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Identities Related to Homo-derivation on Ideal in Prime Rings

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Abstract. This study aims to investigate the commutativity of a prime ring $\mathscr{R}$ with a non-zero ideal $\mathscr{I}$ and a homo-derivation ϑ that satisfies certain algebraic identities. We also provided some examples of why our results hypothesis is essential.

Keywords: prime ring, homo-derivation.

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1. Introduction

Throughout this article, $\mathscr{R}$ denotes a ring with centre Z. A ring $\mathscr{R}$ is said to be a prime ring if $\forall\ a, b \in \mathscr{R}$, $a\mathscr{R}b = \{0\}$ implies $a = 0$ or $b = 0$. For $a, b \in \mathscr{R}$, the symbol $[a, b]$ (resp. $a \circ b$) denotes the (resp. anti-) commutator $ab - ba$ (resp. $ab + ba$). An additive mapping $\vartheta : \mathscr{R} \to \mathscr{R}$ is said to be a homo-derivation if $\vartheta(ab) = \vartheta(a)\vartheta(b) + \vartheta(a)b + a\vartheta(b)\ \forall\ a, b \in \mathscr{R}$, [14]. The only additive map which is both derivation and homo-derivation on prime ring is the zero map. If $\mathscr{D} \subseteq \mathscr{R}$, then a mapping $\vartheta : \mathscr{R} \to \mathscr{R}$ preserves $\mathscr{D}$ if $\vartheta(\mathscr{D}) \subseteq \mathscr{D}$. A map $\vartheta : \mathscr{R} \to \mathscr{R}$ is called zero-power valued on $\mathscr{D}$ if ϑ preserves $\mathscr{D}$ and if for all $a \in \mathscr{D}$, there is a positive integer $m(a) > 1$ such that $\vartheta^{m(a)} = 0$, El Sofy [14].

In 2001, Ashraf and Rehman [13] showed that if $\mathscr{R}$ is a prime ring, $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$ and $\vartheta : \mathscr{R} \to \mathscr{R}$ is a derivation of $\mathscr{R}$, then $\mathscr{R}$ is commutative if $\mathscr{R}$ satisfies any one of the following: $\vartheta(ab) \pm ab \in Z$, $\vartheta(ab) \pm ba \in Z$, $\vartheta(a)\vartheta(b) \pm ab \in Z$, $\vartheta(a)\vartheta(b) \pm ba \in Z$, for all $a, b \in \mathscr{I}$.

In 2016, Asmaa Melaibari et. al. [2] showed that if $\mathscr{R}$ is a prime ring, $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$, and $\vartheta : \mathscr{R} \to \mathscr{R}$ is a non-zero homo-derivation of $\mathscr{R}$, then $\mathscr{R}$ is commutative if $\vartheta([a, b]) = 0$ for all $a, b \in \mathscr{I}$. Moreover, if char($\mathscr{R}$)$\neq 2$, ϑ is zero-power valued on $\mathscr{R}$, and $\vartheta([a, b]) \in Z$ for all $a, b \in \mathscr{R}$, then $\mathscr{R}$ is commutative.

In 2018, Alharfie and Muthana [7] showed that if $\mathscr{R}$ is a prime ring of characteristic $\neq 2$, $\mathscr{I}$ a non-zero left ideal of $\mathscr{R}$, and ϑ a homo-derivation of $\mathscr{R}$, which is a zero-power valued on $\mathscr{I}$, then $\mathscr{R}$ is commutative if $\mathscr{R}$ satisfies any one of the following: $a\vartheta(b) \pm ab \in Z$, $a\vartheta(b) \pm ba \in Z$, $a\vartheta(b) \pm [a, b] \in Z$, $\vartheta(b)a \pm [a, b] \in Z$, $[\vartheta(a), b] \pm ab \in Z$, $[\vartheta(a), b] \pm ba \in Z$, for all $a, b \in \mathscr{I}$.

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In 2019, they showed that if $\mathscr{R}$ is a prime ring, $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$ and $\vartheta : \mathscr{R} \to \mathscr{R}$ is a zero-power valued homo-derivation on $\mathscr{I}$, then $\mathscr{R}$ is commutative if $\vartheta(ab) - ab \in Z$ for all $a, b \in \mathscr{I}$ [6, Theorem 3.1], or $\vartheta(ab) + ab \in Z$ for all $a, b \in \mathscr{I}$ [6, Theorem 3.2]. In fact, we will prove the same result in [6, Theorem 3.1] by replacing the condition "ϑ is zero-power valued" by "char($\mathscr{R}$)$\neq 2$," as in Theorem 1.2(ii). For ([6, Theorem 3.2]), they made a mistake in the proof because they replaced ϑ by $-\vartheta$, given that $-\vartheta$ is homo-derivation, but this is not true in general, take $\vartheta(a) = -a$ of any ring of characteristic $\neq 2$. So, we will prove the previous result as in Theorem 1.2(i) again. Additional references may be found at [1, 3–5, 9, 10, 12, 15–19].

Motivated by these results, we will investigate the commutativity of prime ring $\mathscr{R}$ with homo-derivation ϑ and a non-zero ideal of $\mathscr{I}$ that fulfills specific algebraic identities. We also provided some examples of why our results hypothesis is essential.

Theorem 1.1. *Let $\mathscr{R}$ be a prime ring, $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$ and ϑ a homo-derivation of $\mathscr{R}$ preserves $\mathscr{I}$. If any one of the following holds, then $\mathscr{R}$ is commutative.*

(i) char($\mathscr{R}$)$\neq 2$ and $\vartheta(a)\vartheta(b) + ab \in Z \ \forall \ a, b \in \mathscr{I}$;

(i) $\vartheta(a) \neq -a$ and $\vartheta(a)\vartheta(b) - ab \in Z \ \forall \ a, b \in \mathscr{I}$;

(ii) char($\mathscr{R}$)$\neq 2$ and $\vartheta(a)\vartheta(b) + ba \in Z \ \forall \ a, b \in \mathscr{I}$;

(iii) $\vartheta(a)\vartheta(b) - ba \in Z \ \forall \ a, b \in \mathscr{I}$.

Theorem 1.2. *Let $\mathscr{R}$ be a prime ring, $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$ and ϑ a homo-derivation of $\mathscr{R}$ preserves $\mathscr{I}$. If any one of the following holds, then $\mathscr{R}$ is commutative.*

(i) ϑ is zero-power valued on $\mathscr{I}$ and $\vartheta(ab) + ab \in Z \ \forall \ a, b \in \mathscr{I}$;

(ii) char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab) - ab \in Z \ \forall \ a, b \in \mathscr{I}$;

(iii) $\vartheta(ab) + ba \in Z \ \forall \ a, b \in \mathscr{I}$;

(iv) $\vartheta(ab) - ba \in Z \ \forall \ a, b \in \mathscr{I}$.

The next example demonstrates that the hypothesis ϑ is zero-power valued on $\mathscr{I}$ in Theorem 1.2(i) is essential.

Example 1.1. *Let $\mathbb{Z}$ be the ring of all integers, $\mathscr{R} = M_2(\mathbb{Z})$, $\mathscr{I} = 2\mathscr{R}$, and define $\vartheta : \mathscr{R} \to \mathscr{R}$ by $\vartheta(a) = -a$. Then it is easy to see that ϑ is a homo-derivation such that $\vartheta(ab) + ab \in Z \ \forall \ a, b \in \mathscr{I}$, but ϑ is not zero-power valued on $\mathscr{I}$ and $\mathscr{R}$ is not commutative.*

The next example shows that char($\mathscr{R}$)$\neq 2$ cannot be omitted in the hypothesis of Theorem 1.2(ii).

Example 1.2. *Let $\mathscr{R} = M_2(\mathbb{Z}_2)$, $\mathscr{I} = \mathscr{R}$, and define $\vartheta : \mathscr{R} \to \mathscr{R}$ by $\vartheta(a) = -a$. Moreover, ϑ is a homo-derivation such that $\vartheta(ab) - ab \in Z \ \forall \ a, b \in \mathscr{I}$, but char($\mathscr{R}$)$= 2$ and $\mathscr{R}$ is not commutative.*

Theorem 1.3. *Let $\mathscr{R}$ be a prime ring, $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$ and ϑ a non-zero homo-derivation of $\mathscr{R}$ preserves $\mathscr{I}$. If any one of the following holds, then $\mathscr{R}$ is commutative.*

(i) $\vartheta(a) \neq -a$ and $\vartheta(ab) + \vartheta(a)\vartheta(b) \in Z \ \forall \ a, b \in \mathscr{I}$;

(ii) char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab) - \vartheta(a)\vartheta(b) \in Z \ \forall \ a, b \in \mathscr{I}$;

(iii) char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab) + \vartheta(b)\vartheta(a) \in Z \ \forall \ a, b \in \mathscr{I}$;

(iv) *char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab) - \vartheta(b)\vartheta(a) \in Z \ \forall\ a, b \in \mathscr{I}$.*

The next example demonstrates that Theorem 1.3(ii) cannot be satisfied without char($\mathscr{R}$)$\neq 2$.

Example 1.3. *As in Example 1.2, we see that $\vartheta(ab) - \vartheta(a)\vartheta(b) \in Z$, $\mathscr{R}$ is prime and char($\mathscr{R}$)$= 2$, but $\mathscr{R}$ is not commutative.*

The next example demonstrates how essential primeness is to our results.

Example 1.4. *Let $\mathscr{R} = \left\{ \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} : r, s, t \in \mathbb{Z} \right\}$, $\mathscr{I} = \left\{ \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} : s \in \mathbb{Z} \right\}$, and let us define $\vartheta : \mathscr{R} \to \mathscr{R}$ by $\vartheta(a) = -a$. Then it is easy to see that ϑ is a homo-derivation on $\mathscr{R}$, which satisfies the next conditions $\vartheta(a)\vartheta(b) + ab \in Z$, $\vartheta(a)\vartheta(b) + ba \in Z$, $\vartheta(ab) - ab \in Z$, $\vartheta(ab) + ba \in Z$, $\vartheta(ab) - ba \in Z$, $\vartheta(ab) - \vartheta(a)\vartheta(b) \in Z$, $\vartheta(ab) + \vartheta(b)\vartheta(a) \in Z$, and $\vartheta(ab) - \vartheta(b)\vartheta(a) \in Z$, $\forall\ a, b\mathscr{I}$, but $\mathscr{R}$ is not commutative.*

2. Preliminaries

The following fundamental identities that satisfy $\forall\ a, b, c \in \mathscr{R}$:

$$[ab, c] = a[b, c] + [a, c]b;$$

$$[a, bc] = b[a, c] + [a, b]c;$$

$$a \circ bc = (a \circ b)c - b[a, c] = b(a \circ c) + [a, b]c;$$

$$ab \circ c = a(b \circ c) - [a, c]b = (a \circ c)b + a[b, c],$$

will be applied without being mentioned.

To achieve our aim, we will use the next lemmas.

Lemma 2.1 ([8], Lemma 4)**.** *Let $\mathscr{R}$ a prime ring and $\{b, ab\} \subseteq Z$. Then $a \in Z$ or $b = 0$.*

Lemma 2.2 ([13], Lemma 2.2)**.** *Let $\mathscr{R}$ be a prime ring. If $\mathscr{R}$ contains a non-zero commutative right ideal, then $\mathscr{R}$ is a commutative ring.*

Lemma 2.3 ([11], Lemma 2.5)**.** *Let $\mathscr{R}$ be a prime ring and $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$. If*

(i) *$[a, b] \in Z$;*

(ii) *$a \circ b \in Z \ \forall\ a, b \in \mathscr{I}$.*

Then $\mathscr{R}$ is commutative.

Lemma 2.4. *Let $\mathscr{R}$ be a prime ring and $\mathscr{I}$ a non-zero ideal of $\mathscr{R}$. For $a, b \in \mathscr{R}$, if $a\mathscr{I}b = \{0\}$, then $a = 0$ or $b = 0$.*

Proof. Let $a, b \in \mathscr{R}$ and $a\mathscr{I}b = \{0\}$. Then $a\mathscr{I}\mathscr{R}b = \{0\}$, and so $b = 0$ or $a\mathscr{I} = \{0\}$. Now, if $a\mathscr{I} = \{0\}$, then $a\mathscr{R}\mathscr{I} = \{0\}$. Hence, $a = 0$ or $\mathscr{I} = \{0\}$. Since $\mathscr{I} \neq \{0\}$, we get $a = 0$. □

Remark 2.1. *Let $\mathscr{R}$ be a prime ring, $\mathscr{I}$ an ideal of $\mathscr{R}$ and ϑ a homo-derivation of $\mathscr{R}$. If $\vartheta(a) = a$ $\forall\ a \in \mathscr{I}$, then char($\mathscr{R}$)$= 2$ or $\mathscr{I} = \{0\}$.*

Proof. Suppose that char($\mathscr{R}$)$\neq 2$. From definition of ϑ, we infer that $\vartheta(ab) = \vartheta(a)\vartheta(b) + \vartheta(a)b + a\vartheta(b)$ $\forall\ a, b \in \mathscr{I}$ and so $ab = ab + ab + ab$ and hence $2ab = 0$ thus $ab = 0$, that is, $\mathscr{I}^2 = \{0\}$ and since $\mathscr{R}$ is prime, we see that $\mathscr{I} = \{0\}$. □

3. The main result

Lemma 3.1. *If char($\mathscr{R}$)$\neq 2$ and $\vartheta(a)\vartheta(b)+ab \in Z$ $\forall$ $a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(a)\vartheta(b)+ab \in Z \quad \forall a,b \in \mathscr{I}. \tag{1}$$

Suppose that $\mathscr{I} \cap Z = \{0\}$.

$$\vartheta(a)\vartheta(b)+ab = 0. \tag{2}$$

Replacing b by bt in (2) and using it, where $t \in \mathscr{I}$, we have

$$\vartheta(a)\vartheta(b)\vartheta(t)+\vartheta(a)b\vartheta(t) = 0. \tag{3}$$

Adding $\pm\vartheta(a)bt$ in (3) and applying (2), we get $\vartheta(a)b(\vartheta(t)-t) = 0$, that is, $\vartheta(a)\mathscr{I}(\vartheta(t)-t) = \{0\}$ and by Lemma 2.4, we infer that $\vartheta(a) = 0$ or $\vartheta(t)-t = 0$. If $\vartheta(t)-t = 0$, then $\vartheta(t) = t$ and by Remark 2.1 a contradiction. Now, in case $\vartheta(a) = 0$. Using the last expression in (2), we infer that $ab = 0$, and so $\mathscr{I} = \{0\}$, a contradiction.

So $\mathscr{I} \cap Z \neq \{0\}$. Substituting bz for b in (1) and applying it, where $0 \neq z \in \mathscr{I} \cap Z$, we obtain

$$\vartheta(a)\vartheta(b)\vartheta(z)+\vartheta(a)b\vartheta(z) \in Z. \tag{4}$$

Adding $\pm\vartheta(a)bz$ in (4), we conclude that $\vartheta(a)(\vartheta(b)\vartheta(z)+bz)+\vartheta(a)b(\vartheta(z)-z) \in Z$ and by using (1) in the last relation, we see that $[\vartheta(a)b(\vartheta(z)-z), \vartheta(a)] = 0$. That is,

$$[\vartheta(a),\vartheta(a)]b(\vartheta(z)-z)+\vartheta(a)[b(\vartheta(z)-z),\vartheta(a)] = 0.$$

Hence,

$$\vartheta(a)[b(\vartheta(z)-z),\vartheta(a)] = 0. \tag{5}$$

Writing z instead of a and b by z' in (1), respectively, where $z, z' \in \mathscr{I} \cap Z$, we obtain

$$\vartheta(z)\vartheta(z') \in Z. \tag{6}$$

Taking z by z^2 in (6) and applying it, we get $\vartheta(z)\vartheta(z)\vartheta(z') \in Z$ and by using (6) in the previous expression, we conclude $\vartheta(z)\vartheta(z') = 0$ or $\vartheta(z) \in Z$. If $\vartheta(z) \in Z$, then from (5), we see that $(\vartheta(z)-z)\vartheta(a)[b,\vartheta(a)] = 0$, that is, $(\vartheta(z)-z)\mathscr{I}\vartheta(a)[b,\vartheta(a)] = \{0\}$ and so $\vartheta(z)-z = 0$ or $\vartheta(a)[b,\vartheta(a)] = 0$. If $\vartheta(z)-z = 0$, then

$$\vartheta(z) = z \neq 0. \tag{7}$$

Now, from definition of ϑ, we have $\vartheta(za) = \vartheta(z)\vartheta(a)+\vartheta(z)a+z\vartheta(a)$. Using (7) in the last relation, we get $\vartheta(za) = 2z\vartheta(a)+za$. Taking a by za in (1) and using the previous relation, we infer that $2z\vartheta(a)\vartheta(b)+za\vartheta(b)+zab \in Z$. Applying (1) in the last expression, we find that $z\vartheta(a)\vartheta(b)+za\vartheta(b) \in Z$. Thus,

$$(\vartheta(a)+a)\vartheta(b) \in Z. \tag{8}$$

Taking $b = z$ in (8) and using (7), we see that $\vartheta(a)+a \in Z$. Using the previous relation and Lemma 2.1 in (8), we arrive at $\vartheta(a)+a = 0$ or $\vartheta(b) \in Z$. In case $\vartheta(b) \in Z$. Applying the previous relation in (1), we obtain $ab \in Z$. Putting $b = z$ in the last relation, we have $a \in Z$. Hence, $\mathscr{I} \subseteq \mathscr{R}$, and by Lemma 2.2, $\mathscr{R}$ is commutative. Now, if $\vartheta(a)+a = 0$, then $\vartheta(a) = -a$. Taking $a = z$ in the last relation and by (7), we get a contradiction.

Suppose that

$$\vartheta(a)[b,\vartheta(a)] = 0. \tag{9}$$

Replacing b by sb in (9) and applying it, where $s \in \mathscr{I}$, we get $\vartheta(a)s[b,\vartheta(a)] = 0$, that is, $\vartheta(a)\mathscr{I}[b,\vartheta(a)] = \{0\}$ and so $\vartheta(a) = 0$ or $[b,\vartheta(a)] = 0$. Suppose that $[b,\vartheta(a)] = 0$. Substituting at for a in the last expression and using it, where $t \in \mathscr{I}$, we conclude $\vartheta(a)[b,t] + [b,a]\vartheta(t) = 0$. Taking $b = a$ in the previous expression, we see that $\vartheta(a)[a,t] = 0$. Writing ba instead of t in the last relation and applying it, where $b \in \mathscr{I}$, we find that $\vartheta(a)b[a,t] = 0$, that is, $\vartheta(a)\mathscr{I}[a,t] = \{0\}$ and so $\vartheta(a) = 0$ or $[a,t] = 0$. If $[a,t] = 0$, then $\mathscr{I} \subseteq Z$, and by Lemma 2.2, we get $\mathscr{R}$ is commutative. Now, suppose that

$$\vartheta(a) = 0. \tag{10}$$

Applying (10) in (1), we infer that $ab \in Z$. Putting $0 \neq b = z \in \mathscr{I} \cap Z$ and by Lemma 2.1, we get $a \in Z$, and by Lemma 2.2, we obtain $\mathscr{R}$ is commutative. Now, suppose that $\vartheta(z)\vartheta(z') = 0$. Replacing z' by z in the last expression, we get

$$\vartheta(z)\vartheta(z) = 0. \tag{11}$$

Substituting $b\vartheta(z)$ for b in (5) and using (11), we conclude $\vartheta(a)[-b\vartheta(z)z,\vartheta(a)] = 0$, that is, $\vartheta(a)[b\vartheta(z),\vartheta(a)] = 0$ by applying the last relation in (5), we obtain $\vartheta(a)[-bz,\vartheta(a)] = 0$, that is, $\vartheta(a)[b,\vartheta(a)] = 0$. As in (9), we have $\mathscr{R}$ is commutative. □

Lemma 3.2. *If $\vartheta(a)\vartheta(b) - ab \in Z \;\forall\; a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative or any homo-derivation ϑ is of the form $\vartheta(a) = -a$.*

Proof. Assume that

$$\vartheta(a)\vartheta(b) - ab \in Z \quad \forall a,b \in \mathscr{I}. \tag{12}$$

Suppose that $\mathscr{I} \cap Z = \{0\}$.

$$\vartheta(a)\vartheta(b) - ab = 0. \tag{13}$$

Writing bt instead of b in (13) and using it, where $t \in \mathscr{I}$, we see that $\vartheta(a)\vartheta(b)\vartheta(t)+\vartheta(a)b\vartheta(t) = 0$. Adding $\pm\vartheta(a)bt$ in the previous expression and applying (13), we get $\vartheta(a)b(\vartheta(t) + t) = 0$, that is, $\vartheta(a)\mathscr{I}(\vartheta(t) + t) = \{0\}$ and so $\vartheta(a) = 0$ or $\vartheta(t) + t = 0$. In case $\vartheta(a) = 0$ and using it in (13), we obtain $ab = 0$ and so $\mathscr{I} = \{0\}$, a contradiction. If $\vartheta(t) + t = 0$, then $\vartheta(t) = -t$.

Now, in case $\mathscr{I} \cap Z \neq \{0\}$. Replacing b by bz in (12) and applying it, where $0 \neq z \in \mathscr{I} \cap Z$, we have

$$\vartheta(a)\vartheta(b)\vartheta(z) + \vartheta(a)b\vartheta(z) \in Z. \tag{14}$$

Adding $\pm\vartheta(a)bz$ in (14), we conclude $\vartheta(a)(\vartheta(b)\vartheta(z) - bz) + \vartheta(a)b(\vartheta(z) + z) \in Z$. Using (12) in the last relation, we obtain

$$\vartheta(a)[b(\vartheta(z) + z),\vartheta(a)] = 0. \tag{15}$$

Now as Lemma 3.1 in Eq. (5), we have $\mathscr{R}$ is commutative or $\vartheta(z) + z = 0$. If $\vartheta(z) + z = 0$, then $\vartheta(z) = -z \neq 0$ and by applying the previous expression in (14), we get

$$\vartheta(a)(\vartheta(b) + b) \in Z. \tag{16}$$

Putting $0 \neq a = z \in \mathscr{I} \cap Z$ in (16), this gives $\vartheta(z)(\vartheta(b) + b) \in Z$, that is, $-z(\vartheta(b) + b) \in Z$, and hence $\vartheta(b) + b \in Z$ by Lemma 2.1. Using the previous relation in (16) and by Lemma 2.1, we see that $\vartheta(a) \in Z$ or $\vartheta(b) + b = 0$. If $\vartheta(b) + b = 0$, then $\vartheta(b) = -b$. In case $\vartheta(a) \in Z$ and by (12), we infer that $ab \in Z$. Putting $0 \neq b = z \in \mathscr{I} \cap Z$ and by Lemma 2.1, we conclude that $a \in Z$, and by Lemma 2.2, we get $\mathscr{R}$ is commutative. □

Lemma 3.3. *If char($\mathscr{R}$)$\neq 2$ and $\vartheta(a)\vartheta(b)+ba \in Z \ \forall\ a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(a)\vartheta(b)+ba \in Z \quad \forall a,b \in \mathscr{I}. \tag{17}$$

Suppose that $\mathscr{I} \cap Z = \{0\}$.

$$\vartheta(a)\vartheta(b)+ba = 0. \tag{18}$$

Substituting bt for b in (18) and applying it, where $t \in \mathscr{I}$, we conclude

$$\vartheta(a)\vartheta(b)\vartheta(t)+\vartheta(a)\vartheta(b)t+\vartheta(a)b\vartheta(t)+bta = 0.$$

Adding $\pm bat$ in the last expression and using (18), we get

$$\vartheta(a)\vartheta(b)\vartheta(t)+\vartheta(a)b\vartheta(t)+b[t,a] = 0.$$

Adding $\pm\vartheta(a)tb$ in the above relation and applying (18), we obtain

$$\vartheta(a)b\vartheta(t)-\vartheta(a)tb+b[t,a] = 0. \tag{19}$$

Writing $\vartheta(a)b$ instead of b in (19), this gives

$$\vartheta(a)^2b\vartheta(t)-\vartheta(a)t\vartheta(a)b+\vartheta(a)b[t,a] = 0. \tag{20}$$

Left multiplying (19) by $\vartheta(a)$, we see that

$$\vartheta(a)^2b\vartheta(t)-\vartheta(a)^2tb+\vartheta(a)b[t,a] = 0. \tag{21}$$

Comparing (20) and (21), this gives $\vartheta(a)[\vartheta(a),t]b = 0$, that is, $\vartheta(a)[\vartheta(a),t]\mathscr{I} = 0$ and since $\mathscr{I} \neq 0$, we infer that $\vartheta(a)[\vartheta(a),t] = 0$. Now as in Lemma 3.1 in (9), we have $\vartheta(a) = 0$ or $\mathscr{R}$ is commutative. If $\mathscr{R}$ is commutative, then $\mathscr{I} \cap \mathscr{R} = \{0\}$, and so $\mathscr{I} = \{0\}$, a contradiction. In case $\vartheta(a) = 0$ and from (18), we get $ba = 0$ and so $\mathscr{I} = \{0\}$, contradiction.

So, $\mathscr{I} \cap Z \neq \{0\}$. Replacing b by bz in (17) and using it, where $0 \neq z \in \mathscr{I} \cap Z$, we conclude

$$\vartheta(a)\vartheta(b)\vartheta(z)+\vartheta(a)b\vartheta(z) \in Z.$$

Adding $\pm\vartheta(a)zb$ in the previous expression, we get

$$\vartheta(a)(\vartheta(b)\vartheta(z)+zb)+\vartheta(a)(b\vartheta(z)-zb) \in Z.$$

Using (17) in the last relation, we see that $\vartheta(a)[b(\vartheta(z)-z),\vartheta(a)] = 0$. Now, as in Lemma 3.1 in Eq. (5), we have $\mathscr{R}$ is commutative. □

Lemma 3.4. *If $\vartheta(a)\vartheta(b)-ba \in Z \ \forall\ a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(a)\vartheta(b)-ba \in Z \quad \forall a,b \in \mathscr{I}. \tag{22}$$

In case $\mathscr{I} \cap Z = \{0\}$ as Lemma 3.3, we have a contradiction. So, $\mathscr{I} \cap Z \neq \{0\}$. Substituting bz for b in (22) and applying it, where $0 \neq z \in \mathscr{I} \cap Z$, we conclude

$$\vartheta(a)\vartheta(b)\vartheta(z)+\vartheta(a)b\vartheta(z) \in Z. \tag{23}$$

Adding $\pm\vartheta(a)zb$ in (23), we get

$$\vartheta(a)(\vartheta(b)\vartheta(z)-zb)+\vartheta(a)b(\vartheta(z)+z) \in Z.$$

Using (22) in the above expression, we obtain $\vartheta(a)[b(\vartheta(z)+z),\vartheta(a)] = 0$. Now, as in Lemma 3.2 in Eq. (15), we have $\mathscr{R}$ is commutative or $\vartheta(a) = -a$. Now, in case $\vartheta(a) = -a$. Applying the previous relation in (22), we get $[a,b] \in Z$ and by using Lemma 2.3, we obtain $\mathscr{R}$ is commutative. □

Lemma 3.5. *If ϑ is zero-power valued on $\mathscr{I}$ and $\vartheta(ab)+ab \in Z\ \forall\ a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(ab)+ab \in Z \quad \forall a,b \in \mathscr{I}. \tag{24}$$

Writing bt instead of b in (24), where $t \in \mathscr{I}$, we see that $(\vartheta(ab)+ab)(\vartheta(t)+t) \in Z$ and by applying (24) in the last relation and by Lemma 2.1, we get $\vartheta(ab)+ab=0$ or $\vartheta(t)+t \in Z$. If $\vartheta(ab)+ab=0$, then

$$(\vartheta(a)+a)(\vartheta(b)+b)=0. \tag{25}$$

Replacing b by $\vartheta^{n-1}(b)$ in (25), where $\vartheta^n(b)=0$, we conclude

$$(\vartheta(a)+a)\vartheta^{n-1}(b)=0. \tag{26}$$

Substituting $\vartheta^{n-2}(b)$ for b in (25) and using (26), we see that

$$(\vartheta(a)+a)\vartheta^{n-2}(b)=0$$

and by repeating the previous steps, we conclude that $(\vartheta(a)+a)b=0$. Now as in Eq. (25), we have $ab=0$ and so $\mathscr{I}=\{0\}$, contradiction. So $\vartheta(t)+t \in Z$. As in Eq. (25), we get $t \in Z$ and by Lemma 2.2, $\mathscr{R}$ is commutative. □

Lemma 3.6. *If char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab)-ab \in Z\ \forall\ a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(ab)-ab \in Z \quad \forall a,b \in \mathscr{I}. \tag{27}$$

Writing bt instead of b in (27), where $t \in \mathscr{I}$, we conclude

$$(\vartheta(ab)-ab)(\vartheta(t)+t)+2ab\vartheta(t) \in Z. \tag{28}$$

Thus $2[ab\vartheta(t),\vartheta(t)+t]=0$, that is, $[ab\vartheta(t),\vartheta(t)+t]=0$. Replacing a by sa in the previous expression and applying it, where $s \in \mathscr{I}$, we get $[s,\vartheta(t)+t]ab\vartheta(t)=0$, that is, $[s,\vartheta(t)+t]\mathscr{I}b\vartheta(t)=\{0\}$ and so $[s,\vartheta(t)+t]=0$ or $b\vartheta(t)=0$. If $b\vartheta(t)=0$, then $\mathscr{I}\vartheta(t)=\{0\}$ and hence $\vartheta(t)=0$. Using the last relation in (27), we obtain

$$ab \in Z. \tag{29}$$

Replacing a by b and b by a in (29), we see that $ba \in Z$. From the last expression and (29), we have $[a,b] \in Z$ and by Lemma 2.3(i), $\mathscr{R}$ is commutative. Suppose that $[s,\vartheta(t)+t]=0$. Substituting sr for s in the previous relation and applying it, where $r \in \mathscr{R}$, we get $s[r,\vartheta(t)+t]=0$, that is, $\mathscr{I}[r,\vartheta(t)+t]=\{0\}$ and so $[r,\vartheta(t)+t]=0$ thus $\vartheta(t)+t \in Z$. Again writing ab instead of t in the last expression, we see that $\vartheta(ab)+ab \in Z$. Comparing the last relation and (27), we conclude $a \circ b \in Z$, and by Lemma 2.3(ii), $\mathscr{R}$ is commutative. □

Lemma 3.7. *If $\vartheta(ab)+ba \in Z\ \forall\ a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(ab)+ba \in Z \quad \forall a,b \in \mathscr{I}. \tag{30}$$

Suppose that $\mathscr{I} \cap Z = \{0\}$.

$$\vartheta(ab) + ba = 0. \tag{31}$$

Replacing b by bt in (31), where $t \in \mathscr{I}$, we conclude $(\vartheta(ab)+ab)\vartheta(t)+b[t,a] = 0$. Adding $\pm ba\vartheta(t)$ in the previous expression and using (31), we get $b([t,a] - a\vartheta(t)) = 0$ and since $\mathscr{I} \neq \{0\}$, we infer that

$$[t,a] - a\vartheta(t) = 0. \tag{32}$$

Substituting sa for a in (32), where $s \in \mathscr{I}$, we obtain

$$s[t,a] + [t,s]a - sa\vartheta(t) = 0. \tag{33}$$

Left multiplying (32) by s, we see that

$$s[t,a] - sa\vartheta(t) = 0. \tag{34}$$

Comparing (33) and (34), this gives $[t,s]a = 0$, that is, $[t,s]\mathscr{I} = \{0\}$ and since $\mathscr{I} \neq \{0\}$, we infer that $[t,s] = 0$ and so $\mathscr{I} \subseteq Z$ and by Lemma 2.2, $\mathscr{R}$ is commutative.

Suppose that $\mathscr{I} \cap Z \neq \{0\}$. Writing bz instead of b in (30) and applying it, where $0 \neq z \in \mathscr{I} \cap Z$, we conclude

$$(\vartheta(ab) + ab)\vartheta(z) \in Z. \tag{35}$$

Putting $a = b = z$ in (30), where $0 \neq z \in \mathscr{I} \cap Z$, we get

$$\vartheta(z^2) \in Z. \tag{36}$$

Taking z by z^2 in (35), we have $(\vartheta(ab) + ab)\vartheta(z^2) \in Z$ and by using (36) in the last relation, we see that $\vartheta(ab) + ab \in Z$ or $\vartheta(z^2) = 0$. If $\vartheta(ab) + ab \in Z$ and by applying (30), then $[a,b] \in Z$ and so $\mathscr{R}$ is commutative. Suppose that

$$\vartheta(z^2) = 0. \tag{37}$$

Replacing b by bt in (30), we conclude $\vartheta(ab)\vartheta(t)+\vartheta(ab)t+ab\vartheta(t)+bta \in Z$, where $t \in \mathscr{I}$. Putting $a = b = z$ in the last expression and using (37), where $0 \neq z \in \mathscr{I} \cap Z$, this gives $\vartheta(t) + t \in Z$. Taking t by ab in the previous relation where $a, b \in \mathscr{I}$, we get $\vartheta(ab) + ab \in Z$. Using the last expression in (30), we conclude $[a,b] \in Z$ and so $\mathscr{R}$ is commutative. □

Lemma 3.8. *If $\vartheta(ab) - ba \in Z \ \forall\ a, b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Let $\mathscr{I} \cap Z = \{0\}$. As in Lemma 3.7. Suppose that $\mathscr{I} \cap Z \neq \{0\}$. As in Lemma 3.7 and applying Lemma 2.3(ii). □

Lemma 3.9. *If $\vartheta \neq 0$ and $\vartheta(ab) + \vartheta(a)\vartheta(b) \in Z \ \forall\ a, b \in \mathscr{I}$, then $\mathscr{R}$ is commutative or any homo-derivation ϑ is of the form $\vartheta(a) = -a$.*

Proof. Assume that

$$\vartheta(ab) + \vartheta(a)\vartheta(b) \in Z \quad \forall a, b \in \mathscr{I}. \tag{38}$$

Suppose that $\mathscr{I} \cap Z = \{0\}$.

$$\vartheta(ab) + \vartheta(a)\vartheta(b) = 0. \tag{39}$$

Substituting bt for b in (39) and using it, where $t \in \mathscr{I}$, we see that $(\vartheta(a)+a)b\vartheta(t) = 0$, that is, $(\vartheta(a)+a)\mathscr{I}\vartheta(t) = \{0\}$ and so $\vartheta(a)+a = 0$ or $\vartheta(t) = 0$. In case

$$\vartheta(t) = 0 \quad \forall t \in \mathscr{I}. \tag{40}$$

Writing rt instead of t in the last relation and applying it, where $r \in \mathscr{R}$, we get $\vartheta(r)t = 0$, that is $\vartheta(r)\mathscr{I} = \{0\}$ and so $\vartheta(r) = 0 \; \forall \; r \in \mathscr{R}$, a contradiction. In case $\vartheta(a)+a = 0$, we infer that $\vartheta(a) = -a$.

Suppose that $\mathscr{I} \cap Z \neq \{0\}$. Replacing b by bz in (38) and using it, where $0 \neq z \in \mathscr{I} \cap Z$, we conclude

$$(\vartheta(ab) + \vartheta(a)\vartheta(b))\vartheta(z) + (a+\vartheta(a))b\vartheta(z) \in Z. \tag{41}$$

Using (38) in (41), we obtain

$$[(a+\vartheta(a))b, \vartheta(z)]\vartheta(z) = 0. \tag{42}$$

Substituting $(z+\vartheta(z))b$ for b in (42) and applying it, where $0 \neq z \in \mathscr{I} \cap Z$, we conclude $[a+\vartheta(a), \vartheta(z)]zb\vartheta(z) = 0$ implies that $[a+\vartheta(a), \vartheta(z)]b\vartheta(z) = 0$, that is, $[a+\vartheta(a), \vartheta(z)]\mathscr{I}\vartheta(z) = \{0\}$ and so $[a+\vartheta(a), \vartheta(z)] = 0$ or $\vartheta(z) = 0$. In case

$$\vartheta(z) = 0. \tag{43}$$

Putting $b = z$ in (38) and using (43), where $0 \neq z \in \mathscr{I} \cap Z$, we see that $\vartheta(a)z \in Z$ and hence

$$\vartheta(a) \in Z. \tag{44}$$

Writing ab instead of a in (44) and applying it, where $b \in \mathscr{I}$, we get $\vartheta(a)b + a\vartheta(b) \in Z$ and by using (44) in the previous expression, we have $\vartheta(a)[b,a] = 0$. Replacing b by sb in the last relation and applying it, where $s \in \mathscr{I}$, we obtain $\vartheta(a)s[b,a] = 0$, that is $\vartheta(a)\mathscr{I}[b,a] = \{0\}$ and so $\vartheta(a) = 0$ or $[b,a] = 0$. If $[b,a] = 0$, then $\mathscr{I} \subseteq Z$ and so $\mathscr{R}$ is commutative. In case $\vartheta(a) = 0$ the same as in (40), we infer that $\vartheta = 0$, a contradiction. Now, suppose that $\vartheta(z) \neq 0$ and $[a+\vartheta(a), \vartheta(z)] = 0$. By using the last expression in (42), this gives $(a+\vartheta(a))[b,\vartheta(z)]\vartheta(z) = 0$. Substituting bs for b in the previous relation and applying it, where $s \in \mathscr{I}$, we conclude $(a+\vartheta(a))[b,\vartheta(z)]s\vartheta(z) = 0$, that is, $(a+\vartheta(a))[b,\vartheta(z)]\mathscr{I}\vartheta(z) = \{0\}$ and so $(a+\vartheta(a))[b,\vartheta(z)] = 0$ or $\vartheta(z) = 0$. But $\vartheta(z) \neq 0$ and hence $(a+\vartheta(a))[b,\vartheta(z)] = 0$. Writing sb instead of b in the last expression and using it, where $s \in \mathscr{I}$, we get $(a+\vartheta(a))s[b,\vartheta(z)] = 0$, that is, $(a+\vartheta(a))\mathscr{I}[b,\vartheta(z)] = \{0\}$ and so $a+\vartheta(a) = 0$ or $[b,\vartheta(z)] = 0$. If $a+\vartheta(a) = 0$, then $\vartheta(a) = -a$. Now, suppose that $[b,\vartheta(z)] = 0$. Replacing b by br in the last relation and applying it, where $r \in \mathscr{R}$, we have $b[r,\vartheta(z)] = 0$, that is, $\mathscr{I}[r,\vartheta(z)] = \{0\}$ and so $[r,\vartheta(z)] = 0$ this implies that

$$\vartheta(z) \in Z. \tag{45}$$

Using (45) and (38) in (41), we see that $(a+\vartheta(a))b\vartheta(z) \in Z$ and by applying (45) in the previous expression, we get $(a+\vartheta(a))b \in Z$ or $\vartheta(z) = 0$. In case $\vartheta(z) = 0$ as (43). Now, suppose that $(a+\vartheta(a))b \in Z$. Substituting br for b in the last relation and using it, where $r \in \mathscr{R}$, we obtain $(a+\vartheta(a))b = 0$ or $r \in Z$. If $r \in Z$, then $\mathscr{R}$ is commutative. If $(a+\vartheta(a))b = 0$, then $(a+\vartheta(a))\mathscr{I} = \{0\}$ and so $a+\vartheta(a) = 0$, and hence $\vartheta(a) = -a$. □

Lemma 3.10. *If char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab) - \vartheta(a)\vartheta(b) \in Z \; \forall \; a,b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(ab) - \vartheta(a)\vartheta(b) \in Z \quad \forall a,b \in \mathscr{I}. \tag{46}$$

$$\vartheta(a)b + a\vartheta(b) \in Z. \tag{47}$$

Writing bt instead of b in (47), where $t \in \mathscr{I}$, we conclude

$$(\vartheta(a)b + a\vartheta(b))t + a(\vartheta(b) + b)\vartheta(t) \in Z$$

and by applying (47) in the last expression, we get $[a(\vartheta(b) + b)\vartheta(t), t] = 0$. Replacing a by sa in the previous relation and using it, where $s \in \mathscr{I}$, we obtain $[s,t]a(\vartheta(b) + b)\vartheta(t) = 0$, that is, $[s,t]\mathscr{I}(\vartheta(b) + b)\vartheta(t) = \{0\}$ and so $[s,t] = 0$ or $(\vartheta(b) + b)\vartheta(t) = 0$. If $[s,t] = 0$, then $\mathscr{I} \subseteq Z$ and hence $\mathscr{R}$ is commutative. Suppose that $(\vartheta(b) + b)\vartheta(t) = 0$. Substituting at for t in the last expression and applying it, where $a \in \mathscr{I}$, we see that $(\vartheta(b) + b)a\vartheta(t) = 0$, that is, $(\vartheta(b) + b)\mathscr{I}\vartheta(t) = \{0\}$ and so $\vartheta(b) + b = 0$ or $\vartheta(t) = 0$. In case $\vartheta(t) = 0$, the same as in (40), we infer that $\vartheta = 0$, a contradiction. If $\vartheta(b) + b = 0$, then $\vartheta(b) = -b$ and by using the last relation in (46), we conclude $2ab \in Z$ and so $ab \in Z$. Now as Lemma 3.6 in Eq.(29) we get $\mathscr{R}$ is commutative. □

Lemma 3.11. *If char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab) + \vartheta(b)\vartheta(a) \in Z$ $\forall$ $a, b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(ab) + \vartheta(b)\vartheta(a) \in Z \quad \forall a, b \in \mathscr{I}. \tag{48}$$

Suppose that $\mathscr{I} \cap Z = \{0\}$. Then

$$\vartheta(ab) + \vartheta(b)\vartheta(a) = 0, \tag{49}$$

that is,

$$\vartheta(a) \circ \vartheta(b) + \vartheta(a)b + a\vartheta(b) = 0. \tag{50}$$

Writing ab instead of b in (50), we conclude $\vartheta(a) \circ \vartheta(ab) + \vartheta(a)ab + a\vartheta(ab) = 0$. Using (49) in the previous expression, we get

$$-(\vartheta(a) \circ \vartheta(b)\vartheta(a)) + \vartheta(a)ab - a\vartheta(b)\vartheta(a) = 0.$$

That is,

$$-(\vartheta(a) \circ \vartheta(b))\vartheta(a) + \vartheta(a)ab - a\vartheta(b)\vartheta(a) = 0. \tag{51}$$

Right multiplying (50) by $\vartheta(a)$, this gives

$$(\vartheta(a) \circ \vartheta(b))\vartheta(a) + \vartheta(a)b\vartheta(a) + a\vartheta(b)\vartheta(a) = 0. \tag{52}$$

Comparing (51) and (52), we obtain

$$\vartheta(a)b\vartheta(a) + \vartheta(a)ab = 0. \tag{53}$$

Replacing b by bt in (51), where $t \in \mathscr{I}$, we see that

$$\vartheta(a)bt\vartheta(a) + \vartheta(a)abt = 0. \tag{54}$$

Right multiplying (53) by t, where $t \in \mathscr{I}$, we conclude

$$\vartheta(a)b\vartheta(a)t + \vartheta(a)abt = 0. \tag{55}$$

Comparing (54) and (55), we get

$$\vartheta(a)b[\vartheta(a),t]=0, \tag{56}$$

that is, $\vartheta(a)\mathscr{I}[\vartheta(a),t]=\{0\}$ and so $\vartheta(a)=0$ or $[\vartheta(a),t]=0$. Assume that $\vartheta(a)=0$, the same as in (40), we infer that $\vartheta=0$, a contradiction. In case $[\vartheta(a),t]=0$. Substituting ab for a in the last relation and using it, where $b\in\mathscr{I}$, we obtain $\vartheta(a)[b,t]+[a,t]\vartheta(b)=0$. Putting $t=b$ in the last expression, this gives $[a,b]\vartheta(b)=0$. Writing as instead of a in the previous relation and applying it, where $s\in\mathscr{I}$, we see that $[a,b]s\vartheta(b)=0$, that is, $[a,b]\mathscr{I}\vartheta(b)=\{0\}$ and so $[a,b]=0$ or $\vartheta(b)=0$. If $\vartheta(b)=0$, then $\vartheta=0$, a contradiction. In case $[a,b]=0$, we infer that $\mathscr{R}$ is commutative.

Now, suppose that $\mathscr{I}\cap Z\neq\{0\}$. We can write Eq. (48) as

$$(\vartheta(a)\circ\vartheta(b))+\vartheta(a)b+a\vartheta(b)\in Z. \tag{57}$$

Replacing b by bz in (57) and using it, where $0\neq z\in\mathscr{I}\cap Z$, we get

$$(\vartheta(a)\circ\vartheta(b)\vartheta(z))+(\vartheta(a)\circ b\vartheta(z))+a\vartheta(b)\vartheta(z)+ab\vartheta(z)\in Z. \tag{58}$$

Putting $a=z$ and adding $\pm\vartheta(z)b\vartheta(z)$ in (58), we have

$$\{(\vartheta(z)\circ\vartheta(b))+z\vartheta(b)+\vartheta(z)b\}\vartheta(z)+b\{\vartheta(z)^2+z\vartheta(z)\}\in Z. \tag{59}$$

Taking $a=b=z$ in (57) and since char($\mathscr{R}$)$\neq 2$, we infer that

$$\vartheta(z)^2+z\vartheta(z)\in Z. \tag{60}$$

Using (57) and (60) in (59), we obtain $\{(\vartheta(z)\circ\vartheta(b))+z\vartheta(b)+\vartheta(z)b\}[\vartheta(z),b]=0$. Applying (57) in the last expression, we get $\{(\vartheta(z)\circ\vartheta(b))+z\vartheta(b)+\vartheta(z)b\}\mathscr{I}[\vartheta(z),b]=\{0\}$ and so $(\vartheta(z)\circ\vartheta(b))+z\vartheta(b)+\vartheta(z)b=0$ or $[\vartheta(z),b]=0$.

Suppose that

$$[\vartheta(z),b]=0. \tag{61}$$

Substituting rb for b in (61) and using it, where $r\in R$, we conclude $[\vartheta(z),r]b=0$, that is, $[\vartheta(z),r]\mathscr{I}=\{0\}$ and so $[\vartheta(z),r]=0$ and hence

$$\vartheta(z)\in Z. \tag{62}$$

Adding $\pm\vartheta(a)b\vartheta(z)$ in (58) and applying (57) and (62), we see that $(b\vartheta(a)+ab)\vartheta(z)\in Z$ and by using (62) in the last relation, we get $b\vartheta(a)+ab\in Z$ or $\vartheta(z)=0$. In case

$$\vartheta(z)=0. \tag{63}$$

Putting $b=z$ in (57), where $0\neq z\in\mathscr{I}\cap Z$ and applying the previous expression, we conclude $z\vartheta(a)\in Z$ and so

$$\vartheta(a)\in Z. \tag{64}$$

Writing ab instead of a in (64) and using it, where $b\in\mathscr{I}$, we obtain $\vartheta(a)b+a\vartheta(b)\in Z$ and by applying (64) in the last relation, we get $[\vartheta(a)b+a\vartheta(b),r]=0$, where $r\in R$, by using (64) in the last expression, we see that $\vartheta(a)[b,r]+[a,r]\vartheta(b)=0$. Taking $r=b$ in the previous relation, this gives $[a,b]\vartheta(b)=0$. Replacing a by as in the last expression, where $s\in\mathscr{I}$, we conclude $[a,b]s\vartheta(b)=0$, that is, $[a,b]\mathscr{I}\vartheta(b)=\{0\}$ and so $[a,b]=0$ or $\vartheta(b)=0$. If $[a,b]=0$, then $\mathscr{R}$ is commutative. If $\vartheta(b)=0$, then $\vartheta=0$, a contradiction. Now, suppose that

$$b\vartheta(a)+ab\in Z. \tag{65}$$

Putting $0 \neq b = z \in \mathscr{I} \cap Z$ in (65), we get

$$\vartheta(a) + a \in Z. \tag{66}$$

Adding $\pm ba$ in (65), this gives $b(\vartheta(a) + a) + [a, b] \in Z$. Using 66) in the last relation, we obtain $[[a, b], b] = 0$. Substituting ta for a in the previous expression and applying it, where $t \in \mathscr{I}$, we have $2[t, b][a, b] = 0$ and hence $[t, b][a, b] = 0$. Writing sa instead of a in the last relation and using it, where $s \in \mathscr{I}$, we see that $[t, b]s[a, b] = 0$, that is, $[t, b]\mathscr{I}[a, b] = \{0\}$ and so $[t, b] = 0$ or $[a, b] = 0$. In two cases $\mathscr{R}$ is commutative.

Now, in case $(\vartheta(z) \circ \vartheta(b)) + z\vartheta(b) + \vartheta(z)b = 0$ as in Eq.(50), we conclude as Eq.(56), $\vartheta(z)b[\vartheta(z), t] = 0$, that is, $\vartheta(z)\mathscr{I}[\vartheta(z), t] = \{0\}$ and so $\vartheta(z) = 0$ or $[\vartheta(z), t] = 0$. In case $[\vartheta(z), t] = 0$, as in (61). In case $\vartheta(z) = 0$ as in (63). □

Lemma 3.12. *If char($\mathscr{R}$)$\neq 2$ and $\vartheta(ab) - \vartheta(b)\vartheta(a) \in Z \ \forall\ a, b \in \mathscr{I}$, then $\mathscr{R}$ is commutative.*

Proof. Assume that

$$\vartheta(ab) - \vartheta(b)\vartheta(a) \in Z \quad \forall a, b \in \mathscr{I}. \tag{67}$$

Suppose that $\mathscr{I} \cap Z = \{0\}$. Then

$$\vartheta(ab) - \vartheta(b)\vartheta(a) = 0. \tag{68}$$

That is,

$$[\vartheta(a), \vartheta(b)] + \vartheta(a)b + a\vartheta(b) = 0. \tag{69}$$

Replacing b by ab in (69), we have $[\vartheta(a), \vartheta(ab)] + \vartheta(a)ab + a\vartheta(ab) = 0$. Using (68) in the last relation, we get $[\vartheta(a), \vartheta(b)\vartheta(a)] + \vartheta(a)ab + a\vartheta(b)\vartheta(a) = 0$. Thus,

$$[\vartheta(a), \vartheta(b)]\vartheta(a) + \vartheta(a)ab + a\vartheta(b)\vartheta(a) = 0. \tag{70}$$

Right multiplying (69) by $\vartheta(a)$ this gives

$$[\vartheta(a), \vartheta(b)]\vartheta(a) + \vartheta(a)b\vartheta(a) + a\vartheta(b)\vartheta(a) = 0. \tag{71}$$

Comparing (70) and (71), we obtain

$$\vartheta(a)b\vartheta(a) - \vartheta(a)ab = 0. \tag{72}$$

Substituting bt for b in (72), where $t \in \mathscr{I}$, we see that

$$\vartheta(a)bt\vartheta(a) - \vartheta(a)abt = 0. \tag{73}$$

Right multiplying (72) by t, where $t \in \mathscr{I}$, we conclude

$$\vartheta(a)b\vartheta(a)t - \vartheta(a)abt = 0. \tag{74}$$

Comparing (73) and (74), we get $\vartheta(a)b[\vartheta(a), t] = 0$, that is, $\vartheta(a)\mathscr{I}[\vartheta(a), t] = \{0\}$ and so $\vartheta(a) = 0$ or $[\vartheta(a), t] = 0$. If $\vartheta(a) = 0$, then $\vartheta = 0$, a contradiction. In case $[\vartheta(a), t] = 0$. Writing ab instead of a in the previous relation and using it, where $b \in \mathscr{I}$, we obtain $\vartheta(a)[b, t] + [a, t]\vartheta(b) = 0$. Putting $t = b$ in the last expression, this gives $[a, b]\vartheta(b) = 0$. Replacing a by as in the last relation and applying it, where $s \in \mathscr{I}$, we see that $[a, b]s\vartheta(b) = 0$, that is, $[a, b]\mathscr{I}\vartheta(b) = \{0\}$ and so $[a, b] = 0$ or $\vartheta(b) = 0$. If $\vartheta(b) = 0$, then $\vartheta = 0$, a contradiction. In case $[a, b] = 0$, we infer that $\mathscr{R}$ is commutative.

Now, suppose that $\mathscr{I} \cap Z \neq \{0\}$. Putting $a = b = z$ in (67), where $0 \neq z \in \mathscr{I} \cap Z$, we conclude

$$\vartheta(z) \in Z. \tag{75}$$

Substituting bz for b in (67) and using it and (75), where $0 \neq z \in \mathscr{I} \cap Z$, we get

$$[\vartheta(a), \vartheta(b)]\vartheta(z) + [\vartheta(a), b]\vartheta(z) + a\vartheta(b)\vartheta(z) + ab\vartheta(z) \in Z.$$

Adding $\pm\vartheta(a)b\vartheta(z)$ in the previous expression and suing (67) and (75), we obtain

$$([\vartheta(a), b] + ab - \vartheta(a)b)\vartheta(z) \in Z$$

and so $[\vartheta(a), b] + ab - \vartheta(a)b \in Z$ or $\vartheta(z) = 0$. In case

$$\vartheta(z) = 0. \tag{76}$$

Taking $b = z$ in (67) and applying the above relation, where $0 \neq z \in \mathscr{I} \cap Z$, we see that $\vartheta(a)z \in Z$ and so

$$\vartheta(a) \in Z. \tag{77}$$

By using (77) in (67), we have $\vartheta(a)b + a\vartheta(b) \in Z$ and so $\vartheta(a)[b, a] = 0$. Writing sb instead of b in the last expression, where $s \in \mathscr{I}$, we get $\vartheta(a)s[b, a] = 0$, that is, $\vartheta(a)\mathscr{I}[b, a] = \{0\}$ and so $\vartheta(a) = 0$ or $[b, a] = 0$. If $[b, a] = 0$, then $\mathscr{R}$ is commutative. If $\vartheta(a) = 0$, then $\vartheta = 0$, a contradiction. Now, suppose that

$$[\vartheta(a), b] + ab - \vartheta(a)b \in Z. \tag{78}$$

Putting $b = z$ in (78), where $0 \neq z \in \mathscr{I} \cap Z$, we see that

$$a - \vartheta(a) \in Z. \tag{79}$$

Replacing b by $\vartheta(b)$ in (78), this gives

$$[\vartheta(a), \vartheta(b)] + a\vartheta(b) - \vartheta(a)\vartheta(b) \in Z. \tag{80}$$

Comparing (80) and (67), we conclude $\vartheta(a)(\vartheta(b) + b) \in Z$. Taking $a = z$ in the previous relation, where $0 \neq z \in \mathscr{I} \cap Z$, we get $\vartheta(z)(\vartheta(b) + b) \in Z$ and so $\vartheta(z) = 0$ or $\vartheta(b) + b \in Z$. In case $\vartheta(z) = 0$ as Eq. (76). In case $\vartheta(b) + b \in Z$. Putting $b = a$ in the last expression, where $a \in \mathscr{I}$, we obtain

$$\vartheta(a) + a \in Z. \tag{81}$$

Comparing (80) and (79), we see that $2a \in Z$ and so $a \in Z$, and hence $\mathscr{R}$ is commutative. □

By using Lemmas 3.1–3.12, we get the proof of Theorems 1.1–1.3.

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Тождества, связанные с гомообразованием на идеале в первичных кольцах

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Аннотация. Целью данной работы является исследование коммутативности первичного кольца $\mathscr{R}$ с ненулевым идеалом $\mathscr{I}$ и гомодифференцированием ϑ, удовлетворяющим некоторым алгебраическим тождествам. Мы также привели несколько примеров того, почему наша гипотеза о результатах важна.

Ключевые слова: первичное кольцо, гомопроисхождение.