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Definition of Lifetime due to Wave-Particle Interactions for the Pitch Angle of 90 Degrees

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Abstract. One of very important, but still it is not enough the investigated problems in the theoretical physics of the Earth's magnetosphere is the definition of lifetime of the charged particles due to wave-particle interactions. Therefore for the pitch angle of 90 degrees as mathematical model is offered the ordinary differential equation (ODE) for the analytical description of a perpendicular differential flux of the charged particles in the Earth's magnetosphere which depends on time t and several parameters. Using the analytical solution of the ODE, the new simple formula for definition of lifetime due to wave-particle interactions for the pitch angle of 90 degrees for different geophysical conditions is received. For calculation under this formula it is used correlated observation of enhanced electromagnetic ion cyclotron waves and dynamic evolution of ring current energetic (5–30 keV) proton flux collected by Cluster satellite near the location $L = 4.5$ during March 26–27, 2003, a nonstorm period ($Dst > -10$ nT. In addition are found the perpendicular coefficients of the particle loss function, the particle source function and the pitch angle diffusion. For the first time the modeling dependences of lifetime due to wave-particle interactions for the pitch angle of 90 degrees from the local time LT and the geomagnetic activity Kp-index are received. Mathematical statement of a problem in the general view (the system of two ODEs of the first order with two boundary conditions) is offered, when parameters depend from time, which allows to define the lifetime due to wave-particle interactions for the pitch angle of 90 degrees numerically.

Keywords: Earth's magnetosphere, pitch angle diffusion, data of the Cluster satellite, lifetime due to wave-particle interactions.

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Introduction

Electromagnetic ion cyclotron (EMIC) waves play an important role in precipitation losses of ring current ions [1] via wave-particle interactions during geomagnetic storms in the Earth's magnetosphere. Pitch angle diffusion [2] induced by EMIC waves is considered as an efficient mechanism responsible for the ring current decay during active geomagnetic periods. Variation of radiation belt dynamics is considered as the contribution from wave-particle interactions [3, 4] and from drift resonance associated with enhanced ultra low frequency waves [5–7]. In more detail for the review of the literature look [1–7].

Therefore a lifetime due to wave-particle interactions has very much great value at carrying out of calculations and mathematical modeling. But the works, devoted only to definition of the lifetime due to wave-particle interactions, practically it is not enough. In this connection the purpose of the work is to propose a new variant (a new formula) of definition of the lifetime

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due to wave-particle interactions for the 90° pitch angle for different geophysical conditions and corresponding mathematical statement of a problem in the general view, which allows to define $T_{wp\perp}$ numerically.

1. The mathematical model

The following non-stationary differential equation is used [8, 9] for phase space density, which describes the pitch angle diffusion and losses due to wave-particle interactions in the Earth's magnetosphere for a range of pitch angles from 0° up to 180° :

$$\frac{\partial f}{\partial t} = \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha D_{\alpha\alpha}\frac{\partial f}{\partial\alpha} - \frac{\sin^2\alpha\cos\alpha}{2L}\frac{dL}{dt}f\right) - \frac{f}{T_{wp}} + S_\perp \sin^2\alpha f, \tag{1}$$

where f is the phase space density (or distribution function); t is the time; α is the local pitch angle; $D_{\alpha\alpha}$ is the coefficient of pitch angle diffusion; L is the McIlwain parameter; dL/dt is the radial rate; T_{wp} is the average lifetime due to wave-particle interactions; $S_\perp$ is the perpendicular coefficient of the particle source function.

Equation (1) describes the pitch angle diffusion in the velocity space with losses due to wave-particle interactions. The loss function is conditioned by the fall of charged particles in the so-called "loss cone" as a result of wave-particle interactions. The particle source function can be related, for example, to charged particles that move from the tail of the magnetosphere toward the Earth when affected by magnetospheric convection.

The mathematical model of pitch angle diffusion of charged particles in the Earth's magnetosphere represented by (1) takes into account three physical mechanisms [4]. First, the wave-particle interactions are considered with T_{wp}. Second, the physical mechanism of injection and particle drift is taken into account through the radial drift velocity dL/dt in equation (1). Third, since the electric field potential depends on the geomagnetic activity index Kp [8, 9], we take into account the splitting effect of drift shells of the electric field on the pitch angle distribution of charged particles.

In the further the following relationship will be used between a differential flux of particles j and phase space density f (or function of distribution) $j = 2mEf$, where m is the mass of the charged particle (in the given work of a proton), E is the energy of the charged particle.

Using (1) [10], for the 90° pitch angle as mathematical model is offered the ordinary differential equation (ODE) for the analytical description of a perpendicular differential flux of the charged particles in the Earth's magnetosphere which depends on time and several parameters:

$$\frac{dj_\perp}{dt} + \left(-\frac{1}{2L}\frac{dL}{dt} + \frac{\gamma_\perp - \gamma_{\perp 0}}{\gamma_{\perp 0}\left(\gamma_{\perp 0} + 2\right)T_{wp\perp}}\right)j_\perp = 0, \tag{2}$$

where $j_\perp$ is the perpendicular differential flux of the charged particles, $\gamma_{\perp 0}$ is the well-known (when $j(\alpha) = j_\perp \sin^{\gamma(\alpha)}\alpha$, α is the local pitch angle) parameter of the charged particle pitch angle distribution (or the pitch angle distribution anisotropy index) taken for the 90° pitch angle at $t = 0$, $\gamma_\perp$ is the average parameter of the charged particle pitch angle distribution on a time interval of calculation $t = t_{end}$ (it is supposed, that $\gamma_\perp(t) \approx \text{const}$), $T_{wp\perp}$ is the lifetime due to wave-particle interactions for the 90° pitch angle.

For calculation on experimental data of charged particle pitch angle distributions it are found $j_{\perp 0}(t = 0)$, $\gamma_{\perp 0}(t = 0)$ and $j_{\perp end}(t = t_{end})$, $\gamma_{\perp end}(t = t_{end})$ and then $\gamma_\perp$

$$\gamma_\perp \approx \frac{\gamma_{\perp 0} + \gamma_{\perp end}}{2}. \tag{3}$$

Thus the found average value $\gamma_\perp$ is approximately fair for rather small time intervals and for nonstorm time conditions.

The equation (2) can be solved analytically under the certain geophysical conditions and on a time interval approximately no more than three hours (when a geomagnetic activity index $Kp = \text{const}$) or on a greater time interval, when $Kp \approx \text{const}$.

We shall take the following approached equality, using the bounce-averaged radial drift velocity of charged particles in the Earth's magnetosphere (measured in 1/s) [8–10],

$$\frac{dL}{dt} \approx \left\langle \frac{dL}{dt} \right\rangle = -\Omega \frac{\phi_2}{\phi_0} L^4 \cos\phi = -\frac{2\pi\phi_2 L^4 \cos\phi}{24 \cdot 60 \cdot 60 \cdot 92}, \tag{4}$$

where ϕ is the azimuthal angle (local time LT = 0 h at midnight) or geomagnetic eastern longitude in the magnetic equator plane, Ω is the Earth's angular velocity, $\phi_0 = 92$ kV, and the dependence of ϕ_2 (measured in kV) on geomagnetic activity, i.e. on the Kp-index, is determined from the formula [11]

$$\phi_2 \approx \frac{0.045}{\left(1 - 0.16Kp + 0.01Kp^2\right)^3}. \tag{5}$$

Believing all parameters in the equation (2) in view of (3)–(5) are constants (or approximately constants), we find the analytical solution of the equation (2) exact (or approached):

$$j_\perp(t) = j_{\perp 0} \exp\left(-\left(\frac{\pi\phi_2 L^3 \cos\phi}{7948800} + \frac{\gamma_\perp - \gamma_{\perp 0}}{\gamma_{\perp 0}\left(\gamma_{\perp 0} + 2\right) T_{wp\perp}}\right) \cdot t\right). \tag{6}$$

Here, $j_{\perp 0}$ is the perpendicular differential flux of the charged particles at $t = 0$, and $j_\perp(t)$ is the perpendicular differential flux of the charged particles at the moment of time t.

From the equation (6) we find the new simple formula for definition of the lifetime due to wave-particle interactions for the 90° pitch angle (measured in s) for different geophysical conditions

$$T_{wp\perp} = \frac{\left(\gamma_\perp - \gamma_{\perp 0}\right) \cdot t}{\gamma_{\perp 0}\left(\gamma_{\perp 0} + 2\right)} \left(\ln \frac{j_{\perp 0}}{j_\perp(t)} - \frac{\pi\phi_2 L^3 \cos\phi}{7948800} t\right)^{-1}. \tag{7}$$

Always the condition $T_{wp\perp} > 0$ should be satisfied.

The definition of the lifetime due to wave-particle interactions for the 90° pitch angle (7) also allows to find such very important for modeling quantitative characteristics as the perpendicular coefficient of the particle loss function [8, 9]

$$L_\perp = \frac{1}{T_{wp\perp}}, \tag{8}$$

the perpendicular coefficient of the particle source function [8, 9]

$$S_\perp = \frac{\left(\gamma_{\perp 0} + 3\right)}{\left(\gamma_{\perp 0} + 2\right) T_{wp\perp}}, \tag{9}$$

and the perpendicular coefficient of pitch angle diffusion is offered to be determined [8, 9] as follows

$$D_\perp = \frac{1}{\gamma_{\perp 0}\left(\gamma_{\perp 0} + 2\right) T_{wp\perp}}. \tag{10}$$

If all parameters in the equation (2) in view of (3)–(5) are not constants, then the equation (2) in the general view should be written down so

$$\frac{dj_\perp}{dt} + \left(\frac{\pi\phi_2(t) L^3(t) \cos\phi(t)}{7948800} + \frac{\gamma_\perp(t) - \gamma_{\perp 0}}{\gamma_{\perp 0}\left(\gamma_{\perp 0} + 2\right) T_{wp\perp}}\right) j_\perp = 0. \tag{11}$$

The equation (11) allows to enter into calculation a trajectory of the satellite, for example, in a parametrical form $L(t)$, $\phi(t)$, change of potential of a magnetospheric convection electric field $\phi_2(Kp(t))$ and change of a anisotropy index of the charged particle pitch angle distribution $\gamma_\perp(t)$. But in this equation $T_{wp\perp} = \text{const}$, therefore the definition of the lifetime due to wave-particle interactions for the 90° pitch angle can be offered as follows.

The problem connected with ODE of the first order (11) can be considered as a boundary value problem with undetermined parameter $T_{wp\perp}$, if to set two boundary conditions for a perpendicular differential flux: $j_{\perp 0}$ and $j_{\perp end}$. By introducing the parameter $T_{wp\perp}$ as dependent variable, the problem can be written as a boundary value problem in standard form. To find the solution $j_\perp(t)$ and the value of $T_{wp\perp}$, just add the equation

$$\frac{dT_{wp\perp}}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{1}{T_{wp\perp}}\right) = 0. \tag{12}$$

This system of ODEs of the first order (11) and (12) solves a problem with two boundary conditions for definition $j_\perp(t)$ and $T_{wp\perp}$.

Thus, at use of the offered system of the equations (11), (12) with two boundary conditions $j_{\perp 0}$ and $j_{\perp end}$, when parameters $L(t)$, $\phi(t)$, $\phi_2(Kp(t))$, $\gamma_\perp(t)$ depend from time, the lifetime due to wave-particle interactions for the 90° pitch angle $T_{wp\perp}$ is defined numerically.

2. Results of calculations

Further it will be used correlated observation of enhanced electromagnetic ion cyclotron (EMIC) waves and dynamic evolution of ring current proton flux collected by Cluster satellite near the location $L = 4.5$ during March 26–27, 2003, a nonstorm period ($Dst > -10$ nT) [1]. It is shown, that energetic (5–30 keV) proton fluxes are found to drop rapidly (e.g., a half hour) at lower pitch angles, corresponding to intensified EMIC wave activities.

At the moment of time 00:00 RT = 23:56 UT on 26 March, 2003 (RT is the "running", current time of modeling) Kp-index of geomagnetic activity was equaled 2.66 or 3- and further within almost 3 hours remained constant. As Kp-index is the planetary three-hour index. Time of modeling for comparison with experimental data has been taken 30 minutes, i.e. 00:30 RT = 00:26 UT on 27 March, 2003.

The initial condition in the moment of time 00:00 RT = 23:56 UT on 26 March, 2003 we shall take from work [1], where the pitch angle distribution measured on the Cluster satellite for energy $E = 17.1$ keV, $L = 4.17$, LT = 22.58 h (nonstorm conditions) is presented. In this work the initial pitch angle distribution (Fig. 1) is designated squares [12]. In the same figure experimental data (circles) are shown for energy $E = 17.1$ keV also in the moment of time 00:30 RT = 00:26 UT on 27 March, 2003 (again nonstorm conditions).

For the approached analytical description of experimental data it is used the pitch angle distribution

$$j(\alpha) = j_\perp \sin^\gamma \alpha, \quad \gamma = \gamma_\perp = const. \tag{13}$$

For example, the good consent turns out, when the initial perpendicular differential flux of protons is equal $j_{\perp 0} \approx 560360$ (cm^2 s sr keV)$^{-1}$, and the parameter of pitch angle distribution $\gamma = \gamma_{\perp 0} = 0.5157$.

For calculation we shall take following data on the interval $t = (0\text{–}1800)$ s, $t_{end} = 1800$ s (Fig. 1): $j_{\perp 0}(t = 0 \text{ s}) = 5.6036 \cdot 10^5$ (cm^2 s sr keV)$^{-1}$, $j_{\perp end}(t_{end} = 1800 \text{ s}) = 2.1165 \cdot 10^5$ (cm^2 s sr keV)$^{-1}$, $Kp = 2.66$ or 3-, $L = 4.17$, $\phi \equiv \text{LT} = 22.58$ h, $\gamma_{\perp 0}(t = 0 \text{ s}) = 0.5157$, $\gamma_\perp = 0.580$, $E = 17.1$ keV. Then using (5) and (7) the lifetime due to wave-particle interactions for the 90° pitch angle is received $T_{wp\perp} = 92.5356$ s. If necessary it is possible to present dependence $j_\perp(t)$ (6) on the interval $t = (0\text{–}1800)$ s.

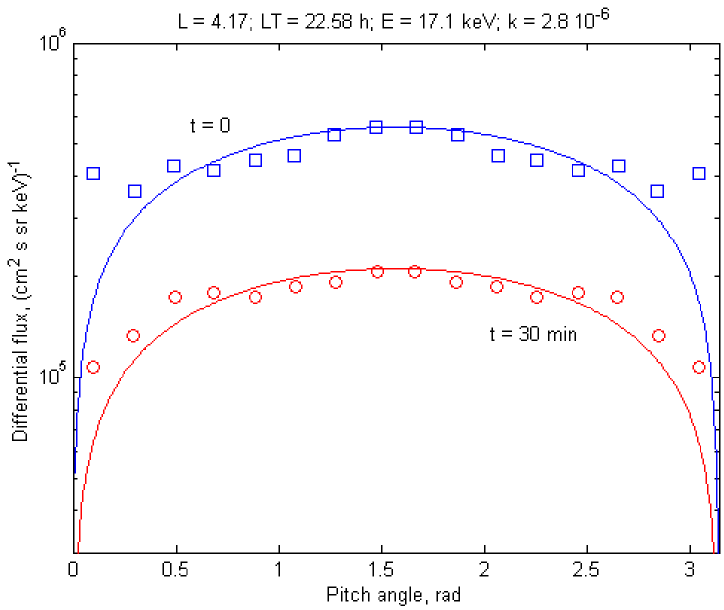

Fig. 1. The pitch angle distributions of protons measured of the Cluster satellite for energy $E = 17.1$ keV. Squares and circles specify a differential flux for 00:00 RT = 23:56 UT on 26 March, 2003 and for 00:30 RT = 00:26 UT on 27 March, 2003 (nonstorm conditions) respectively. Blue and red lines specify a modeling differential flux for 00:00 RT and 00:30 RT ($L = 4.17$, LT = 22.58 h, $k = 2.8 \cdot 10^{-6}$ [12]) respectively

Accuracy of definition of the lifetime due to wave-particle interactions for the 90° pitch angle $T_{wp\perp}$ is connected with accuracy of definition of the data entering into the formula (7).

In addition in this case we receive following concrete results: the perpendicular coefficient of the particle loss function (8) $L_\perp = 0.0108$ 1/s, the perpendicular coefficient of the particle source function (9) $S_\perp = 0.0151$ 1/s and the perpendicular coefficient of pitch angle diffusion (10) $D_\perp = 0.0083$ 1/s.

Using the previous data and assuming, that they remain constants or approximately constants, under the formula (7) we can find a modeling dependence $T_{wp\perp}$ on azimuthal angle (local time LT = 0 h at midnight) or geomagnetic eastern longitude in the magnetic equator plane $\phi \equiv$ LT (Fig. 2).

As it is well visible (Fig. 2), the presented modeling dependence corresponds to the law $\cos\phi$ (7).

Again using the previous data and assuming, that they remain constants or approximately constants, under the formula (7) we can find a modeling dependence $T_{wp\perp}$ from the geomagnetic activity Kp-index (Fig. 3).

As a result (Fig. 3) it is received nonlinear dependence the lifetime due to wave-particle interactions for the 90° pitch angle from the geomagnetic activity Kp-index with the maximal value $T_{wp\perp}$ at $Kp = 8$.

Dependences $T_{wp\perp}$ from the magnetic local time LT and the geomagnetic activity Kp-index are received for the first time.

Thus for the offered variant (the formula (7)) it is enough to have only the experimental data on two charged particle pitch angle distributions to find the lifetime due to wave-particle interactions for the 90° pitch angle, and then it is possible to find the perpendicular coefficient of the particle loss function (8), the perpendicular coefficient of the particle source function (9)

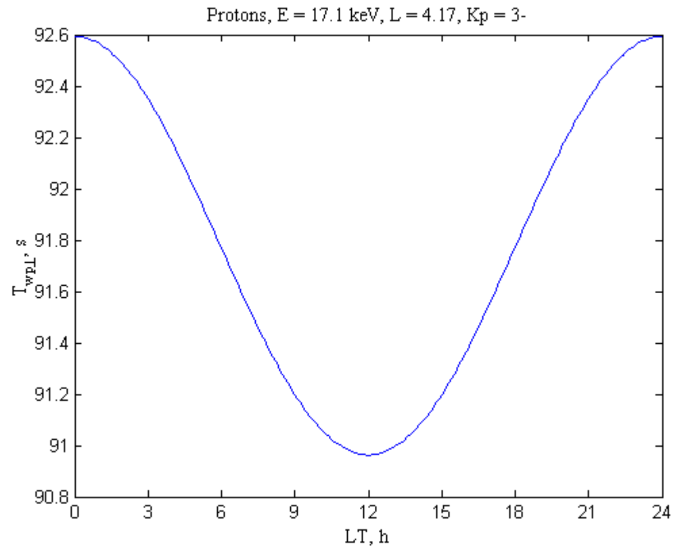

Fig. 2. The modeling dependence the lifetime due to wave-particle interactions for the 90° pitch angle $T_{wp\perp}$ on ϕ or the local time LT

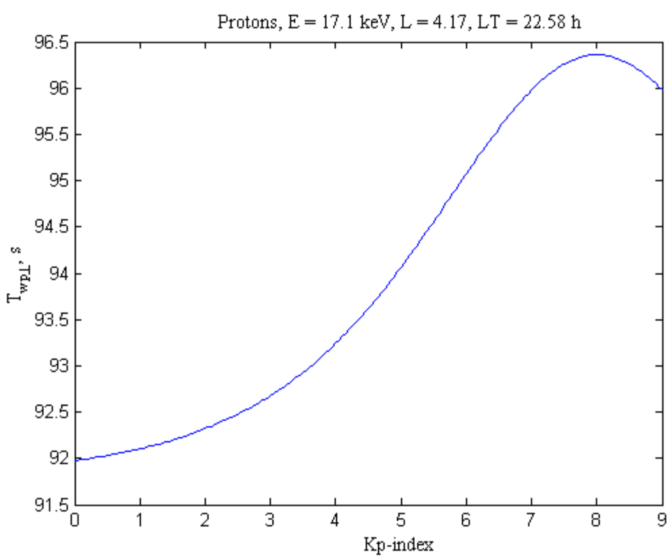

Fig. 3. The modeling dependence the lifetime due to wave-particle interactions for the 90° pitch angle $T_{wp\perp}$ from the geomagnetic activity Kp-index

and the perpendicular coefficient of pitch angle diffusion (10) for the subsequent mathematical modeling.

Further we use other mathematical statement of a problem, namely system of ODEs of the first order (11) and (12) with two boundary conditions. For an example of such calculation we shall take the previous data on a time interval (0–1800) s and we shall find $T_{wp\perp}$. As a result as well as should be numerically we receive (for example using the Mathematica software program) the same result $T_{wp\perp} = 92.5356\,\text{s}$.

Thus in the further by means of system of two equations (11) and (12), when parameters depend from time on experimental data, it will be possible to receive more exact dependences $T_{wp\perp}$ from the local time LT and the geomagnetic activity Kp-index.

Conclusion

1. For the 90° pitch angle as the mathematical model the ordinary differential equation for the analytical description of a perpendicular differential flux of the charged particles in the Earth's magnetosphere which depends on time and several parameters is offered.
2. The new simple formula for definition of the lifetime due to wave-particle interactions for the 90° pitch angle $T_{wp\perp}$ for different geophysical conditions is found.
3. The definition of the lifetime due to wave-particle interactions for the 90° pitch angle also allows to find such very important quantitative characteristics as the perpendicular coefficient of the particle loss function, the perpendicular coefficient of the particle source function and the perpendicular coefficient of pitch angle diffusion for the subsequent mathematical modeling.
4. For concrete calculations data of energetic (5–30 keV) proton pitch angle distributions collected by Cluster satellite near the location $L = 4.5$ during March 26–27, 2003, a nonstorm period ($Dst >$ -10 nT) are used.
5. For the first time the modeling dependences $T_{wp\perp}$ from the local time LT and the geomagnetic activity Kp-index are received.
6. For the offered variant (the new simple formula (7)) it is enough to have only the experimental data on two charged particle pitch angle distributions to find the lifetime due to wave-particle interactions for the 90° pitch angle analytically.
7. Mathematical statement of a problem in the general view (the system of two ODEs of the first order (11) and (12) with two boundary conditions) is offered, when parameters depend from time, which allows to define $T_{wp\perp}$ numerically.
8. By means of system of two ODEs, when parameters depend from time on experimental data, it will be possible to receive more exact dependences $T_{wp\perp}$ from the local time LT and the geomagnetic activity Kp-index.

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Определение времени жизни вследствие взаимодействий волна-частица для питч-угла 90 градусов

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Аннотация. Одной из очень важных, но еще недостаточно исследованных задач в теоретической физике магнитосферы Земли является определение времени жизни заряженных частиц вследствие взаимодействий волна-частица. Поэтому для питч-угла 90 градусов как математическая модель предлагается обыкновенное дифференциальное уравнение (ОДУ) для аналитического описания перпендикулярного дифференциального потока заряженных частиц в магнитосфере Земли, которое зависит от времени и нескольких параметров. Используя аналитическое решение ОДУ, получена новая простая формула для определения времени жизни вследствие взаимодействий волна-частица для питч-угла 90 градусов для разных геофизических условий. Для расчета по этой формуле используется коррелированное наблюдение усиленных электромагнитных ионно-циклотронных волн и динамической эволюции потока энергичных (5–30 кэВ) протонов кольцевого тока, собранное спутником Cluster около положения $L = 4.5$ в течение 26–27 марта 2003 г. в магнитоспокойный период ($Dst > -10$ nT). В дополнение найдены перпендикулярные коэффициенты функции потерь частиц, функции источника частиц и питч-угловой диффузии. Впервые получены модельные зависимости времени жизни вследствие взаимодействий волна-частица для питч-угла 90 градусов от местного времени LT и Kp-индекса геомагнитной активности. Предложена математическая постановка задачи в общем виде (система двух ОДУ первого порядка с двумя граничными условиями), когда параметры зависят от времени, которая позволяет определять время жизни вследствие взаимодействий волна-частица для питч-угла 90 градусов численно.

Ключевые слова: магнитосфера Земли, питч-угловая диффузия, данные спутника Cluster, время жизни вследствие взаимодействий волна-частица.