will probably be useful to consider simple non-Abelian groups and to exclude an extensive list from possible autotopism subgroups.

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Подгруппы линейных автотопизмов полуполевых проективных плоскостей

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Аннотация. Изучается известная гипотеза Д. Хьюза 1959 г. о разрешимости полной группы коллинеаций недезарговой полуполевой проективной плоскости конечного порядка (также вопрос 11.76 Н. Д. Подуфалова в Коуровской тетради). Эта гипотеза редуцируется к группе автотопизмов, состоящей из коллинеаций, фиксирующих треугольник. В работе описаны элементы порядка 4 и диэдральные либо кватернионные подгруппы порядка 8 в группе линейных автотопизмов полуполевой плоскости ранга 2 над ядром. Основные доказанные результаты являются техническими и необходимы для дальнейшего изучения подгрупп четного порядка в группе автотопизмов конечной недезарговой полуполевой плоскости. Результаты могут быть использованы для изучения полуполевых плоскостей, допускающих подгруппы автотопизмов из списка Д. Г. Томпсона минимальных простых групп.

Ключевые слова: полуполевая плоскость, автотопизм, гомология, бэровская инволюция, проблема Хьюза.