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On the Convergence Exponent of the Special Integral of the Tarry Problem for a Quadratic Polynomial

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Abstract. In this paper it is considered the summation problem for trigonometric integrals with quadratic phase. This problem was considered in the papers [7–9] in particular cases. Our results generalize the results of those papers to multidimensional trigonometrical integrals.

Keywords: trigonometrical integral, exponent, sums, phase, polynomial.

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Introduction

Let $P(x, s) \in \mathbb{R}[x]$ (where $x \in \mathbb{R}^k$) be a polynomial with real coefficients $s \in \mathbb{R}^N$. We consider the trigonometric integral given by

$$T(s) = \int_Q \exp(iP(x, s))dx, \tag{1}$$

where $Q \subset \mathbb{R}^k$ is a compact set.

Problems related to such kind of integrals arise in mathematical physics (see [1]), harmonic analysis (see [2–5]), analytic number theory (see [6–11]) and so on. Surely, the given references are not complete.

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One of the well known problems related to the trigonometric integrals is the issue on convergence of the special integral of the Tarry problem, which is given by the following:

$$\theta = \int_{\mathbb{R}^N} |T(s)|^p ds, \text{ with } Q = [0,1]^k. \tag{2}$$

The integral θ arises as the coefficient of asymptotic representation for a number of integer solutions of a Diophantine system [2, 6, 7]. Therefore, it is important to find a minimal value of the parameter p, where the special integral is convergent, which is also essential in the Fourier restriction problem in harmonic analysis [3].

Definition. *A real number γ is called to be a convergence exponent of the special integral if for every $p > \gamma$ the integral (2) is convergent and for every $p < \gamma$ it is divergent. In other words $\gamma = \inf\{p : T \in L_p(\mathbb{R}^N)\}$.*

It should be noted that the convergence exponent essentially depends on the form of the polynomials $P(x, s)$. Thus the main problem can be formulated as:

Problem: Find the number γ.

This problem was considered by I. M. Vinogradov [11] in connection with the problems of analytic number theory. He obtained an upper bound for the number γ in the case $k = 1$. This bound was improved in [10].

The exact value of γ was indicated in [6] for the case $k = 1$. It is interesting to note that in one-dimensional case depending on form of the polynomial $P(x, s)$ the exact value of γ can be expressed by the sum of exponents of the non-trivial terms of the polynomial $P(x, s)$. Moreover, it was proved un upper bound for the number γ in multidimensional cases.

It should be noted that, in [12] a lower bound was found for the number γ. Moreover, it was found the number γ provided that the coefficients of the polynomial vary in some subspace of $\mathbb{R}^N$. Similar problems were considered in the works [13, 14, 15].

In [7] a lower bound was obtained for γ and also, it was investigated analogical problem for more subtle object trigonometric sums in the case $k = 2$. In [7] and [9] a similar problem was considered in the case $k = 2$. Moreover, in [7], it is shown that if P is a homogeneous quadratic polynomial and $k = 2$, then $\gamma = 4$ in the case when $Q = [0, 1]^2$, more precisely, the special integral θ is convergent if $p > 4$ and divergent if $p \leqslant 4$.

It was interesting to extend the results proved by L. G .Arkhipova, V. N. Chubarikov related to trigonometric integrals to multidimensional case.

In this paper we study the problem in the classical setting. In other words, P is a quadratic polynomial function and $Q = [0, 1]^k$ is the unit cube and also for the case when Q is a compact domain. Analogical problem was considered by J. Makenhaupt [2], who obtain the number γ in the case when the polynomial $P(x, s)$ satisfies some "non-degeneracy" condition.

It should be noted that the condition of J. Makenhaupt does not hold for the general case (see [2]). Actually, J. Mokenhaupt used an interesting approach. He computed the multidimensional trigonometric integral, for which the amplitude function is the gauss function. Then he be able to get the sharp value of the convergence exponent for some cases. It should be noted that using the gauss functions to investigate behavior of oscillatory integrals goes back to E. M. Stein [1]. We obtain the exact value of γ, whenever P is a homogeneous polynomial of degree two.

We use the idea of J. Makenhaupt and then we able to investigate the obtained integrals. We observe that the integral over $\mathbb{R}^N$ can be written as an iterated integral over the orbit of the orthogonal group and then over the corresponding fundamental domain. It is interesting that the integrant in the trigonometric integrals with quadratic phase with special amplitude function, more precisely gauss functions, is invariant under action of the orthogonal group. Thus, our approach is natural in this case. Unfortunately, it seems such approach does not work for trigonometric integrals with more general polynomial phase functions.

The paper is organized as follows in the next Section 1 we formulate our main results. In the next Section 2 we give some auxiliary results on integrals. In particular, we obtain transformation of the volume form under the natural action of the orthogonal group. Then we give a proof of our main results in the next Section 3. Finally, we give some results related to two-dimensional integrals in the last Section 4.

1. Formulation of the main results

Let P be the polynomial given by

$$P(x, A, b) = (Ax, x) + (b, x),$$

where $A = (a_{lm})^k{}_{l,m=1}$ is a symmetric $k \times k$ matrix with real entries, $b := (b_1, b_2, \ldots, b_k) \in \mathbb{R}^k$ and $(\cdot, \cdot)$ is the inner product of the corresponding vectors. Consider the trigonometric integral

$$T(A, b) = \int_{\mathbb{R}^k} \exp\left(iP(x, A, b)\right)\chi_Q(x)dx,$$

where Q is a compact set and $\chi_Q(x)$ is its characteristic function.

Consider the integral

$$\theta = \int_{\mathbb{R}^N} |T(A, b)|^p db\, da,$$

where $db = db_1 db_2 \ldots db_k$ and $da = \prod\limits_{1 \leqslant l \leqslant m \leqslant k} da_{lm}$.

The following is true:

Theorem 1.1. *Let Q be a compact set, then the integral θ converges, whenever $p > 2k + 2$ and if Q contains an interior point x^0 and there exists a line l passing through point x^0 such that the boundary of the set $\{l \cap Q\}$ contains only a finite number of points, then the integral diverges provided $p \leqslant 2k + 2$. In particular, if $Q = [0, 1]^k$, then $\gamma = 2k + 2$.*

1. The case when P is a homogeneous polynomial of the second order

Now suppose that $P(x, A) = (Ax, x)$. In [9] it has been proved that if Q is a quadratic polynomial in $\mathbb{R}^2$, then for $p > 4$ the θ integral converges and when $p \leqslant 4$ the θ integral diverges. In this paper we extend those results to the case when Q is a polyhedron in $\mathbb{R}^k$.

By polyhedron we mean a finite union of nondegenerate simplexes [5].

Theorem 1.2. *If $P(x, A) = (Ax, x)$ and Q is a polyhedron, then for $p > 2k$ the integral θ converges. If $Q = [0, 1]^k$, then for $p \leqslant 2k$ the integral θ diverges.*

Remark 1. In this case, we cannot apply the results of [3] as the corresponding set $\{x_i x_j\}_{i \leqslant j=1}^n$ is not a smooth surface.

Remark 2. Depending on the set Q, the exponent p may be smaller than $2k$. For example, if $k = 2$ and Q is a sufficiently small square centered at $(1, 1)$, then it can be proved that for $p > 3$ the integral θ converges.

2. Preliminaries

Consider the following integral

$$T_\infty(A,b) = \int_{\mathbb{R}^k} \exp\left(iP(x,A,b) - (x,x)\right)dx.$$

It is easy to check that this integral, whose calculation details are given in [2], is absolutely and uniformly converges with respect to the parameters A and b.

Lemma 2.1. *The following equality holds*

$$T_\infty(A,b) = (2\pi)^{\frac{k}{2}}(det(I-iA))^{-\frac{1}{2}} \exp\left(-\frac{((I-iA)^{-1}b,b)}{4}\right),$$

where the square root is determined in the following way

$$(det(I-iA))^{-\frac{1}{2}} := (1-i\lambda_1)^{-\frac{1}{2}} \cdot (1-i\lambda_2)^{-\frac{1}{2}} \cdot \ldots \cdot (1-i\lambda_k)^{-\frac{1}{2}},$$

with $\lambda_1, \ldots, \lambda_k$ being eigenvalues of A. The branch cut of the multiply-valued function $z^{-\frac{1}{2}}$ is taken on the complex plane by cutting the negative part of the real axis and $1^{-\frac{1}{2}} = 1$.

Lemma 2.1 is proved by reducing A to the diagonal form. Consequently, the calculation of the integral is reduced to a one-dimensional integral and it is explicitly calculated (see. [1]).

Obviously, the following equations are satisfied:

$$\left|\exp\left(-\frac{((I-iA)^{-1}b,b)}{4}\right)\right|^p = \exp\left(-\frac{((I+A^2)^{-1}b,b)p}{4}\right),$$

$$\int_{\mathbb{R}^k} \exp\left(-\frac{((I+A^2)^{-1}b,b)p}{4}\right)db = \frac{(8\pi)^{\frac{k}{2}}(det(I+A^2))^{\frac{1}{2}}}{p^{\frac{k}{2}}}.$$

Let us introduce the following notation:

$$\theta_\infty = \int_{\mathbb{R}^N} |T_\infty(A,b)|^p db\, da,$$

where $N = \dfrac{k(k+2)}{2}$.

Proposition 1. *The integral θ_∞ converges when $p > 2k+2$ and diverges when $p \leqslant 2k+2$.*

Due to Lemma 2.1, the proof of the Proposition 1 comes by studying the following integral

$$\theta_\infty = c(p) \int_{\mathbb{R}^{N-k}} \frac{da}{(det(I+A^2))^{\frac{p-2}{4}}}, \tag{3}$$

where $c(p)$ is some positive number.

As the determinant is an invariant of the orthogonal group, it is convenient to integrate it first by the orbits of the orthogonal group and then by the quotient space, e.g. over fundamental domain with respect to action of the orthogonal group.

Let M be the set of symmetric matrices with real entries and $G = SO_k$ be a special subgroup of orthogonal matrices. This group naturally acts in the space M as $g(A) = g^tAg$, where $g \in SO_k$ and $A \in M$.

It is known that for any real symmetric matrix A, there exists $g\in G$ such that $g(A) = diag(\lambda_1, \ldots, \lambda_k)$, where $diag(\lambda_1, \ldots, \lambda_k)$ is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_k$. In other words for any matrix A there exists $g\in G$ such that $A = g^t\Lambda g$, where $\Lambda = diag(\lambda_1, \ldots, \lambda_k)$. Hence it is possible to define a surjective smooth map

$$\Phi : \mathbb{R}^k \times G \mapsto M$$

which is defined by the formula $\Phi(\Lambda, g) = g^t\Lambda g$.

Let $da = da_{11}\wedge \ldots \wedge da_{kk}$ be the standard volume form in the space M. We can define the image of this form under the map Φ, denoted by $\Phi^* da \in \wedge^{N-k}(\mathbb{R}^k \times SO_k)$.

Lemma 2.2. *The following equality holds*

$$\Phi^* da = \prod_{1\leqslant l<m\leqslant k} (\lambda_m - \lambda_l) d\lambda_1\wedge \ldots \wedge d\lambda_k \wedge \omega,$$

where ω is the volume form on the orthogonal group SO_k.

Lemma 2.2 can be proved by using the zero sets of the Jacobian of the map Φ. Note that the equality $\prod\limits_{1\leqslant l<m\leqslant k} (\lambda_m - \lambda_l)^2 = \rho_A(\lambda)$ holds, where $\rho_A(\lambda)$ is the discriminant of the characteristic polynomial of the matrix A.

By Lemma 2.1 the integral (3) can be rewritten as

$$\int\limits_{\mathbb{R}^{N-k}} \frac{da}{(det(I + A^2))^{\frac{p-2}{4}}} = \int\limits_{\mathbb{R}^k} \frac{\prod\limits_{1\leqslant l<m\leqslant k} |\lambda_m - \lambda_l|}{\prod\limits_{1\leqslant l\leqslant k} (1 + {\lambda_l}^2)^{\frac{p-2}{4}}} d\lambda_1\wedge \ldots \wedge d\lambda_k \int\limits_{SO_k} \omega.$$

From the last equality, it follows that the convergence of the integral (3) comes from the investigation of the convergence of the following integral

$$\int\limits_{\mathbb{R}^k} \frac{\prod\limits_{1\leqslant l<m\leqslant k} |\lambda_m - \lambda_l|}{\prod\limits_{1\leqslant l\leqslant k} (1 + {\lambda_l}^2)^{\frac{p-2}{4}}} d\lambda_1\wedge \ldots \wedge d\lambda_k.$$

Note that this integral converges when $p > 2k + 2$ and diverges when $p \leqslant 2k + 2$ and this proves the Proposition 1. $\square$

3. Proofs of the main results

Proof of the Theorem 1.1. The upper bound for γ follows from the main Theorem 1.1 of paper [3]. Consider the following subset $\Omega(a_{11})$ in $\mathbb{R}^{N-1}$:

$$|a_{12}| + |a_{13}| + \cdots + |a_{1k}| < c_1 a_{11}, \quad -\frac{1}{2} < \frac{b_1}{a_{11}} < -\frac{1}{4}, \quad |a_{lj} - \frac{a_{1l}a_{1j}}{a_{11}}| \leqslant c_2, \quad |b_l - \frac{2b_l a_{1l}}{a_{11}}| \leqslant c_2,$$

where $l = 2, \ldots, n$ and c_1, c_2 are sufficiently small fixed positive numbers and $a_{11} > 1$.

Lemma 3.1. *There is a positive number c such that, the following equality holds:*

$$\mu(\Omega(a_{11})) = c \cdot a_{11}^k,$$

for the Lebesgue measure of μ of the set $\Omega(a_{11})$.

Proof. Consider the following maps:

$$\xi_{1l}(A, b_1, \dots, b_k) = a_{1l},$$
$$\xi^1(A, b_1, \dots, b_k) = b_1,$$
$$\xi^l(A, b_1, \dots, b_k) = b_l - \frac{2b_1 a_{1l}}{a_{11}},$$
$$\xi_{lj}(A, b_1, \dots, b_k) = a_{lj} - \frac{a_{1l}a_{1j}}{a_{11}},$$
$$j \leqslant l, \quad j, l = 2, 3, \dots, k.$$

Jacobian of this map is equal to ± 1.

Denote by $\Omega(\xi_{11})$ the image of the map. Since the Jacobian is ± 1, then we have

$$\mu(\Omega(a_{11})) = \mu(\Omega(\xi_{11})).$$

It is easy to verify that for the set $\Omega(\xi_{11})$ with

$$\mid \xi_{12} \mid + \mid \xi_{13} \mid + \dots + \mid \xi_{1k} \mid < c_1 \cdot a_{11},$$
$$-\frac{1}{2} < \frac{\xi^1}{\xi_{11}} < -\frac{1}{4},$$
$$\mid \xi^l \mid \leqslant c_2, \quad \mid \xi_{lj} \mid \leqslant c_2, \quad j \leqslant l, \quad j, l = 2, 3, \dots, k$$

we have

$$\mu(\Omega(\xi_{11}) = c \cdot \xi_{11}^k = c \cdot a_{11}^k.$$

Hence,

$$\mu(\Omega(a_{11})) = c \cdot a_{11}^k.$$

□

Lemma 3.2. *There exists a positive number L such that when $a_{11} > L$ and $(A, b) \in \Omega(a_{11})$ for the integral $T(A, b)$ the following asymptotic equality holds*

$$T(A, b) = \frac{c(A, b)}{a_{11}^{\frac{1}{2}}} + O\left(\frac{1}{a_{11}}\right) \quad as \quad a_{11} \to +\infty.$$

Moreover, there exists a positive number δ such that for any $(A, b) \in \Omega(a_{11})$, the following inequality holds:

$$|c(A, b)| > \delta.$$

Proof. Lemma 3.2 is proved by the method of stationary phases. Note that for the sufficiently small c_1, c_2 and for the sufficiently large L, the phase has oscillation only in the x_1 direction on the set $(A, b) \in \Omega(a_{11})$. Consequently, for fixed values of $x_2, \dots, x_n \in [0, 1]$, the non-degenerated critical point $x_1(A, b, x_2, \dots, x_n)$ lies in $(0, 1)$. □

Finally, for integral θ we have the following lower bound:

$$\theta \geqslant \int_L^\infty \int_{\Omega(a_{11})} |T(A, b)|^p db\, da \geqslant \delta c \int_L^\infty a_{11}^{k - \frac{p}{2}} da_{11}.$$

Thus, when $p \leqslant 2k + 2$ the last integral diverges, which proves the Theorem 1.1. □

Proof of the Theorem 1.2. We use the classical Young inequality.

Let $f\in L_p(\mathbb{R}^k)$ and $g\in L_r(\mathbb{R}^k)$ be arbitrary functions. The following inequality holds:

$$\|f{*}g\|_{L_q} \leqslant \|f\|_{L_p}\|g\|_{L_r},$$

where $f{*}g$ is a convolution of the functions f and g. Moreover, constants $1\leqslant p, q, r\leqslant\infty$ are related by

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}.$$

Let Q be a compact polyhedron in $\mathbb{R}^k$ and

$$h(b)=\int_{\mathbb{R}^k} e^{|x|^2}\chi_Q(x)e^{-2\pi i(b,x)}dx.$$

Lemma 3.3. *The following relation $h\in L_{1+0}(\mathbb{R}^k)$ holds true, where $L_{1+0}(\mathbb{R}^k):=\cap_{p>1}L_p(\mathbb{R}^k)$.*

Proof. Note that, for any $\varepsilon>0$, $\hat{\chi}_Q\in L_{1+\varepsilon}(\mathbb{R}^k)$ (see. [4]). Then the statement of Lemma 2.1 easily follows from the Young's inequality.

Now let us return to the proof of Theorem 1.2. According to the Plancherel theorem we have:

$$T(A)=\int_Q e^{i(Ax,x)}dx=\int_{\mathbb{R}^k} e^{i(Ax,x)}\chi_Q(x)dx=\int_{\mathbb{R}^k} e^{i(Ax,x)-|x|^2}e^{|x|^2}\chi_Q(x)dx=\int_{\mathbb{R}^k}\widehat{f}(A,b)\overline{g}(b)db,$$

where $\widehat{f}(A,b)=\int\limits_{\mathbb{R}^k} e^{i(Ax,x)-|x|^2-2\pi i(x,b)}dx$ and $\widehat{g}(b)=\int\limits_{\mathbb{R}^k} e^{|x|^2}e^{-2\pi i(x,b)}dx$.

Let $q>1$ be a fixed number. Then, using the Hölder inequality, we have:

$$|T(A)|\leqslant\|\widehat{f}(A,\cdot)\|_{L_{q'}(\mathbb{R}^k)}\|g\|_{L_q(\mathbb{R}^k)},$$

where $\dfrac{1}{q}+\dfrac{1}{q'}=1$.

According to Lemma 2.1 , we have

$$|T(A)|\leqslant\frac{c_q}{(det(I+A^2))^{\frac{p}{4}-\frac{1}{2q'}}}.$$

Thus, if $p>2k$, then we can choose $q'>1$ such that $\dfrac{p}{4}-\dfrac{1}{2q'}>\dfrac{k}{2}$. It follows that if $\dfrac{p}{4}-\dfrac{1}{2q'}>\dfrac{k}{2}$, then $T\in L_p(\mathbb{R}^k)$.

It remains to prove the sharpness of the result. Consider the following subset $\Omega^+(a_{11})$ in $\mathbb{R}^{N-1}$, where $N=\dfrac{k(k+1)}{2}$.

$$a_{11}>0, |a_{12}|+|a_{13}|+\cdots+|a_{1k}|<c_1a_{11},\quad \left|a_{lj}-\frac{a_{1l}a_{1j}}{a_{11}}\right|\leqslant c_2, a_{1l}<0$$

where $l\leqslant j=\overline{2,n}$, $l=2,\ldots,n$ and c_1, c_2 are sufficiently small fixed positive numbers.

According to the Lemma 3.1 there exist positive numbers c_1 and c_2 such that the following equality holds for the Lebesgue measure of $\Omega^+(a_{11})$:

$$\mu(\Omega^+(a_{11}))=c\cdot a_{11}^{k-1}. \qquad \square$$

Lemma 3.4. *There exists a positive number L such that when $a_{11} > L$ and $(A, b) \in \Omega(a_{11})$ for the integral $T(A)$ the following asymptotic equality holds*

$$T(A) = \frac{c(A)}{a_{11}^{\frac{1}{2}}} + O\left(\frac{1}{a_{11}}\right) \quad as \quad a_{11} \to +\infty.$$

Moreover, there exists a positive number δ such that for any $(A, b) \in \Omega^+(a_{11})$ the inequality

$$|c(A)| > \delta > 0$$

holds true.

Lemma 3.4 is proved by the method of stationary phases. Note that if $\delta_2 > 0$ and $\delta_1 < 0$ are fixed numbers then the following relation holds true

$$\int\limits_{\delta_1\sqrt{\lambda}}^{\delta_2\sqrt{\lambda}} \cos y^2 dy = c(\delta_1, \delta_2, \lambda)$$

and there exist λ_0, $\varepsilon > 0$ such that the inequality $c(\delta_1, \delta_2, \lambda) \geqslant \varepsilon > 0$ holds for all $\lambda \geqslant \lambda_0$.

Indeed, we have the following relation

$$\lim_{\lambda \to +\infty} \int\limits_{\delta_1\sqrt{\lambda}}^{\delta_2\sqrt{\lambda}} \cos y^2 dy = \frac{\sqrt{2\pi}}{2}.$$

Note that, for sufficiently small c_1, c_2 at $A \in \Omega^+(a_{11})$ and for sufficiently large L, the phase has oscillations only in the x_1 direction. Also, for fixed values $x_2, \dots, x_n \in [0, 1]$, the nondegenerate critical point $x_1(A, b, x_2, \dots, x_n)$ lies inside $(0, 1)$.

Finally, for the integral θ, we have the following lower bound:

$$\theta \geqslant \int_L^\infty \int_{\Omega(a_{11})} |T(A)|^p da \geqslant \delta c \int_L^\infty a_{11}^{k - \frac{p}{2} - 1} da_{11}.$$

Thus, the last integral diverges, whenever $p \leqslant 2k$. The Theorem 1.2 is proved. □

4. Two-dimensional case

Note that in the homogeneous case the results of [3] are not applicable. The proof of Theorem 1.2 essentially uses the property $\widehat{\chi}_Q \in L_{1+0}(\mathbb{R}^k)$.

In Lebedev's paper, it is given an example of the domain $\partial D \in C^{1,\omega}$, where ω is the continuity module of the gradient φ that locally defines ∂D, such that $\widehat{\chi}_Q \in L_{1+0}(\mathbb{R}^k)$. Therefore, we can assume that D is a compact domain with sufficiently smooth boundary.

The following is true

Theorem 4.1. *Let D be a compact domain such that $\widehat{\chi}_D \in L_q(\mathbb{R}^2)$ and $T(A) = \int\limits_D e^{i(Ax,x)} dx$. Then $T \in Lp(\mathbb{R}^3)$ for $p > 6 - \dfrac{2}{q}$. Moreover, if $\widehat{\chi}_D \in L_{1+0}(\mathbb{R}^2)$, then for any $p > 4$, the inclusion $T \in L_p(\mathbb{R}^3)$ is valid.*

Remark 3. From the results given in [4] it follows that there exists a domain D other than a polygon such that $\widehat{\chi}_Q \in L_{1+0}(\mathbb{R}^2)$.

Corollary 1. If $D \subset \mathbb{R}^2$ is a compact set such that $\partial D \subset C^1$, then for $p > 4.5$ the relation $T \in L_p(\mathbb{R}^3)$ holds.

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О показателях сходимости особого интеграла проблемы Терри для квадратичного многочлена

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Аннотация. В статье рассматривается проблема суммируемости для тригонометрических интегралов с квадратичной фазой. Аналогичная задача рассмотрена в работах [7–9] в частных случаях. Наши результаты обобщают результаты этих работ на кратные тригонометрические интегралы.

Ключевые слова: тригонометрический интеграл, экспонент, сумма, фаза, многочлен.