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Contact Mappings of Jet Spaces

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Abstract. In this paper we consider mappings of jet spaces that preserve the module of canonical Pfaffian forms, but are not generally invertible. These mappings are called contact. A lemma on the prolongation of contact mappings is proved. Conditions are found under which these mappings transform solutions of some partial differential equations into solutions of other equations. Examples of contact mappings of differential equations are given. We consider contact mappings depending on a parameter and give example of differential equation invariant under the maps.

Keywords: jets, canonical differential forms, invariant solutions.

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Introduction

As is well known, contact transformations are used to solve problems of classical mechanics and equations of mathematical physics [1–3]. The most known examples of such transformations are the Legendre and Ampere transformations. The theory of contact transformations was developed by S. Lie. At present, there are numerous sources devoted to these issues [4–7]. The contact transformations are diffeomorphisms of the jet space that preserve the contact structure. To integrate differential equations, it is useful to find contact transformations that leave these equations invariant.

However, not only contact transformations are applied to integrate differential equations. Leonhard Euler started using differential substitutions, which are not diffeomorphisms, to integrate linear partial differential equations [8]. Now these substitutions are called the Euler-Darboux transformation [9] or simply the Darboux transformation [10].

In this paper, we consider analytic mappings of jet spaces that preserve the modulus of canonical differential forms and call these mappings contact. We prove a lifting lemma that shows how to construct a contact mapping. For applications to differential equations, the mappings are required to transform solutions of the equations into solutions of other equations or act on solutions of given equations. Examples of second-order partial differential equations connected by contact mappings are given.

We also study contact mappings depending on a parameter. It is easier to look for such mappings in the form of series in powers of the parameter. As an example, we consider the Burgers equation. Parametric contact mappings are found that act on solutions of this equation. These mappings have no inversional maps.

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1. Contact mappings of the jet space

We begin with notation and definitions. Denote by $\mathbb{Z}_{\geqslant 0}$ the non-negative integer numbers and by $\mathbb{N}_n$ the set of natural numbers $1, \ldots, n$. The p-th-order jet space [5] with coordinates $\{x_i, u^j_\alpha : i \in \mathbb{N}_n, j \in \mathbb{N}_m, \alpha \in \mathbb{Z}^n_{\geqslant 0}\}$ is denoted by $J^p(\mathbb{R}^n, \mathbb{R}^m)$ or simply J^p. We suppose that $J^0 = \mathbb{R}^n(x) \times \mathbb{R}^m(u)$.

Denote by J^∞ the space of infinite jets and by π_p the projection $\pi_p : J^\infty \longrightarrow J^p$. Consider a point $a \in J^\infty$ and a ring A^p_a of convergent power series centered at the point $a_p = \pi_p(a)$. We write

$$A_a = \bigcup_{p=0}^{\infty} A^p_a.$$

Recall that an operator D on a ring is called a derivation operator if it satisfies the conditions:

$$D(a + b) = Da + Db, \qquad D(ab) = D(a)b + aD(b)$$

for all elements a, b of the ring. We say that a derivation operator D_k ($k \in \mathbb{N}_n$) on the ring A_a is the total derivative when the following conditions is satisfied

$$D_k(x_i) = \delta_{ik}, \qquad D_k(u^j_\alpha) = u^j_{\alpha+1_k},$$

where δ_{ik} is the Kronecker delta and $1_k = (\delta_{1k}, \ldots, \delta_{nk}) \in \mathbb{Z}^n_{\geqslant 0}$. So A_a is a differential ring with set Δ of derivation operators $D_1, \ldots, D_n$; its elements are called differential power series. Any ideal of A_a stable under Δ is called a differential ideal of A_a. The differential ideal of the ring A_a generated by the set $E \subset A_a$ is denoted by $< E >$.

The set of differentials

$$\{dx_i, du^j_\alpha : i \in \mathbb{N}_n, j \in \mathbb{N}_m; \alpha \in \mathbb{Z}^n_{\geqslant 0}, |\alpha| < p\}$$

generates a left module Ω_{a_p} of differential 1-forms over the ring A_{a_p}. As usual, we say that differential forms

$$\omega^j_\alpha = du^j_\alpha - \sum_{i=1}^{n} D_i(u^j_\alpha)dx_i, \qquad j \in \mathbb{N}_m, \quad \alpha \in \mathbb{Z}^n_{\geqslant 0} \tag{1}$$

are canonical.

Definition. A submodule of the left module Ω_{a_p} generated by canonical forms ω^j_α (where $|\alpha| \leqslant p$) is denoted by $\mathcal{C}^p_a$ and is called the contact submodule.

We describe below a dual transformation of forms [11]. Let $\mathcal{A}(W)$ be the ring of analytic functions on an open set $W \subset \mathbb{R}^k$ and let $\Omega(W)$ be the left module of differential 1-forms on W. Suppose $W_1 \subset \mathbb{R}^{k_1}$, $W_2 \subset \mathbb{R}^{k_2}$ are open sets. Then any analytic mapping $\phi : W_1 \longrightarrow W_2$ induces a homomorphism

$$\phi^\star : \mathcal{A}(W_2) \to \mathcal{A}(W_1), \qquad \phi^\star(f) = f(\phi),$$

and a linear map $\hat{\phi}^\star : \Omega(W_2) \to \Omega(W_1)$, given by

$$\hat{\phi}^\star\left(\sum_{i=1}^{k_2} f_i(y)dy_i\right) = \sum_{i=1}^{k_2} \phi^\star(f_i)d\phi_i,$$

where ϕ_i is component of ϕ. It is convenient to think of $\hat{\phi}^\star$ as a module homomorphism over different rings connected by the homomorphism of rings. The maps $\phi^\star$ and $\hat{\phi}^\star$ are usually not distinguished.

We now generalize the classical contact transformations [3].

Definition. Let U be a neighborhood of a point $a \in J^p$ and let V be a neighborhood of a point $b \in J^q$ ($q \leqslant p$). An analytic mapping $\phi : U \to V$ is called a contact mapping if the module homomorphism $\hat{\phi}^\star$ maps the contact submodule $\mathcal{C}_b^q$ into the contact submodule $\mathcal{C}_a^p$.

The following lifting lemma shows how to construct a contact mapping.

Lemma. *Let U be a neighborhood of the point $a \in J^p$ and let $\phi : U \to J^0$ be an analytic mapping of the form*

$$y = f(x, \dots, u_\alpha), \qquad u = g(x, \dots, u_\alpha), \qquad |\alpha| \leqslant p$$

such that the matrix $Df = (D_i f_j)_{1 \leqslant i,j \leqslant n}$ is invertible at the point a. Then there exists an open set $U^1 \subset J^{p+1}$ and a unique contact mapping $\phi^1 : U^1 \to J^1$ coinciding with ϕ on U.

Proof. In what follows, we use the following notation

$$dx = (dx_1, \dots, dx_n), \quad du_\alpha = (du_\alpha^1, \dots, du_\alpha^m), \quad u_{\alpha+1} = (D_i(u_\alpha^j))_{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m}.$$

According to the definition of a contact mapping, the differential form $dv - v_1 dx$ must be represented as

$$dg - v_1 df = B_0(du - u_1 dx) + \dots + B_p(du_\alpha - u_{\alpha+1} dx), \quad |\alpha| = p, \tag{2}$$

where $B_0, \dots, B_p$ are $m \times m$ matrices.

The left-hand side of equation (2) is written as

$$g_x dx + g_u du + \dots + g_{u_\alpha} du_\alpha - v_1(f_x dx + f_u du + \dots + f_{u_\alpha} du_\alpha),$$

where $g_x = \left(\frac{\partial g_j}{\partial x_i}\right)$, $\left(\frac{\partial g_j}{\partial u^i}\right), \dots, f_{u_\alpha} = \left(\frac{\partial f_j}{\partial u^i}\right)$ are the corresponding Jacobian matrices. We collect together the coefficients of similar differential terms in (2) and set all of them equal to zero. The result is a system of matrix equations

$$g_x - v_1 f_x + B_0 u_1 + \dots + B_p u_{\alpha+1} = 0,$$

$$g_u - v_1 f_u = B_0, \quad \cdots, \quad g_{u_\alpha} - v_1 f_{u_\alpha} = B_p$$

Substituting $B_0, \dots, B_p$ into the first equation of this system, we have

$$Dg = v_1 Df,$$

with matrices $Dg = (D_i g_j)$, $Df = (D_i f_k)$ where $j \in \mathbb{N}_m$ and $i, k \in \mathbb{N}_n$. By the hypotheses of our lemma, the matrix Df is invertible, so the lifting formula (first prolongation) is

$$v_1 = (Dg) \circ (Df)^{-1}. \tag{3}$$

The lifting to $J^2, \dots, J^{k+1}$ is carried out in a similar way. The recurrent formula has the form

$$v_{k+1} = (Dv_k) \circ (Df)^{-1}. \tag{4}$$

These formulas are generalizations of the well-known formulas for the lifting (prolongation) of point transformations [1,9].

Definition. If $E = \{f_i\}_{i \in \mathbb{N}_k}$ is a family of differential series of the ring A_a^p, then the expression

$$f_i = 0, \qquad 1 \leqslant i \leqslant k$$

is called a system of differential equations and denoted by $sys[E]$.

Definition. Let E be a family of differential series in the ring A_a^p and let V be an open set in $\mathbb{R}^n$. We say that a smooth mapping $s : V \to J^p$ annihilates the family E if

$$s^\star(E) = 0, \qquad \hat{s}^\star(\mathcal{C}_a^p) = 0 \tag{5}$$

and the rank of s is n at every point of V. If π_0^p is the projection of J^p onto J^0, then the composition $\pi \circ s$ is called a solution of $sys[E]$.

The mapping s is lifted so that it annihilates the canonical forms (1) as described above. The lifted map is denoted by $\tilde{s}$.

Proposition. *Let E_1, E_2 be two families of differential series of rings A_a^p and A_b^p respectively. Let U be a neighborhood of the point $a_p \in J^p$ and let V be an open set in $\mathbb{R}^n$. Assume that a mapping $s : V \to U$ annihilates E_1 and $\phi : U \to J^q$ $(q \leqslant p)$ is a contact mapping such that $\phi^\star(E_2) \subset < E_1 >$, then $\phi \circ s$ annihilates E_2.*

Proof. Since

$$\phi^\star(E_2) \subset < E_1 >, \tag{6}$$

it is clear that $\tilde{s}^\star(\phi^\star(E_2)) = 0$. It follows that

$$\tilde{s}^\star(E_2(\phi)) = E_2(\phi \circ s) = (\phi \circ s)^\star(E_2) = 0.$$

The equality $(\phi \circ s)^\star(\mathcal{C}_b^p) = 0$ follows in the same way.

Remarks. To put it simply, the contact mapping ϕ maps solutions of $sys[E_1]$ to solutions of $sys[E_2]$ if $\phi^\star(E_2) \subset < E_1 >$. If we extend the homomorphism $\phi^\star$ to the ideal $< E_2 >$, then the condition (6) can be written more invariantly

$$\tilde{\phi}^\star(< E_2 >) \subset < E_1 > .$$

Definition. *Let E be a family of differential series of the ring A_a^p and let U be a neighborhood of the point $a_p \in J^p$. A contact mapping $\phi : U \to J^q$ $(q \leqslant p)$ such that $\phi^\star(E) \subset < E >$ is called a symmetry of $sys[E]$.*

Let us give examples of contact mappings connecting partial differential equations. We now use the classical notation. Consider two equations

$$u_{tt} = x^n u_{xx}, \qquad n \in \mathbb{N}, \tag{7}$$

$$v_{tt} = v_{yy} + \frac{m}{y} v_y, \qquad m \in \mathbb{R}. \tag{8}$$

We want to find a contact mapping that transforms solutions of the equation (7) to solutions of the equation (8). Consider a mapping $\phi : J^1 \to J^0$ of the form

$$t' = t, \qquad y = h(x), \qquad v = f(x)u_x + g(x)u, \tag{9}$$

where h, f, g are some smooth functions. We will lift this mapping according to the formulas (3), (4)

$$v_t = D_t v = f u_{tx} + g u_t, \qquad v_{tt} = f u_{ttx} + g u_{tt},$$
$$v_y = \frac{D_x(v)}{D_x h} = \frac{D_x(f u_x + g u)}{h'}, \qquad v_{yy} = \frac{D_x(v_y)}{h'}. \tag{10}$$

Substituting the found expression for v_{tt} into (8), we have

$$f u_{ttx} + g u_{tt} = v_{yy} + \frac{m}{y} v_y.$$

We can express u_{tt}, u_{ttx} by using (10) and obtain a new equation

$$(x^n u_{xx})_x f + x^n u_{xx} g + \frac{1}{h'} D_x \left(\frac{D_x(f u_x + g u)}{h'} \right) + \frac{m D_x(f u_x + g u)}{h h'} = 0. \tag{11}$$

The left-hand side of this equation is a polynomial in u_{xxx}, u_{xx}, u_x, u. Collecting the coefficients of similar terms in the polynomial and setting all of them equal to zero, we obtain four equations for the functions f, h, g. The two shortest equations are

$$x^n(h')^2 = 1, \qquad m(h')^2g' + hh'g'' - hh''g' = 0.$$

Integrating these equations for $n \neq 2$, we find

$$h = \pm\frac{2}{2-n}x^{\frac{2-n}{2}} + c_0, \qquad g = c_1 + c_2h^{1-m},$$

where c_0, c_1, c_2 are arbitrary constants. The remaining two equations for the function f are easy to integrate. The following two cases arise: $c_1 \neq 0, c_2 = 0$ and $c_1 = 0$, $c_2 \neq 0$. In the first case, the function f is equal to ax ($a \in \mathbb{R}$). Then the transformation

$$y = \pm\frac{2}{2-n}x^{\frac{2-n}{2}}, \qquad v = a(xu_x + (n-1)u), \quad a \in \mathbb{R}$$

maps solutions of the equation (7) into ones of equation

$$v_{tt} = v_{yy} + \frac{3n-4}{2-n}v_y.$$

In the second case, the transformation

$$y = \pm\frac{2}{2-n}x^{\frac{2-n}{2}}, \qquad v = ax^{2n-3}(xu_x + (n-1)u)$$

maps solutions of the equation (7) into ones of equation

$$v_{tt} = v_{yy} + \frac{5n-8}{n-2}v_y.$$

2. Parametric contact mappings

It is well known that finding symmetries of differential equations can be simplified if we restrict ourselves to the search for one-parameter groups of transformations that leave the equations invariant. In this section, it is assumed that contact mappings depend on the parameter a. More precisely, we seek an expansion of the mappings in powers of a.

Next we restrict ourselves to to the case $n = 2$, $m = 1$ and use the classical notation for coordinates in the jet spaces $J^0(x, y, u)$, $J^1(x, y, u, p, q)$,
$J^2(x, y, u, p, q, r, s, t)$.

Consider a mapping of the form

$$\begin{aligned} \bar{x} &= x + ax_1 + a^2x_2 + a^3x_3 + \dots\ , \\ \bar{y} &= y + ay_1 + a^2y_2 + a^3y_3 + \dots\ , \\ \bar{u} &= u + au_1 + a^2u_2 + a^3u_3 + \dots\ , \end{aligned} \tag{12}$$

where $x_1, x_2, x_3, y_1, \dots, u_3$ are functions of $x, y, \dots, u_\alpha$. To find the first prolongation of the mapping (12)

$$\begin{aligned} \bar{p} &= p + ap_1 + a^2p_2 + a^3p_3 + \dots\ , \\ \bar{q} &= q + aq_1 + a^2q_2 + a^3q_3 + \dots\ , \end{aligned}$$

it is necessary that the differential form

$$\bar{\omega}_0 = d\bar{u} - \bar{p}d\bar{x} - \bar{q}d\bar{y} \tag{13}$$

vanishes when the Pfaff equation

$$\omega_0 = du - pdx - qdy = 0 \tag{14}$$

is satisfied.

Substituting the expressions (12) into the form $\bar{\omega}_0$, by using the equality (14), and collecting together all terms that contain a, we obtain the well-known the first prolongation formulas [1]

$$p_1 = D_x(u_1) - pD_x(x_1) - qD_x(y_1), \quad q_1 = D_y(u_1) - pD_y(x_1) - qD_y(y_1).$$

Collecting together all terms that contain a^2, we find that

$$p_2 = D_x(u_2) - pD_x(x_2) - p_1D_x(x_1) - qD_x(y_2) - q_1D_x(y_1),$$
$$q_2 = D_y(u_2) - pD_y(x_2) - p_1D_y(x_1) - qD_y(y_2) - q_1D_y(y_1).$$

It is important to remark that x_2, y_2, u_2 are an arbitrary functions. When we collect together all terms that contain a^3 this leads to

$$p_3 = D_x(u_3) - pD_x(x_3) - p_1D_x(x_2) - p_2D_x(x_1) - qD_x(y_3) - q1D_x(y_2) - q_2D_x(y_1),$$
$$q_3 = D_y(u_3) - pD_y(x_3) - p_1D_y(x_2) - p_2D_y(x_1) - qD_y(y_3) - q_1D_y(y_2) - q_2D_y(y_1).$$

Similar formulas are valid for p_n, q_n $(n > 3)$.

It is easy to find formulas for the second prolongation

$$\bar{r} = r + ar_1 + a^2r_2 + a^3r_3 + \dots\ , \qquad \bar{s} = s + as_1 + a^2s_2 + a^3s_3 + \dots\ ,$$
$$\bar{t} = t + at_1 + a^2t_2 + a^3t_3 + \dots\ .$$

For this to be accomplished, it is necessary that the differential forms

$$\bar{\omega}_{10} = d\bar{p} - \bar{r}d\bar{x} - \bar{s}d\bar{y} \qquad \bar{\omega}_{01} = d\bar{q} - \bar{s}d\bar{x} - \bar{t}d\bar{y}$$

vanish if

$$\omega_{10} = dp - rdx - sdy = 0, \qquad \omega_{01} = dq - sdx - tdy = 0.$$

Using arguments similar to those given above, it is easy to obtain the following formulas

$$r_1 = D_x(p_1) - rD_x(x_1) - sD_x(y_1), \quad s_1 = D_y(p_1) - rD_y(x_1) - sD_y(y_1),$$
$$t_1 = D_y(q_1) - sD_y(x_1) - tD_y(y_1),$$
$$r_2 = D_x(p_2) - rD_x(x_2) - r_1D_x(x_1) - sD_x(y_2) - s_1D_x(y_1),$$
$$s_2 = D_y(p_2) - rD_y(x_2) - r_1D_y(x_1) - sD_y(y_2) - s_1D_y(y_1),$$
$$t_2 = D_y(q_2) - sD_y(x_2) - s_1D_y(x_1) - tD_y(y_2) - t_1D_y(y_1).$$

As example, consider the Burgers equation

$$u_y - u_{xx} - uu_x = 0. \tag{15}$$

We look for contact mappings such that (15) is invariant under the ones. The symmetry condition implies that the expression

$$\bar{u}_{\bar{y}} - \bar{u}_{\bar{x}\bar{x}} - \bar{u}\bar{u}_{\bar{x}}$$

lies in the ideal $< u_t - u_{xx} - uu_x >$.

The simplest of these mappings has the form

$$\bar{x} = x, \qquad \bar{y} = y, \qquad \bar{u} = u + \frac{2au_x}{au + 1}, \quad a \in \mathbb{R}.$$

This mapping satisfies the second-order differential equation

$$\bar{u}_{aa} = 2\frac{\bar{u}_a(a\bar{u}_a - \bar{u})}{a\bar{u} + 1}$$

with initial conditions: $\bar{u}(0) = u, \bar{u}_a(0) = u_x$. Recall that in Lie theory, symmetry transformations satisfy first order ordinary differential equations [1].

A more general symmetry mapping is given by the formulas

$$\bar{x} = x, \qquad \bar{y} = y, \qquad \bar{u} = u + 2D_x(\log h),$$

where the function h satisfies the condition

$$D_y h - D_x^2 h - uD_x h \in < u_y - u_{xx} - uu_x > . \tag{16}$$

More precisely, the following statement is true.

Proposition. *Let u be a solution to the equation (15), and let the differential series h satisfy the condition (16). Then the function*

$$v = u + 2D_x(\log h) \tag{17}$$

is also a solution to the Burgers equation

$$v_t - v_{xx} - vv_x = 0.$$

Indeed, substituting the function v given by (17) into the left-hand side of the last equation, we obtain an expression that can be represented as

$$u_y - u_{xx} - uu_x + 2D_x\left(\frac{D_y h - D_{xx}h - uD_x h}{h}\right).$$

Thus the Proposition follows from (16).

It is important to note that if h satisfies the condition (16), then $\eta = D_x h$ is a solution of the determining equations for the symmetry generator. Therefore, knowing the symmetries of the equation it is easy to find h.

In particular, the condition (16) is satisfied by h of the form

$$h = s_0 + a[s_1(2u_x + u^2) + s_2 u + s_3(yu + x) + s_4(2yu_x + yu^2 + xu) + \\ + s_5(y^2(4u_x + 2u^2) + 2xyu + x^2 + 2y)],$$

where $a, s_0, \ldots, s_5$ are arbitrary constants. If $s_0 \neq 0$, then the function $\bar{u}$ is represented by power series in a. The condition (16) is equivalent to a new determining equation

$$D_y h - D_x^2 h - uD_x h = 0.$$

In this case, the last equation should follow from Eq. (15).

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Контактные отображения пространства джетов

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Аннотация. В работе рассматриваются отображения пространств джетов, сохраняющие контактную структуру — канонические дифференциальные формы Пфаффа. В общем случае они не являются обратимыми, и мы называем их контактными отображениями. Доказывается лемма о поднятии контактных отображений. Найдены условия, гарантирующие, что контактные отображения переводят решения одних уравнений с частными производными в решения других уравнений. Рассматриваются контактные отображения, зависящие от параметра. Приводятся примеры контактных отображений, связывающих решения дифференциальных уравнений, и примеры новых симметрий уравнений.

Ключевые слова: пространства джетов, канонические дифференциальные формы, инвариантные решения.