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Uniform Estimates for Mittag-Leffler Functions with Smooth Phase

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Abstract. In this paper we consider the problem on uniform estimates for Mittag–Leffler functions with the smooth phase functions having singularities D_∞, $D_4^\pm$ and A_r. The generalisation is that we replace the exponential function with the Mittag–Leffler-type function, to study oscillatory type integrals.

Keywords: Mittag–Leffler functions, phase function, amplitude.

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1. Introduction and preliminaries

The function $E_\alpha(z)$ is named after the great Swedish mathematican Gösta Magnus Mittag–Leffler (1846–1927) who defined it by a power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad Re\alpha > 0, \tag{1}$$

and studied its properties in 1902-1905 in five subsequent notes [9–12] in connection with his summation method for divergent series.

A classic generalizations of the Mittag–Leffler function, namely the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad Re\alpha > 0, \tag{2}$$

which was deeply investigated independently by Humbert and Agarval in 1953 [13–15] and by Dzherbashyan in 1954 [16–18] and other properties has in [19].

In this paper we consider natural generalization of exponential function is the Mittag–Leffler function defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R},$$

with the property [7] that

$$E_{1,1}(x) = e^x. \tag{3}$$

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We consider the following integral with phase f and amplitude ψ is an integral of the form:

$$I_{\alpha,\beta} = \int_a^b E_{\alpha,\beta}(i\lambda f(x))\psi(x)dx \tag{4}$$

where $0 < \alpha \leqslant 1,\ \beta > 0$ and $\lambda > 0$.

The main result of the work is the following.

Theorem 1.1. *Let $-\infty < a < b < \infty$. Let phase function has the a homogenous polynomial third degree with two variables and let $\psi \in L^p[a,b]^2$, $1 < p \leqslant \infty$. If $0 < \alpha < 1$, $\beta > 0$ and $\lambda > 0$ then we have following estimates*

$$\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^2 x_2)\psi(x)dx \right| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{2}-\frac{1}{2p}}}, \tag{5}$$

$$\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda (x_1^2 x_2 \pm x_2^3))\psi(x)dx \right| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}, \tag{6}$$

$$\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^3)\psi(x)dx \right| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3}-\frac{1}{3p}}}, \tag{7}$$

where constant C is depend only p.

If $\alpha = \beta = 1$ in the integral (4), so this type of integral is called oscillatory integrals. Estimates for oscillatory with polynomial phase we can consider in the following authors paper [2–4,6,20,21].

2. Auxiliary statements

Let we consider homogenous polynomial third degree with two variables. First we give auxiliary statements. Proposition (see. [1] page 189) A homogenous polynomial third degree with two variables may be reduced by a R-linear transformation to one of the forms: 1) $x_1^2x_2$; 2) $x_1^2x_2 \pm x_2^3$; 3) x_1^3; 4) 0.

Definition 1. *Given $\mu \in (1, \infty]$, a critical point, equivalent to the critical point of the function $x_1^2x_2 \pm x_2^{\mu-1}$ is said to be a critical point of type $D_\mu^\pm$, where $x_2^{\mu-1} \equiv 0$ for $\mu = \infty$.*

Definition 2. *A critical point, equivalent to the critical point of the function x_1^{r+1} is said to be a critical point of type A_r where $r \geqslant 1$.*

Proposition 1 ([8]). *If $0 < \alpha < 2, \beta$ is an arbitrary real number, μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, then there is $C > 0$, such that we have*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \quad z \in \mathbb{C}, \quad \mu \leqslant |\arg(z)| \leqslant \pi. \tag{8}$$

Proposition 2 ([7]). *Let $\alpha, \beta > 0$ and $f : [a, b] \to \mathbb{C}$. Then for all $\lambda \in \mathbb{C}$ we have*

$$E_{\alpha,\beta}(i\lambda f(x)) = E_{2\alpha,\beta}(-\lambda^2 f^2(x)) + i\lambda f(x)E_{2\alpha,\beta+\alpha}(-\lambda^2 f^2(x)). \tag{9}$$

3. Proof of the main results

Proof of Theorem 1.1 As for small λ the integral (4) is just bounded, we consider proof of theorem for $\lambda \geqslant 1$. Without loss of generality, we can consider the integral on $[0, 1]^2$ square otherwise we reduce to square with form linear transformation. As we given homogenous polynomial third degree with two variables, so from Proposition 2. we can represent it one of the following form 1) $x_1^2x_2$; 2) $x_1^2x_2 \pm x_2^3$; 3)x_1^3 4) 0. If phase function is $f(x) \equiv 0$ it is clear that integral will be exactly zero. So we will consider the three cases separately.

Using inequalities (8) and (9) we obtain:

$$|E_{\alpha,\beta}(i\lambda f(x))| \leqslant |E_{2\alpha,\beta}(-\lambda^2 f^2(x))| + \lambda|f(x)||E_{2\alpha,\beta+\alpha}(-\lambda^2 f^2(x))| \leqslant \\ \leqslant \frac{C}{1+\lambda^2 f^2(x)} + \frac{C\lambda|f(x)|}{1+\lambda^2 f^2(x)} \leqslant \frac{C(1+\lambda|f(x)|)}{1+\lambda^2 f^2(x)} \leqslant \frac{C}{1+\lambda|f(x)|}. \tag{10}$$

Case I. First we assume that the phase function has critical point of type D_∞ so $f(x) = x_1^2x_2$.

We consider integral (4) with following form:

$$I_{\alpha,\beta} = \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda x_1^2 x_2)\psi(x)dx. \tag{11}$$

We use inequality (10) for the integral (11) and we obtain:

$$|I_{\alpha,\beta}| = \left| \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda x_1^2 x_2)\psi(x)dx \right| \leqslant \int_{[0,1]^2} \left|E_{\alpha,\beta}(i\lambda x_1^2 x_2)\right| |\psi(x)|\, dx \leqslant \\ \leqslant C\int_0^1 dx_1 \int_0^1 \frac{|\psi(x)|dx_2}{1+\lambda x_1^2 x_2}. \tag{12}$$

Let us take q such that $\dfrac{1}{p}+\dfrac{1}{q}=1$. Assume that $p \neq \infty$, so that $q>1$. Then using the Hölder inequality in the inner integral we get

$$J_{in1} := \int_0^1 \frac{|\psi(x)|dx_2}{1+\lambda x_1^2 x_2} \leqslant \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\int_0^1 \frac{dx_2}{|1+\lambda x_1^2 x_2|^q}\right)^{\frac{1}{q}} = \\ = \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\right)^{\frac{1}{q}}.$$

Thus,

$$|I_{\alpha,\beta}| \leqslant C\int_0^1 \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\right)^{\frac{1}{q}} dx_1.$$

Then using again the Hölder inequality for this integral we obtain

$$|I_{\alpha,\beta}| \leqslant C\left(\int_0^1\int_0^1 |\psi(x)|^p dx_2 dx_1\right)^{\frac{1}{p}} \left(\int_0^1 \frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\, dx_1\right)^{\frac{1}{q}} \leqslant \\ \leqslant C\|\psi\|_{L^p} \left(\int_0^1 \frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\, dx_1\right)^{\frac{1}{q}}.$$

Let

$$K := \int_0^1 \frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\, dx_1.$$

Since $(1+\lambda x_1^2)^{1-q} = 1 + O(\lambda x_1^2)$ near $x_1 = 0$ and $q > 1$, so, this integral is convergent. Now we estimate it. First using the change of variables $t = \sqrt{\lambda} x_1$ we get

$$K = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{\sqrt{\lambda}} \frac{1-(1+t^2)^{1-q}}{t^2} dt = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{\sqrt{\lambda}} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt =$$
$$= \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{1} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt + \frac{1}{(q-1)\sqrt{\lambda}} \int_1^{\sqrt{\lambda}} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt =: K_1 + K_2.$$

Since $q - 1 \leqslant [q]$, where $[q] \geqslant 1$ is the integer part of $q > 1$, by the Newton's binomial formula

$$(1+t^2)^{q-1} \leqslant (1+t^2)^{[q]} = 1 + [q]t^2 + \frac{[q]([q]-1)}{2} t^4 + \ldots + t^{2[q]},$$

and hence

$$K_1 = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{1} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt \leqslant \frac{C_q}{\sqrt{\lambda}},$$

where

$$C_q := \frac{1}{q-1} \int_0^1 \frac{[q] + \frac{[q]([q]-1)}{2} t^2 + \ldots + t^{2[q]-2}}{(1+t^2)^{q-1}} dt.$$

Moreover, since $\dfrac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} < \dfrac{1}{t^2}$,

$$K_2 = \frac{1}{(q-1)\sqrt{\lambda}} \int_1^{\sqrt{\lambda}} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt < \frac{1}{(q-1)\sqrt{\lambda}} \int_1^{\sqrt{\lambda}} \frac{1}{t^2} dt =$$
$$= \frac{1}{(q-1)\sqrt{\lambda}} \Big(1 - \frac{1}{\sqrt{\lambda}}\Big) < \frac{1}{(q-1)\sqrt{\lambda}}.$$

Hence,

$$K \leqslant \frac{C'_q}{\sqrt{\lambda}}, \qquad C'_q := C_q + \frac{1}{q-1},$$

and

$$|I_{\alpha,\beta}| \leqslant \frac{C''_q \|\psi\|_{L^p}}{\lambda^{\frac{1}{2q}}},$$

where C''_q is some coefficient depending only on q.

Now we consider the case $q = 1$. Notice that the coefficient $C''_q \to +\infty$ as $q \to 1$, therefore we cannot directly conclude from this estimate the one for $q = 1$. As $q = 1$ so $p = \infty$ and $\psi \in \mathbb{L}^\infty$. In view of (12), first we'll estimate inner integral and we get

$$|J_{in1}| = \int_0^1 \frac{|\psi(x)| dx_2}{1+\lambda x_1^2 x_2} \leqslant \sup_{x_2\in[0,1]} |\psi(x)| \int_0^1 \frac{dx_2}{1+\lambda x_1^2 x_2} \leqslant$$
$$\leqslant \frac{\sup_{x_2\in[0,1]} |\psi(x)|}{\lambda x_1^2} \ln(1+\lambda x_1^2 x_2)\Big|_0^1 = \frac{\sup_{x_2\in[0,1]} |\psi(x)| \ln(1+\lambda x_1^2)}{\lambda x_1^2}.$$

So

$$|I_{\alpha,\beta}| \leqslant \int_0^1 \frac{\sup_{x_2\in[0,1]} |\psi(x)| \ln(1+\lambda x_1^2)}{\lambda x_1^2} dx_1 \leqslant C\|\psi\|_{L^\infty} \int_0^1 \frac{\ln(1+\lambda x_1^2)}{\lambda x_1^2} dx_1.$$

Now we use change variables as $\lambda x_1^2 = y$ in the last integral then we have

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^\infty}}{\lambda^{\frac{1}{2}}}\int_0^\lambda \frac{\ln(1+y)}{y^{\frac{3}{2}}}dy \leqslant \frac{C\|\psi\|_{L^\infty}}{\lambda^{\frac{1}{2}}}\int_0^\infty \frac{\ln(1+y)}{y^{\frac{3}{2}}}dy.$$

We consider convergence of the last integral. Using formula of integration by part for the last integral and we obtain

$$\int_0^\infty \frac{\ln(1+y)}{y^{\frac{3}{2}}}dy = -\lim_{N_1\to 0, N_2\to\infty} \frac{2\ln(1+y)}{y^{\frac{1}{2}}}\Bigg|_{N_1}^{N_2} + \int_0^\infty \frac{2dy}{(1+y)y^{\frac{1}{2}}} =$$

$$= \int_0^\infty \frac{4dy^{\frac{1}{2}}}{1+y} = 4\arctan y|_0^\infty = 2\pi.$$

Since the last integral converges and we get for $p\to\infty$

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^\infty}}{\lambda^{\frac{1}{2}}}.$$

Case II. We consider second case form as phase function has critical point of type $D_4^\pm$ so $f(x) = x_1^2x_2 \pm x_2^3$. We estimate integral (4) when phase function has critical point of type D_4^+ the case D_4^- will consider analogically.

We consider following integral

$$I_{\alpha,\beta} = \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda(x_1^2x_2 + x_2^3))\psi(x)dx. \tag{13}$$

Using inequality (10) for the integral (13) we get

$$|I_{\alpha,\beta}| = \left|\int_{[0,1]^2} E_{\alpha,\beta}(i\lambda(x_1^2x_2 + x_2^3))\psi(x)dx\right| \leqslant \int_{[0,1]^2} \left|E_{\alpha,\beta}(i\lambda(x_1^2x_2 + x_2^3))\right| |\psi(x)|\, dx \leqslant$$

$$\leqslant \int_0^1 dx_1 \int_0^1 \frac{|\psi(x)|dx_1}{1+\lambda(x_1^2x_2+x_2^3)} = \int_0^1 dx_1 \int_0^1 \frac{|\psi(x)|dx_1}{1+\lambda x_2^3 + \lambda x_1^2 x_2}.$$

We use Hölder inequality for the last inner integral we obtain

$$|J_{in2}| := \int_0^1 \frac{|\psi(x)|dx_1}{|1+\lambda x_2^3 + \lambda x_2 x_1^2|} \leqslant \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\int_0^1 \frac{dx_1}{|1+\lambda x_2^3+\lambda x_2 x_1^2|^q}\right)^{\frac{1}{q}}.$$

Then using again the Hölder inequality for this integral we have

$$|I_{\alpha,\beta}| \leqslant \left(\int_0^1\int_0^1 |\psi(x)|^p dx_2 dx_1\right)^{\frac{1}{p}} \left(\int_0^1\int_0^1 \frac{dx_1}{|1+\lambda x_2^3+\lambda x_2 x_1^2|^q}dx_2\right)^{\frac{1}{q}}.$$

We change the variables as $x_1 = \left(\dfrac{1+\lambda x_2^3}{\lambda x_2}\right)^{\dfrac{1}{2}} t$ then we have

$$|I_{\alpha,\beta}| \leqslant \|\psi\|_{L^p}\left(\int_0^1\int_0^1 \frac{dx_1dx_2}{|1+\lambda x_2^3+\lambda x_2 x_1^2|^q}\right)^{\frac{1}{q}} = \|\psi\|_{L^p}\left(\int_0^1 \frac{(1+\lambda x_2^3)^{\frac{1}{2}-q}}{(\lambda x_2)^{\frac{1}{2}}}dx_2 \int_0^A \frac{dt}{(1+t^2)^q}\right)^{\frac{1}{q}},$$

where $A = \left(\dfrac{\lambda x_2}{1+\lambda x_2^3}\right)^{\frac{1}{2}}$ and $\int\limits_0^A \dfrac{dt}{(1+t^2)^q} < C$ as $A \to \infty$. So we get

$$|I_{\alpha,\beta}| \leqslant C\|\psi\|_{L^p}\left(\int_0^1 \frac{(1+\lambda x_2^3)^{\frac{1}{2}-q}}{(\lambda x_2)^{\frac{1}{2}}}dx_2\right)^{\frac{1}{q}}.$$

Replacing x_2 by $\lambda^{-\frac{1}{3}}\tau$ and $\dfrac{1}{q} = 1 - \dfrac{1}{p}$ and we get

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}\left(\int_0^{\lambda^{\frac{1}{3}}} \frac{d\tau}{\tau^{\frac{1}{2}}(\tau^3+1)^{q-\frac{1}{2}}}\right)^{\frac{1}{q}} \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}\left(\int_0^{\infty} \frac{d\tau}{\tau^{\frac{1}{2}}(\tau^3+1)^{q-\frac{1}{2}}}\right)^{\frac{1}{q}}.$$

Since the last integral is covergence. Thus

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}.$$

Case III. And now we consider the case when phase function has critical point of type A_2 so $f(x) = x_1^3$. We estimate the integral (4) with phase function $f(x) = x_1^3$

$$|I_{\alpha,\beta}| \leqslant \int_0^1\int_0^1 |E_{\alpha,\beta}(i\lambda x_1^3)||\psi(x)|dx_1dx_2.$$

First we use inequality (10) for the last inner integral we obtain

$$|J_{in3}| := \int_0^1 \frac{|\psi(x)|dx_1}{1+\lambda x_1^3}.$$

Then we use Hölder inequality for the last integral $I_{\alpha,\beta}$ twise and we get:

$$|I_{\alpha,\beta}| \leqslant \left(\int_0^1\int_0^1 |\psi(x)|^p dx_1dx_2\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 \frac{dx_1}{|1+\lambda x_1^3|^q}dx_2\right)^{\frac{1}{q}}.$$

Replacing $\lambda^{-\frac{1}{3}}x_1$ by t in the above inequality, we obtain

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3q}}}\left(\int_0^{\lambda^{\frac{1}{3}}} \frac{dt}{|1+t^3|^q}\right)^{\frac{1}{q}} \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3q}}}\left(\int_0^{\infty} \frac{dt}{|1+t^3|^q}\right)^{\frac{1}{q}}.$$

Since $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and the last integral convergence so we have

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3}-\frac{1}{3p}}}.$$

The proof is complete. □

Remark 1. *If* $\alpha = \beta = 1$ *in the integral* (4) *integral called oscillatory integral and theorem holds for it.*

Declaration of competing interest

This work does not have any conflicts of interest.

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Равномерная оценка для функции Миттаг-Леффлера с гладкой фазой

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Аннотация. В статье рассматривается задача о равномерных оценках функций Миттаг-Леффлера с гладкими фазовыми функциями, имеющими особенности D_∞, $D_4^{\pm}$ и A_r. Обобщение состоит в том, что мы заменяем экспоненциальную функцию функцией типа Миттаг–Леффлера для изучения типа осцилляторного интеграла.

Ключевые слова: функция Миттаг–Леффлера, фаза, функция, амплитуда.