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Green Function of Quantum Particle Moving in Two-dimensional Annular Potential

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Abstract. In this work, we present a new result which concerns the obtainment of the Green function relative to the time-independent Schrodinger equation in two dimensional space. The system considered in this work is a particle that have an energy E and moves in an axi-symmetrical potential. Precisely, we have assumed that the potential ($V(r)$), in which the particle moves, to be equal to zero inside an annular region (radius b) and to be equal a positive constant (V_0) in a crown of internal radius b and external radius a ($b < a$) and equal zero outside the crown ($r > a$). We have explored the bounded states regime for which ($E < V_0$). We have used, to obtain the Green function, the continuity of the solution and of its derivative at ($r = b$) and ($r = a$): We have obtained the associate Green function and the discrete spectra of the Hamiltonian in the region ($r < b$).

Keywords: quantum mechanics, Schrodinger equation, Green's function, bounded states.

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Introduction

The method of Green's function is considered as an important tool that allows to solve many problems encountered in general physics, mechanics, fluid mechanics, quantum mechanics, acoustics, electromagnetism and mathematical physics etc. The Green function is attributed to the distribution theory that was introduced by Green [1] in electromagnetism, and later used by

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Neuman and by Helmholtz in the theory of Newtonian potential [2] and acoustics consecutively. There are usually several Green functions associated with the same equation. These different functions are distinguished from each other by the boundary conditions. Thus it is important, when we calculate the Green function of the linear differential equation to specify the boundary conditions. Before dealing with the description of our problem we include some works that are closely related to our problem. In [3, 4] the authors have treated the problem of a thin circular Kirchhoff Poisson-plate. The plate edge is assumed to be elastically supported so that the boundary values are that the radial bending moment equals zero, whereas the strength is proportional to the function of the deflection on the boundary. The Green function is also studied by [5] in circular, annular and exterior circular domain. In [6, 7] the Green function was studied for the elliptic domain. The quantum problem relative to the scattering in two dimensions was also treated in [8], and the problem of the Dirac particle in a spherical scalar potential well in 3-D was treated by [9, 10]. In [11–14] the Green function problem is treated in an approximative context. In our work, we address the problem of the Schrodinger equation in two dimensions: the Shrodinger operator is defined to be piecewise operator on three connected circular domains ($0 < r \leqslant b$; $b \leqslant r \leqslant a$; $a \leqslant r < \infty$) but with new boundary conditions. These boundaries conditions are useful in quantum mechanics to solve the diffusion problems and also the bound states. In quantum mechanics, if the potential is constant in the crown and is zero outside (or vice versa) the solution of the Schroedinger equation and the derivative of the solution are continuous on the boundary (the edge) of the crown. Specify clearly our problem: the Schrodinger equation takes different forms depending on whether it is inside the crown ($b \leqslant r \leqslant a$) or outside. This type of problem matches in quantum mechanics to the study of a particle subjected to a potential which is a positive constant inside the crown ($b \leqslant r \leqslant a$) and zero outside the crown. None of the cited works, in our knowledge, the explicit Green's function for a piecewise continuous potential has been calculated in two dimensions for this type of problem. The physical phenomenon that we want to describe in this work is related to the well-known tunnel effect in one dimension, by extending it to two dimensions. It is therefore, a question of studying the propagation of the waves associated with particles (electrons for example) emitted from a source that is located at the space origin, in a homogeneous two-dimensional medium. During propagation, the particles (waves) enter a coronal region (barrier) in which they are subjected to a constant potential V_0. Then they cross this region to go to infinity (r tends to infinity). To be clear and more precise, the particles cannot cross this region where the potential V_0 reigns if their energy E is less than V_0. This reasoning is purely classical. In quantum mechanics, we show that the particle, even for $E < V_0$, has a non-zero probability (which is proportional to the Green function) to cross this coronal region. So our paper will be organized as it follows: in the next section (Sec. 1), we expose the problem we will solve. In section four (Sec. 2), we will calculate the Green function for the bounded states. It turns out that the energies spectra is obtained from the poles of the Green function in the region $r < b$. We end our paper by a conclusion in Sec. 3.

1. Axi-symmetric two dimensional quantum problem

Consider a quantum particle moving in a symmetrical potential (independent of the angle θ) defined as (see Fig. 1):

$$V(r,\theta) = \begin{cases} 0 & 0 \leqslant r \leqslant b \quad \text{region1} \\ V_0 & b \leqslant r \leqslant a \quad \text{region2} \\ 0 & r \geqslant a \quad \text{region3} \end{cases} . \tag{1}$$

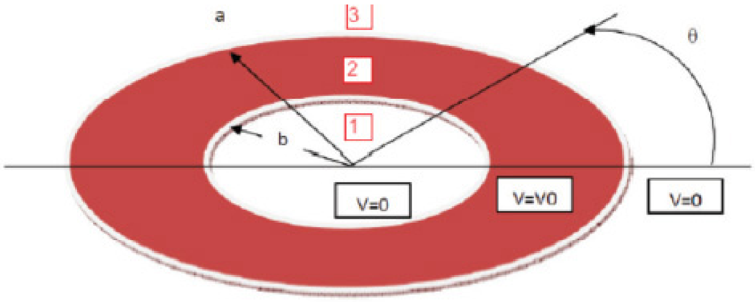

Fig. 1. A scheme of the coronal potential in two dimensions

The dynamics of this particle is governed by the time-independent Schroedinger equation:

$$\hat{H}\left(r,\theta\right)\Psi(r,\theta)=E\Psi\left(r,\theta\right) \tag{2}$$

which is written in the natural polar coordinates (r,θ) and where $\hat{H}\ (r,\theta)$ is the hamiltonien of the particle with a mass M, moving in this potential. The equation (2) is merely an eigenvalues E and eigenfuntions equation $\Psi\left(r,\theta\right)$. The explicit form of the hamiltonien of the system is:

$$\hat{H}=-\frac{\hbar^2}{2M}\triangle_{r,\theta}+V\left(r,\theta\right) \tag{3}$$

where

$$\triangle_{r,\theta}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{4}$$

is the well known laplacian in polar coordinates. The equation (2) writes as:

$$\left(-\frac{\hbar^2}{2M}\triangle_{r,\theta}+V\left(r,\theta\right)-E\right)\Psi\left(r,\theta\right)=0 \tag{5}$$

or, with respect of the definition of $V\left(r,\theta\right)$ in the formula (1)

$$\begin{cases}\left(\frac{\hbar^2}{2M}\triangle_{r,\theta}+E\right)\Psi_3\left(r,\theta\right)=0 & r>a\\ \left(\frac{\hbar^2}{2M}\triangle_{r,\theta}-V_0+E\right)\Psi_2\left(r,\theta\right)=0 & b\leqslant r\leqslant a\\ \left(\frac{\hbar^2}{2M}\triangle_{r,\theta}+E\right)\Psi_1\left(r,\theta\right)=0 & 0\leqslant r\leqslant b\end{cases}. \tag{6}$$

This system is subjected to the boundary conditions defined as $\Psi\left(r,\theta\right)$ and $\frac{d}{dr}\Psi\left(r,\theta\right)$ are to be continous at $r=b$ and $r=a$ for all values of the azimutal angle θ. The separation variables method leads to transform the last equations (6) as

$$\begin{cases}\frac{d}{dr}\left(r\frac{d}{dr}\Psi_3\right)+\left(\frac{2M}{\hbar^2}Er-\frac{l^2}{r}\right)\Psi_3(r)=0 & r>a\\ \frac{d}{dr}\left(r\frac{d}{dr}\Psi_2\right)+\left(\frac{2M}{\hbar^2}(E-V_0)r-\frac{l^2}{r}\right)\Psi_2(r)=0 & b\leqslant r\leqslant a\\ \frac{d}{dr}\left(r\frac{d}{dr}\Psi_1\right)+\left(\frac{2M}{\hbar^2}Er-\frac{l^2}{r}\right)\Psi_1(r)=0 & 0\leqslant r\leqslant b\end{cases} \tag{7}$$

whose solutions are combination of two linear independent Bessel's functions of order l ($l \in Z$). The solution must obey to the boundary conditions at $r = b$ and $r = a$

$$\Psi_3(a) = \Psi_2(a), \quad \left(\frac{d\Psi_3(r)}{dr}\right)_{r=a} = \left(\frac{d\Psi_2(r)}{dr}\right)_{r=a} \tag{8}$$

and

$$\Psi_2(b) = \Psi_1(b), \quad \left(\frac{d\Psi_2(r)}{dr}\right)_{r=b} = \left(\frac{d\Psi_1(r)}{dr}\right)_{r=b} \tag{9}$$

where $l = \cdots - 2, -1, 0, +1, +2, \ldots$. The Green function of the problem (7) augmented by the boundary conditions (8–11) is given by

$$G(\overrightarrow{r}, \overrightarrow{r}', E) = G(r, \theta, r', \theta', E) = \sum_{l=-\infty}^{+\infty} G(l; r, r', E) \exp(il(\theta - \theta')) \tag{10}$$

where $G(l; r, r', E) \equiv G(l; r, r')$ is the radial Green function that we shall calculate in the subsequent sections. To calculate the Green function we will study the case $0 < E < V_0$ for which corresponds the bounded states regime. In our next investigation, we follow the Krasnov approach to calculate Green function [15, 16].

2. The bounded states regime 0<E<V_0

We consider in this work that the particle is described by a wave that comes from the origin O (source O) and spreads out in two dimensions space. It crosses a coronal region (region2) and escape to infinity (region3).

1. The region $a \leqslant (r, r') < \infty$

By using the first equation of (7), in the region3 ($r > a$), the corresponding radial Green function can be written as the following

$$G^{3,3}(l : r, r') = \begin{cases} C_3(r')[Y_l(kr) - B_3(r')J_l(kr)] & a \leqslant r \leqslant r' \\ A_3(r')\, J_l(kr) & r' \leqslant r < \infty \end{cases} \tag{11}$$

where $k^2 = \dfrac{2M}{\hbar^2}E$. the continuity of the Green function at $r = r'$

$$G^{3,3}\left(l; r'_+, r'\right) - G^{3,3}\left(l; r'_-, r'\right) = 0$$

gives

$$[A_3(r') + B_3(r')C_3(r')]\, J_l(kr') - C_3(r')\, Y_l(kr') = 0 \tag{12}$$

and the discontinuity of the first derivative with respect r at $r = r'$, gives:

$$\frac{d}{dr}G^{3,3}\left(l; r'_+, r'\right) - \frac{d}{dr}G^{3,3}\left(l; r'_-, r'\right) = \frac{2}{\pi r'}$$

therefore

$$-C_3(r')\, Y'_l(kr') + [A_3(r') + B_3(r')C_3(r')]\, J'_l(kr') = \frac{2}{\pi k r'}. \tag{13}$$

By following (12) and (13) we check that

$$A_3(r') + B_3(r')C_3(r') = -Y_l(kr'). \tag{14}$$

After substituting (14) in (12):

$$C_3(r')\,Y_l(kr') + Y_l(kr')\,J_l(kr') = 0 \tag{15}$$

we find:

$$C_3(r') = -J_l(kr') \tag{16}$$

and after replacing (16) in (14) we check that

$$A_3(r') = B_3(r')J_l(kr') - Y_l(kr') \tag{17}$$

and after substituting (16) and (17) in (11) we find

$$G^{3,3}(l:r,r') = -\begin{cases} J_l(kr')\,[Y_l(kr) - B_3(r')J_l(kr)] & a \leqslant r \leqslant r' \\ [Y_l(kr') - B_3(r')J_l(kr')]\,J_l(kr) & r' \leqslant r < \infty \end{cases}. \tag{18}$$

It remains to determine the coefficient $B_3(r')$. To do this, we use the symmetry properties of $G(l:r,r')$ which states that, by reversing the roles of r and r' in the first expression of $G^{3,3}(l:r,r')$, we must find the second expression that is to say

$$[Y_l(kr') - B_3(r')J_l(kr')]J_l(kr) = J_l(kr)[Y_l(kr') - B_3(r)J_l(kr')]. \tag{19}$$

By identifying in the last equation we find

$$B_3(r') = B_3(r) = B_3 = constant. \tag{20}$$

Then the Green function in this region $(r > a)$ is given by:

$$G^{3,3}(l;r,r') = -\begin{cases} J_l(kr')\,[Y_l(kr) - B_3 J_l(kr)] & a \leqslant r \leqslant r' \\ [Y_l(kr') - B_3 J_l(kr')]\,J_l(kr) & r' \leqslant r < \infty \end{cases}. \tag{21}$$

The constant B_3 will be determined later.

2. The region $b \leqslant (r, r') \leqslant a$

To highlight, in the region 2, that there are forward and backward waves, the Green's function in the region2 is written as

$$G^{2,2}(l;r,r') = \begin{cases} E_2(r')\,[K_l(\mu r) - \delta_2(r')I_l(\mu r)] & b \leqslant r \leqslant r' \\ F_2(r')\,[K_l(\mu r) - \gamma_2(r')\,I_l(\mu r)] & r' \leqslant r < a \end{cases}$$

where: $\mu^2 = \dfrac{2M}{\hbar^2}(V_0 - E)$. To calculate the coefficients $E_2(r')$, $F_2(r')$, $\gamma_2(r')$ and $\delta_2(r')$ we use the continuity of the Green function at $r = r'$:

$$G^{2,2}\left(l; r'_+, r'\right) - G^{2,2}\left(l; r'_-, r'\right) = 0$$

then

$$K_l(\mu r')\,[F_2(r') - E_2(r')] - I_l(\mu r')\,[\gamma_2(r')\,F_2(r') - \delta_2(r')E_2(r')] = 0 \tag{22}$$

and the use of the discontinuity of the first derivative with respect r at $r = r'$ gives

$$\frac{d}{dr}G^{2,2}\left(l;r'_{+},r'\right)-\frac{d}{dr}G^{2,2}\left(l;r'_{-},r'\right)=\frac{2}{\pi r'}$$

then

$$K'_{l}\left(\mu r'\right)\left[F_{2}\left(r'\right)-E_{2}\left(r'\right)\right]-I'_{l}\left(\mu r'\right)\left[\gamma_{2}\left(r'\right)F_{2}\left(r'\right)-\delta_{2}(r')E_{2}\left(r'\right)\right]=\frac{2}{\pi\mu r'}. \tag{23}$$

By combining (22) and (23) it is easy to obtain:

$$F_{2}\left(r'\right)=\frac{E_{2}\left(r'\right)\left[K_{l}\left(\mu r'\right)-\delta_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}{\left[K_{l}\left(\mu r'\right)-\gamma_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}$$

and

$$\begin{gathered}K'_{l}\left(\mu r'\right)\left[\frac{E_{2}\left(r'\right)\left[K_{l}\left(\mu r'\right)-\delta_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}{\left[K_{l}\left(\mu r'\right)-\gamma_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}-E_{2}\left(r'\right)\right]-I'_{l}\left(\mu r'\right)\times\\ \times\left[\gamma_{2}\left(r'\right)\frac{E_{2}\left(r'\right)\left[K_{l}\left(\mu r'\right)-\delta_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}{\left[K_{l}\left(\mu r'\right)-\gamma_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}-\delta_{2}(r')E_{2}\left(r'\right)\right]=\frac{2}{\pi\mu r'}.\end{gathered} \tag{24}$$

By using the Bessel Wronksian for the pair $(I_l(\mu r), K_l(\mu r))$

$$W\left(I_{l}\left(\mu r'\right),K_{l}\left(\mu r'\right)\right)=I_{l}\left(\mu r'\right)K'_{l}\left(\mu r'\right)-K_{l}\left(\mu r'\right)I'_{l}\left(\mu r'\right)=\frac{1}{\mu r'} \tag{25}$$

we get the coefficients:

$$E_{2}\left(r'\right)=\frac{2\left[K_{l}\left(\mu r'\right)-\gamma_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}{\pi g_{2}(r')} \tag{26}$$

where

$$g_{2}(x)=\gamma_{2}\left(x\right)-\delta_{2}\left(x\right) \tag{27}$$

and

$$F_{2}\left(r'\right)=\frac{2\left[K_{l}\left(\mu r'\right)-\delta_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]}{\pi g_{2}(r')}. \tag{28}$$

Then, the Green function in the region $b \leqslant r, r' \leqslant a$ is given by:

$$G^{2,2}\left(l;r,r'\right)=\frac{2}{\pi g_{2}(r')}\begin{cases}\left[K_{l}\left(\mu r'\right)-\gamma_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]\left[K_{l}\left(\mu r\right)-\delta_{2}\left(r'\right)I_{l}\left(\mu r\right)\right]\\ \left[K_{l}\left(\mu r'\right)-\delta_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]\left[K_{l}\left(\mu r\right)-\gamma_{2}\left(r'\right)I_{l}\left(\mu r\right)\right]\end{cases}$$

for $b \leqslant r \leqslant r' \leqslant a$ and $b \leqslant r' \leqslant r \leqslant a$ respectively.
It remains to determine the coefficients $\delta_2(r')$, $\gamma_2(r')$ and $g_2(r')$. To do this, we use the symmetry properties of $G(l;r,r')$

$$G^{2,2}(l;r,r')=G^{2,2}(l;r',r)$$

then

$$\begin{gathered}\left[K_{l}\left(\mu r'\right)-\gamma_{2}\left(r'\right)I_{l}\left(\mu r'\right)\right]\left[K_{l}\left(\mu r\right)-\delta_{2}\left(r'\right)I_{l}\left(\mu r\right)\right]=\\ =\left[K_{l}\left(\mu r\right)-\delta_{2}\left(r\right)I_{l}\left(\mu r\right)\right]\left[K_{l}\left(\mu r'\right)-\gamma_{2}\left(r\right)I_{l}\left(\mu r'\right)\right].\end{gathered} \tag{29}$$

By identifying in the last equation we find

$$\delta_{2}\left(r\right)=\delta_{2}\left(r'\right)=\delta_{2}=constant, \tag{30}$$

$$\gamma_{2}\left(r\right)=\gamma_{2}\left(r'\right)=\gamma_{2}=constant,$$

$$g_{2}\left(r\right)=g_{2}\left(r'\right)=g_{2}=constant. \tag{31}$$

These constants we must to determine later.

3. The coefficients γ_2 and δ_2 determination

To find the coefficients γ_2 and δ_2 we use the continuity of the Green function and the continuity of its derivative at $r = a$:

$$G^{3,3}(l;r,a)\rfloor_{r=a} = G^{2,2}(l;r,a)\rfloor_{r=a}$$

then

$$\frac{2}{\pi g_2}\left[K_l(\mu a) - \delta_2 I_l(\mu a)\right]\left[K_l(\mu a) - \gamma_2 I_l(\mu a)\right] = -J_l(ka)\left[Y_l(ka) - B_3 J_l(ka)\right] \tag{32}$$

and

$$\frac{d}{dr}G^{3,3}(l;r,a)\rfloor_{r=a} = \frac{d}{dr}G^{2,2}(l;r,a)\rfloor_{r=a}$$

then

$$\frac{2\mu}{\pi g_2}\left[K_l'(\mu a) - \gamma_2 I_l'(\mu a)\right]\left[K_l(\mu a) - \delta_2 I_l(\mu a)\right] = kJ_l(ka)\left[B_3 J_l'(ka) - Y_l'(ka)\right]. \tag{33}$$

By dividing (33) over (32)

$$\frac{\mu\left[K_l'(\mu a) - \gamma_2 I_l'(\mu a)\right]}{\left[K_l(\mu a) - \gamma_2 I_l(\mu a)\right]} = \frac{k\left[Y_l'(ka) - B_3 J_l'(ka)\right]}{\left[Y_l(ka) - B_3 J_l(ka)\right]} \tag{34}$$

and after simplifications we get the coefficient

$$\gamma_2 = V_2(k,\mu,a,b)/U_2(k,\mu,a,b) \tag{35}$$

such that

$$V_2 = kK_l(\mu a)[Y_l'(ka) - B_3 J_l'(ka)] - \mu K_l'(\mu a)[Y_l(ka) - B_3 J_l(ka)], \tag{36}$$

$$U_2 = kI_l(\mu a)\left[Y_l'(ka) - B_3 J_l'(ka)\right] - \mu I_l'(\mu a)\left[Y_l(ka) - B_3 J_l(ka)\right]. \tag{37}$$

Using the fact $g_2 = \gamma_2 - \delta_2$, and by using (33) we obtain the constant δ_2 as

$$\delta_2 = \frac{2\mu K_l(\mu a)\left[K_l'(\mu a) - \gamma_2 I_l'(\mu a)\right] + k\pi\gamma_2 J_l(ka)\left[Y_l'(ka) - B_3 J_l'(ka)\right]}{2\mu I_l(\mu a)\left[K_l'(\mu a) - \gamma_2 I_l'(\mu a)\right] + k\pi J_l(ka)\left[Y_l'(ka) - B_3 J_l'(ka)\right]} \tag{38}$$

and by replacing (35) in (38) it is easy to see that

$$\delta_2 = \frac{2\mu K_l(\mu a)\left[U_2 K_l'(\mu a) - V_2 I_l'(\mu a)\right] + k\pi V_2 J_l(ka)\left[Y_l'(ka) - B_3 J_l'(ka)\right]}{2\mu I_l(\mu a)\left[U_2 K_l'(\mu a) - V_2 I_l'(\mu a)\right] + k\pi U_2 J_l(ka)\left[Y_l'(ka) - B_3 J_l'(ka)\right]}. \tag{39}$$

The last formula can be more simplified. To check this statment, it suffices to replace U_2 and V_2 by their above expressions and the result is straightly obtained

$$U_2 K_l'(\mu a) - V_2 I_l'(\mu a) = \frac{k\left[Y_l'(ka) - B_3 J_l'(ka)\right]}{\mu a}. \tag{40}$$

This term factorizes in all terms of δ_2 as

$$\delta_2 = \frac{2kK_l(\mu a)\left[Y_l'(ka) - B_3 J_l'(ka)\right] + ka\pi V_2 J_l(ka)\left[Y_l'(ka) - B_3 J_l'(ka)\right]}{2kI_l(\mu a)\left[Y_l'(ka) - B_3 J_l'(ka)\right] + ka\pi U_2 J_l(ka)\left[Y_l'(ka) - B_3 J_l'(ka)\right]} \tag{41}$$

and we get the wanted expression of δ_2

$$\delta_2 \equiv \delta_2(k,\mu,a,b) = \frac{2K_l(\mu a) + \pi a J_l(ka)\,V_2(k,\mu,a,b)}{2I_l(\mu a) + \pi a J_l(ka)\,U_2(k,\mu,a,b)} \tag{42}$$

from which we find the Green function in the region2

$$G^{2,2}(l;r,r') = \frac{2}{\pi g_2(k,\mu,a,b)} \begin{cases} \left[K_l(\mu r') - \dfrac{2K_l(\mu a) + \pi a J_l(ka) V_2(k,\mu,a,b)}{2I_l(\mu a) + \pi a J_l(ka) U_2(k,\mu,a,b)} I_l(\mu r')\right] \times \\ \times \left[K_l(\mu r) - \dfrac{V_2(k,\mu,a,b)}{U_2(k,\mu,a,b)} I_l(\mu r)\right] \\ r \longleftrightarrow r' \end{cases} \tag{43}$$

4. The region $0 \leqslant (r,r') \leqslant b$

In this region, there are forward and backward waves. Then, the Green function can be written as:

$$G^{1,1}(l;r,r') = \begin{cases} A(r')(Y_l(kr) + q(r') J_l(kr)) & 0 \leqslant r \leqslant r' \\ B_1(r')[Y_l(kr) - \alpha_1(r') J_l(kr)] & r' \leqslant r \leqslant b \end{cases}.$$

Because $Y_l(kr)$ diverges at $r=0$, we must discard it from the first combinaison, then the Green's function is

$$G^{1,1}(l;r,r') = \begin{cases} A_1(r') J_l(kr) & 0 < r \leqslant r' \\ B_1(r')[Y_l(kr) - \alpha_1(r') J_l(kr)] & r' \leqslant r \leqslant b \end{cases} \tag{44}$$

where: $k^2 = \dfrac{2M}{\hbar^2}E$. To calculate the coefficients $A_1(r')$, $B_1(r')$ and $\alpha_1(r')$, we use the continuity of the Green function at $r = r'$:

$$G^{1,1}(l;r'_+,r') - G^{1,1}(l;r'_-,r') = 0$$

then

$$B_1(r') Y_l(kr') - [A_1(r') + \alpha_1(r') B_1(r')] J_l(kr') = 0 \tag{45}$$

and we use the discontinuity of the first derivative with respect r at $r = r'$:

$$\frac{d}{dr}G^{1,1}(l;r'_+,r') - \frac{d}{dr}G^{1,1}(l;r'_-,r') = \frac{2}{\pi r'}$$

then

$$B_1(r') Y'_l(kr') - [A_1(r') + \alpha_1(r') B_1(r')] J'_l(kr') = \frac{2}{\pi k r'}. \tag{46}$$

By combining (45) and (46) we obtain

$$A_1(r') = \frac{B_1(r')[Y_l(kr') - \alpha_1(r') J_l(kr')]}{J_l(kr')} \tag{47}$$

and

$$B_1(r') \frac{[Y'_l(kr') J_l(kr') - Y_l(kr') J'_l(kr')]}{J_l(kr')} = \frac{2}{\pi k r'}. \tag{48}$$

Using the Bessel Wronksian for the pair $(J_l(kr), Y_l(kr))$

$$W(J_l(kr'), Y_l(kr')) = J_l(kr') Y'_l(kr') - Y_l(kr') J'_l(kr') = \frac{2}{\pi k r'} \tag{49}$$

we get the coefficients

$$B_1(r') = J_l(kr') \tag{50}$$

and

$$A_1\left(r'\right)=\left[Y_l\left(kr'\right)-\alpha_1(r')J_l\left(kr'\right)\right]. \tag{51}$$

Then, the Green function in this region ($r \leqslant b$) is given by:

$$G^{1,1}\left(l;r,r'\right)=\begin{cases}\left[Y_l\left(kr'\right)-\alpha_1(r')J_l\left(kr'\right)\right]J_l\left(kr\right) & 0<r\leqslant r'\\ \left[Y_l\left(kr\right)-\alpha_1(r')J_l\left(kr\right)\right]J_l\left(kr'\right) & r'\leqslant r\leqslant b\end{cases}. \tag{52}$$

It remains to determine the coefficient $\alpha_1(r')$. To do this, we use the symmetry properties of $G(l;r,r')$

$$G^{1,1}(l;r,r')=G^{1,1}(l;r',r),$$

$$[Y_l(kr')-\alpha_1(r')J_l(kr')]J_l(kr)=[Y_l(kr')-\alpha_1(r)J_l(kr')]J_l(kr).$$

By identifying in the last equation we find

$$\alpha_1(r')=\alpha_1(r)=\alpha_1=constant. \tag{53}$$

Then the Green function in the region1 is given by:

$$G^{1,1}\left(l;r,r'\right)=\begin{cases}\left[Y_l\left(kr'\right)-\alpha_1J_l\left(kr'\right)\right]J_l\left(kr\right) & 0<r\leqslant r'\\ \left[Y_l\left(kr\right)-\alpha_1J_l\left(kr\right)\right]J_l\left(kr'\right) & r'\leqslant r\leqslant b\end{cases}$$

the constant α_1 must be determined in the following subsection.

5. The coefficient α_1 determination

To find the coefficient α_1, we use the continuity of the Green function and the continuity of its derivative at $r=b$:

$$G^{1,1}\left(l;r,b\right)=G^{2,2}\left(l;r,b\right)$$

then

$$\begin{aligned}&\left[K_l\left(\mu b\right)-\frac{2Y_l\left(\mu a\right)+\pi aJ_l\left(ka\right)V_2(k,\mu,a,b)}{2J_l\left(\mu a\right)+\pi aJ_l\left(ka\right)U_2(k,\mu,a,b)}I_l\left(\mu b\right)\right]\times\\ &\times\left[\frac{K_l\left(\mu b\right)}{g_2(k,\mu,a,b)}-\frac{V_2(k,\mu,a,b)}{U_2(k,\mu,a,b)}\frac{I_l\left(\mu b\right)}{g_2(k,\mu,a,b)}\right]=-\left[Y_l\left(kb\right)-\alpha_1J_l\left(kb\right)\right]J_l\left(kb\right)\end{aligned} \tag{54}$$

and

$$\frac{d}{dr}G^{1,1}\left(l;r,b\right)\rfloor_{r=b}=\frac{d}{dr}G^{2,2}\left(l;r,b\right)\rfloor_{r=b}$$

or:

$$\begin{aligned}&\frac{\mu}{g_2(k,\mu,a,b)}\left[K_l'\left(\mu b\right)-\frac{2K_l\left(\mu a\right)+\pi aJ_l\left(ka\right)V_2(k,\mu,a)}{2I_l\left(\mu a\right)+\pi aJ_l\left(ka\right)U_2(k,\mu,a)}I_l'\left(\mu b\right)\right]\times\\ &\times\left[K_l\left(\mu b\right)-\gamma_2(k,\mu,a,b)I_l\left(\mu b\right)\right]=-k\left[Y_l\left(kb\right)-\alpha_1J_l\left(kb\right)\right]J_l'\left(kb\right).\end{aligned} \tag{55}$$

By dividing (55) and (54) we obtain

$$\frac{\mu\left[K_l'\left(\mu b\right)-\frac{2K_l(\mu a)+\pi aJ_l(ka)V_2(k,\mu,a,b)}{2I_l(\mu a)+\pi aJ_l(ka)U_2(k,\mu,a,b)}I_l'\left(\mu b\right)\right]}{\left[K_l\left(\mu b\right)-\frac{2K_l(\mu a)+\pi aJ_l(ka)V_2(k,\mu,a,b)}{2I_l(\mu a)+\pi aJ_l(ka)U_2(k,\mu,a,b)}I_l\left(\mu b\right)\right]}=\frac{kJ_l'\left(kb\right)}{J_l\left(kb\right)} \tag{56}$$

after replacing U_2 and V_2 in (56) by their expressions (36) and (37), we find

$$B_3(k,\mu,a,b) = \Gamma(k,\mu,a,b)/\Phi(k,\mu,a,b) \tag{57}$$

where

$$\begin{aligned}\Gamma(k,\mu,a,b) &= \mu\pi a J_l(ka)Y_l(ka)[\mu J_l(kb)\Phi_2(k,\mu,a,b) + kJ_l'(kb)\Gamma_2(k,\mu,a,b)]+\\ &+ \mu J_l(kb)[2 + k\pi a J_l(ka)Y_l'(ka)]\Gamma_1(k,\mu,a,b)+\\ &+ kJ_l'(kb)[2 + k\pi a J_l(ka)J_l'(ka)]\Phi_1(k,\mu,a,b)\end{aligned} \tag{58}$$

and

$$\begin{aligned}\Phi(k,\mu,a,b) &= \pi a k\mu J_l\,(ka) \times\\ &\times [J_l'\,(ka)\,J_l\,(kb)\,\Gamma_1(k,\mu,a,b) - J_l\,(ka)\,J_l'\,(kb)\,\Gamma_2(k,\mu,a,b)] - \pi a J_l\,(ka) \times\\ &\times \left[\mu^2 J_l\,(ka)\,J_l\,(kb)\,\Phi_2(k,\mu,a,b) + k^2 J_l'\,(ka)\,J_l'\,(kb)\,\Phi_1(k,\mu,a,b)\right]\end{aligned} \tag{59}$$

such that

$$\Gamma_1(k,\mu,a,b) = I_l\,(\mu a)\,K_l'\,(\mu b) - K_l\,(\mu a)\,I_l'\,(\mu b)\,, \tag{60}$$
$$\Gamma_2(k,\mu,a,b) = I_l\,(\mu b)\,K_l'\,(\mu a) - K_l\,(\mu b)\,I_l'\,(\mu a)\,, \tag{61}$$
$$\Phi_1(k,\mu,a,b) = I_l\,(\mu a)\,K_l\,(\mu b) - K_l\,(\mu a)\,I_l\,(\mu b)\,, \tag{62}$$
$$\Phi_2\,(a,b,k,\mu) = I_l'\,(\mu b)\,K_l'\,(\mu a) - K_l'\,(\mu b)\,I_l'\,(\mu a)\,. \tag{63}$$

After a minor simplifications we get the coefficient α_1 equal to

$$\begin{aligned}\alpha_1(k,\mu,a,b) &= \left[\frac{K_l\,(\mu b)}{g_2(k,\mu,a,b)} - \frac{V_2(k,\mu,a,b)}{U_2(k,\mu,a,b)}\frac{I_l\,(\mu b)}{g_2(k,\mu,a,b)}\right] \times\\ &\times \frac{[K_l\,(\mu b) - \delta_2(k,\mu,a,b)I_l\,(\mu b)]}{J_l^2\,(kb)} + \frac{Y_l\,(kb)}{J_l\,(kb)} = \psi_1(k,\mu,a,b) + \frac{Y_l\,(kb)}{J_l\,(kb)}\end{aligned} \tag{64}$$

such that

$$\begin{aligned}\psi_1(k,\mu,a,b) &= \left[\frac{K_l\,(\mu b)}{g_2(k,\mu,a,b)} - \frac{V_2(k,\mu,a,b)}{U_2(k,\mu,a,b)}\frac{I_l\,(\mu b)}{g_2(k,\mu,a,b)}\right] \times\\ &\times \frac{[K_l\,(\mu b) - \delta_2(k,\mu,a,b)I_l\,(\mu b)]}{J_l^2\,(kb)}.\end{aligned} \tag{65}$$

Finally, the Green function in this region ($r \leqslant b$) is given by:

$$G^{1,1}\,(l;r,r') = \begin{cases} \left[Y_l\,(kr') - \left[\psi_1\,(a,b,k,\mu) + \dfrac{Y_l\,(kb)}{J_l\,(kb)}\right] J_l\,(kr')\right] J_l\,(kr) & 0 < r \leqslant r' \\ \left[Y_l\,(kr) - \left[\psi_1\,(a,b,k,\mu) + \dfrac{Y_l\,(kb)}{J_l\,(kb)}\right] J_l\,(kr)\right] J_l\,(kr') & r' \leqslant r \leqslant b \end{cases}. \tag{66}$$

The energies spectra corresponding to this case ($E < V_0$) is determined by the poles (see Fig. 2) of Green's function in this region, that is to say by the poles of $\psi_1(k,\mu,a,b)$ that is to say $g_2(k,\mu,a,b) = 0$ or

$$\gamma_2(k,\mu,a,b) = \delta_2(k,\mu,a,b). \tag{67}$$

From (35,42) and (36,37) we respectively get

$$K_l(\mu a)\, U_2(k,\mu,a,b) = I_l(\mu a)\, V_2(k,\mu,a,b), \tag{68}$$

$$B_3(a,b,k,\mu) = Y_l(ka)/J_l(ka) \tag{69}$$

and from (57) we obtain

$$Y_l(ka)\, \Phi(k,\mu,a,b) = J_l(ka)\, \Gamma(k,\mu,a,b) \tag{70}$$

where $\Gamma(k,\mu,a,b)$ and $\Phi(k,\mu,a,b)$ are given above (58) and (59). Finally, once we have find B_3, the Green's function in the region $(r > a)$ is given by:

$$G^{3,3}(l;r,r') = -\begin{cases} J_l(kr')\left[Y_l(kr) - \dfrac{Y_l(ka)}{J_l(ka)} J_l(kr)\right] & a \leqslant r \leqslant r' \\ \left[Y_l(kr') - \dfrac{Y_l(ka)}{J_l(ka)} J_l(kr')\right] J_l(kr) & r' \leqslant r < \infty \end{cases}.$$

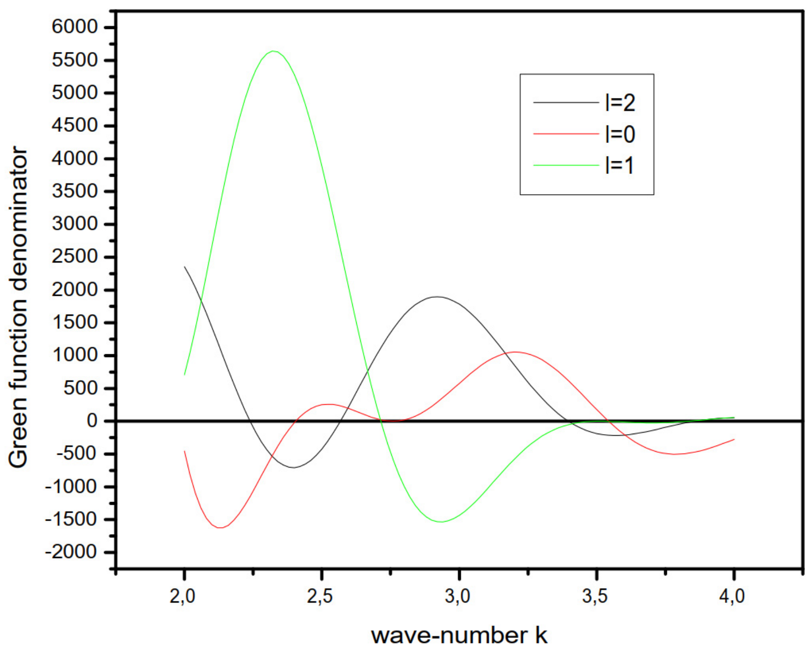

Fig. 2. The energies spectra, given by the intersection ef the curve with k-axis for different angular momentum l=0,1,2

Conclusion

In this work, we have calculated the Green function for the time-independent Schrodinger equation in two dimensional space. The system considered in this work is a particle that have an energy E and moves in an axi-symmetrical potential. We have assumed that the Hamiltonian operator is a piecewise continue operator: the potential $V(r)$, in which the particle moves, is equal to zero in the regions ($r < b$ and $r > a$) and equal a positive constant V_0 in a crown of internal radius b and external radius a ($b < a$). Our study was focused on the bounded states regime for which $E < V_0$. We have used, to derive the Green funtion, the continuity of the solution and of its first derivative at $r = b$ and $r = a$. We have obtained the associate Green function and the discrete spectra (for the case $E < V_0$) of the Hamiltonian in the region $r < b$.

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Функция Грина квантовой частицы, движущейся в двумерном кольцевом потенциале

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Аннотация. В этой работе мы представляем новый результат, который касается получения функции Грина относительно не зависящего от времени уравнения Шредингера в двумерном пространстве. Система, рассматриваемая в этой работе, представляет собой частицу, обладающую энергией E и движущуюся в осесимметричном потенциале. Точнее, мы предположили, что потенциал $(V(r))$, в котором движется частица, равен нулю внутри кольцевой области (радиус b) и равен положительной постоянной (V_0) в кольце внутреннего радиуса b и внешнего радиуса $(b < a)$ и равен нулю за пределами кольца $(r > a)$. Мы исследовали режим ограниченных состояний, для которого $(E < V_0)$. Для получения функции Грина мы использовали непрерывность решения и его производной в точках $(r = b)$ и $(r = a)$. Мы получили ассоциированную функцию Грина и дискретные спектры гамильтониана в области $(r < b)$.

Ключевые слова: квантовая механика, уравнение Шрёдингера, функция Грина, ограниченные состояния.