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Green's Function on a Parabolic Analytic Surface

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Abstract. The class of plurisubharmonic functions on a complex parabolic surface is considered in this paper. The concepts of the Green's function and pluripolar set are also introduced and their potential properties are studied.

Keywords: parabolic manifold, parabolic surface, regular parabolic surface, Green's function, pluripolar set.

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1. Introduction and preliminaries

This paper is devoted to plurisubharmonic (*psh*) functions on a complex parabolic manifold and surfaces embedded in the space $\mathbb{C}^N$. The concepts of Green's function and pluripolar set are introduced and a number of their potential properties are studied.

Parabolic manifolds presumably were considered for the first time by P. Griffiths, J. King [1] and by W. Stoll [2, 3]. They were used in the construction of the multidimensional Nevanlinna theory for holomorphic map $f: X \to P$, where X is a parabolic manifold $\dim X = n$, and P is a compact Hermitian manifold, $\dim P = m$. Various types of parabolicity were classified by A. Aytuna and A. Sadullaev [4–6].

Definition 1. *A Stein manifold $X \subset \mathbb{C}^N$, $\dim X = n$ is called parabolic if it does not contain different from a constant plurisubharmonic function bounded from above, i.e., if $u(z)$ is plurisubharmonic on X and $u(z) \leqslant C$ then $u(z) \equiv const$.*

It is called S-parabolic manifold if it contains a special exhaustion function $\rho(z)$ that satisfies the following conditions

a) $\rho(z) \in psh(X), \{\rho \leqslant c\} \subset\subset X \ \ \forall c \in \mathbb{R}$;

b) $(dd^c\rho)^n = 0$ outside some compact $K \subset\subset X$, i.e., function ρ is maximal function on $X \setminus K$.

X is called S^*-parabolic if there is a continuous special exhaustion function $\rho(z)$ on it.

It is clear that S^*-parabolic manifold is S-parabolic and, in turn, it is easy to prove that S-parabolic manifold is parabolic. It was noted [5, 6] that for $n = 1$ all these 3 concepts coincide (see [7]). However, for $n > 1$ the equivalence of these three definitions is still an open problem.

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The purpose of this paper is to study holomorphic and plurisubharmonic functions on an analytic surface. Concepts of parabolic surfaces $X \subset \mathbb{C}^N$, plurisubharmonic functions and Green function on them are introduced (Section 2). A series of properties of plurisubharmonic on X functions are proved. Some of these properties are non-trivial due to the presence of singular (critical) points of X. When proving properties in neighbourhoods of such points, the local principle of analytic covering is used. In Section 3, the concepts of polynomials are defined, the class of polynomials on parabolic surfaces is studied, and a number of examples of surfaces of this type are given. Theorem 3.1 states that complement $X = \mathbb{C}^N \backslash A$ of an arbitrary pure $(n-1)$-dimensional algebraic set $A = \{p(z) = 0\} \subset \mathbb{C}^N$ is *regular* S^*-parabolic surface.

2. Parabolic surfaces

In this section, parabolic surfaces, their classification and the Green's functions on them are we studied.

2.1. Plurisubharmonic functions on analytic surfaces.

Let us consider an analytic surface, i.e., an irreducible analytic set X, $\dim X = n$ embedded in a complex space $\mathbb{C}^N$, $X \subset \mathbb{C}^N$ such that for any ball $B(0,r) \subset \mathbb{C}^N$ the intersection $X \cap B(0,r)$ lies compactly on X, $X \cap B(0,r) \subset\subset X$. To define plurisubharmonic functions on X we denote the set of regular points of the set X by the $X^0 \subset X$. Then the set of critical (singular) points $X \backslash X^0$ is an analytic set of lower dimension $\dim X \backslash X^0 < n$. Set $X \backslash X^0$ does not split X, and set X^0 is a complex n dimensional submanifold in $\mathbb{C}^N$ (see [8, 9]).

Definition 2 ([10]). *Function $u(z)$ defined in a domain $D \subset X$ is called plurisubharmonic (psh) in D if it is locally bounded from above in this domain and plurisubharmonic on the manifold $D \cap X^0$, $u(z) \in psh(D \cap X^0)$.*

The class of plurisubharmonic functions in D is denoted by $psh(D)$. In practice, at critical points $z \in X \backslash X^0$ function $u^*(z) = \overline{\lim\limits_{\substack{w \to z \\ w \in X^0 \cap D}}} u(w)$, $z \in D$ is usually considered, and in studies of plurisubharmonic functions $u^*(z)$ is studied. Function $u^*(z)$ is assumed to be upper semicontinuous in D, the set $\{z \in D : \ u^*(z) < C\}$ is open for all $C \in \mathbb{R}$ and $u^*(z) = u(z)$, $\forall z \in X^0 \cap D$.

Let us consider several properties of plurisubharmonic functions on X that are needed below.

1) *A linear combination of finite number of plurisubharmonic functions in $D \subset X$ with positive coefficients is a plurisubharmonic function, i.e., if $u_j^*(z) \in psh(D)$, $\alpha_j \geqslant 0$, $\quad j = 1, 2, \ldots, s$ then*

$$\alpha_1 u_1^*(z) + \ldots + \alpha_s u_s^*(z) \in psh(D).$$

2) *The uniform limit or the limit of a monotonically decreasing sequence $\{u_j^*(z)\}$ of plurisubharmonic functions is plurisubharmonic function, i.e., if $u_j^*(z) \in psh(D)$, $\quad j = 1, 2, \ldots$, $u_j^*(z) \rightrightarrows u^*(z)$ or if $u_j^*(z) \searrow u^*(z)$ then $u^*(z) \in psh(D)$.*

The following property is not trivial due to the presence of critical points on the surface X.

3) **Maximum principle.** *For $u^*(z) \in psh(D)$, $D \subset X$ the maximum principle holds, i.e., if at some interior point $z^0 \in D$ the value $u^*(z^0) = \sup\limits_D u^*(z)$ then $u^* \equiv const$.*

Now let us assume that u^* has a maximum at an interior point $z^0 \in D$ (without loss of generality one can assume $z^0 = 0$) and $u^*(0) \geqslant u^*(z)$ $\forall z \in D$. If $0 \in D$ is regular point then it is obvious that $u(z) \equiv const$ in $D\backslash S$, where $S = X\backslash X^0$, because for plurisubharmonic functions on a manifold the maximum principle is valid. Therefore, $u^*(z) \equiv const$ in D. If 0 is a critical point then there is a complex plane $0 \in \Pi \subset \mathbb{C}^N$ such that $\dim \Pi = N - n$, $X \cap \Pi$ is discrete. Hence, there exists a ball $B(''0, r) \subset \Pi$, $r > 0$ such that

$$X \cap B(''0, r) = \{0\}, \ X \cap \partial B(''0, r) = \emptyset. \tag{1}$$

Let us set $z = ('z, ''z)$, $'z = (z_1, \ldots, z_n)$, $''z = (z_{n+1}, \ldots, z_N)$. Let $\Pi = \{\, 'z = (z_1, \ldots, z_n) = 0\,\}$. Since X is closed then according to (1), there exists a neighbourhood $'U \ni '0 : X \cap \partial B('z, r) = \emptyset$, $\forall 'z \in 'U$. Therefore, $\pi : X \cap ['U \times ''U] \to 'U$ is a k-sheeted analytic covering, $1 \leqslant k < \infty$.

Let $J \subset 'U$ be the set of critical points of this covering. It means that

$$\pi \ : \ \{X \cap ['U \times ''U]\}\backslash \pi^{-1}(J) \ \to \ 'U\backslash J$$

is a regular k-sheeted covering

$$\pi^{-1}('z) \cap \{X \cap ['U \times ''U]\}\backslash \pi^{-1}(J) \ = \{\alpha_1('z), \ldots, \alpha_k('z)\} \ \ \forall 'z \in 'U\backslash J. \tag{2}$$

Moreover, for each point $'z^0 \in 'U\backslash J$ locally, in some neighbourhood of $W \ni 'z^0$, the inverse-image $\pi^{-1}(W) \cap X \cap ['U \times ''U]$ is split into k pieces of disjoint complex manifolds $M_1, M_2, \ldots, M_k$ (see, for example, [8, 11]). Function $u^*(z) = u(z)$ is plurisubharmonic function on every piece of manifolds M_j, $j = 1, 2, \ldots, k$.

It follows from (2) that $w('z) = \sum_{j=1}^{k} u^*(\alpha_j('z))$ is plurisubharmonic function in $'U\backslash J$ locally bounded in $'U$. Since $J \subset 'U$ is an analytic set then $w('z)$ is plurisubharmonically extended to $'U$. (Recall that if $w('z) \in psh(D\backslash P)$ is locally bounded in D, P is closed pluripolar set then $w('z)$ is plurisubharmonically extended to D (see [12] and also [13, 14]).

Thus, $w('z) \in psh('U)$ and by assumption it reaches its maximum at the point $0 \in U$. This is a contradiction. □

2.2. Holomorphic functions

It is convenient for us to define holomorphic functions on an analytic surface X in the sense of H. Cartan [9].

Definition 3. *Function $f(z)$ defined in a domain $D \subset X$ is called holomorphic in D, if:*
a) it is holomorphic on the manifold $D \cap X^0$;
b) it is locally bounded in D, i.e., for each point $z^0 \in D$ there exists a neighborhood $W \ni z^0$, $W \subset D$ such that $|f(z)| \leqslant const \quad , \forall z \in W \cap X^0$.

The class of holomorphic functions in D is denoted by $\mathcal{O}(D)$. Holomorphic functions in space X have many properties of holomorphic functions of several complex variables. In particular, a linear combination of holomorphic functions with constant coefficients is holomorphic function. In other words, if $f_1, \ldots, f_m \in \mathcal{O}(D)$ then $c_1 f_1 + \ldots + c_m f_m \in \mathcal{O}(D)$; the product of two holomorphic functions is also holomorphic function, i.e., if $f, g \in \mathcal{O}(D)$ then $f \cdot g \in \mathcal{O}(D)$. In addition, the theorem of uniqueness holds, i.e., if $f \in \mathcal{O}(D)$ and $f \equiv 0$ in some non-empty neighbourhood of $W \subset D$ then $f \equiv 0$ in domain $D \subset X$. Since holomorphic functions are defined only at regular points then $f \equiv 0$ in some neighbourhood $W \subset X$ means that $f \equiv 0 \ \forall z \in W \cap X^0$.

The following theorem of H. Cartan is very useful in the study of holomorphic functions.

Theorem 2.1 (H. Cartan [9]). *Let function $f(z)$ defined in a domain $D \subset X$ is continuous on $D \cap X^0$ and has the property that for each point $z^0 \in D$ there exist a neighbourhood $W \ni z^0$, $W \subset D$, and holomorphic in W functions $g_1, \ldots, g_m \in \mathcal{O}(W) : f^m(z) + g_1(z) f^{m-1}(z) + \ldots + g_m(z) = 0$ $\forall\, z \in W \cap X^0$. Then $f(z) \in \mathcal{O}(D)$.*

There is an intimate connection between holomorphic and plurisubharmonic functions.

Theorem 2.2. *If $f(z) \in \mathcal{O}(D)$ then function $u(z) = \ln |f(z)|$ is plurisubharmonic in domain D, $u(z) \in psh(D)$.*

2.3. S-parabolic analytic surfaces

Let $X \subset \mathbb{C}^N$ be an analytic surface embedded in $\mathbb{C}^N$, i.e., X is an irreducible analytic set in $\mathbb{C}^N$ for which the intersections $B(0, r) \cap X \subset\subset X$, $\forall\, r > 0$. The concept of parabolicity of surface X is introduced similarly to the parabolicity of manifolds.

Definition 4. *An analytic surface X is called parabolic if it does not contain a bounded plurisubharmonic function that is different from a constant.*

Analytic surface X is called S-parabolic if it has a special exhaustion function $\rho(z)$ satisfying the following conditions

a) $\rho(z) \in psh(X), \{\rho \leqslant c\} \subset\subset X \,\forall\, c \in \mathbb{R}$;

b) function ρ^* is a maximal function on $X \backslash K$ for some compact set $K \subset\subset X$. This is equivalent to $(dd^c \rho^*)^n = 0$ on $X^0 \backslash K$ (see [15]).

Analytic surface X is called S^*-parabolic if there exists a continuous special exhaustion function $\rho(z) \in C(X^0)$.

It is clear that S^*-parabolic analytic surface is S-parabolic. As we noted above, the converse assumption remains open even for a complex manifold.

The main result of Section 2 is the following theorem.

Theorem 2.3. *S-parabolic surface X is parabolic, i.e., on the S-parabolic surface X there is no bounded from above plurisubharmonic function $u^*(z)$ different from a constant.*

Proof. Let X be a S-parabolic analytic surface with special exhaustion function $\rho(z) \in psh(X)$, and ρ is a maximal plurisubharmonic function on $X \backslash K$, where $K \subset\subset X$ is some compact set. Suppose that there exists function $u(z) \in psh(X)$, $u(z) \leqslant M$ but $u(z) \not\equiv const$. Consider a ball $B_r = \{z \in X : \rho(z) < \ln r\} \subset\subset X$. Let us put $\rho_r = \max\limits_{\overline{B}_r} \rho(z)$, $u_r = \max\limits_{\overline{B}_r} u^*(z)$, $u_r \leqslant M$. Let us fix the numbers $r < r' < R < \infty$, $B_r \supset K$. Then for P-measure (see [15]) we have

$$\omega^*(z, \overline{B}_r, B_R) = \frac{\rho(z) - \rho_R}{\rho_R - \rho_r}. \tag{3}$$

Let us note that $u^*(z) \leqslant u_r$, $z \in \overline{B}_r$ and $u^*(z) \leqslant u_R$, $z \in \overline{B}_R$. Therefore, by the theorem on two constants [15] we have

$$u^*(z) \leqslant u_R \cdot (1 + \omega^*(z, \overline{B}_r, B_R)) - u_r \cdot \omega^*(z, \overline{B}_r, B_R).$$

Substituting (3) into the last inequality, we obtain for $z \in \overline{B}_r$

$$u_{r'} \leqslant \Big(1 + \frac{\rho_{r'} - \rho_R}{\rho_R - \rho_r}\Big) u_R - \frac{\rho_{r'} - \rho_R}{\rho_R - \rho_r} u_r.$$

Since function $u(z)$ is bounded on X then $u_R \leqslant M$, and when $R \to \infty$ we have $u_{r'} \leqslant u_r$. Hence, according to the maximum principle, $u^*(z) \equiv const$ in the ball B_r. Since $r < \infty$ is an arbitrary fixed number then $u^*(z) \equiv const$ on X. Theorem 2.3 is proved. □

2.4. Green's function on parabolic surfaces

In this subsection the Green's function on S-parabolic analytic surfaces is introduced.

Let (X, ρ) be a S-parabolic surface. Let us denote the class of plurisubharmonic functions $u \in psh(X)$ satisfying the condition

$$u(z) \leqslant c_u + \rho^+(z),\ z \in X,$$

by the $\mathfrak{A}_\rho(X)$. Here c_u is some constant that depends on function u and $\rho^+(z) = \max\{0,\ \rho(z)\}$. Class $\mathfrak{A}_\rho(X)$ is called the Lelong class of plurisubharmonic functions on X. For a fixed compact set $K \subset\subset X$, we define

$$V_\rho(z, K) = \sup\{u(z) :\ u \in \mathfrak{A}_\rho(X),\ u|_K \leqslant 0\}.$$

Then the regularization $V_\rho^*(z, K) = \overline{\lim_{w \to z}} V_\rho(w, K)$ is called ρ-*Green's function* of the compact $K \subset\subset X$.

Similarly, in the classical case there are

1. *Either* $V_\rho \in \mathfrak{A}_\rho(X)$ or $V_\rho \equiv +\infty$. $V_\rho(z, K) \equiv +\infty$ *if and only if* K *is pluripolar set on* X, *i.e., there exists a function* $u^* \in psh(X) :\ u^* \not\equiv -\infty,\ u^*(z) = -\infty\, \forall\, z \in K$.

2. Let $K \subset\subset X$ *be a non-pluripolar compact set. Then the Green's function* $V_\rho(z, K)$ *is maximal in* $X \backslash K$. *In particular,* $[dd^c V_\rho(z, K)]^n = 0$ *on the complex manifold* $X^0 \backslash K$.

The proofs of these important properties of the $\rho-$ Green's function are identical to the proofs of the corresponding properties of the Green's function in space $\mathbb{C}^n$, and they are omitted.

Definition 5. *A compact set* $K \subset X$ *is called regular at a point* $z^0 \in X$ *if* $V_\rho^*(z^0,\ K) = 0$. *If all points of* $K \subset X$ *are regular then compact* $K \subset X$ *is called regular.*

Note that if compact set $K \subset X$ is regular then the open set $G_\varepsilon = \{z \in X : V_\rho^*(z, K) < \varepsilon\}$ contains K, $G_\varepsilon \supset K$.

2.5. Regular parabolic surfaces

2.5.1. Polynomials on parabolic analytic surfaces

Let $X \subset \mathbb{C}^N$ be a S-parabolic surface and $\rho(z)$ is a special exhaustion function.

Definition 6. *If function* $f \in \mathcal{O}(X)$ *satisfies the inequality*

$$\ln|f(z)| \leqslant d\rho^+(z) + c_f\ \forall\, z \in X, \tag{4}$$

where c_f *and* d *are positive real numbers (constant) then* f *is called the* ρ*-polynomial. The smallest value* d *that satisfies condition (4) is called the degree of polynomial* f.

Let us denote the set of all ρ-polynomials of degree less than or equal to d by $\mathcal{P}_\rho^d(X)$ and the union $\mathcal{P}_\rho(X) = \bigcup_{d \geqslant 1} \mathcal{P}_\rho^d(X)$ by $\mathcal{P}_\rho(X)$. Then it is easy to prove (see [6,16]) that $\mathcal{P}_\rho^d(X)$ is a linear space of finite dimension $\dim \mathcal{P}_\rho^d(X) \leqslant C\,(d+1)^n$.

However, a parabolic manifold was constructed [6] where there are no non-trivial polynomials, i.e., any polynomial $P(z)$ on X is equal to a constant, $\mathcal{P}_\rho \simeq \mathbb{C}$.

Definition 7. *If space of all* ρ*-polynomials* $\mathcal{P}_\rho(X) = \bigcup_{d \geqslant 1} \mathcal{P}_\rho^d(X)$ *is dense in space* $\mathcal{O}(X)$ *then* S*-parabolic surface* X *is called regular.*

2.6. Examples

Example 1. *Let $A \subset \mathbb{C}^N$ be irreducible, n dimensional,* $\dim A = n$*, $n < N$ algebraic set. According to the well-known criterion of W. Rudin [17] (see also [18]) and after corresponding linear transformation, algebraic set A can be included in a special cone*

$$A \subset \{w = ('w, ''w) = (w_1, \dots, w_n, w_{n+1}, \dots, w_N) : \ \|''w\| < C\,(1 + \|'w\|)\},$$

where C is constant.

Let us consider projection $\pi('w, ''w) = 'w : A \to \mathbb{C}^n$. If $('w^0, ''w^0)$ is a regular point of A, i.e., $('w^0, ''w^0) \in A^0$ then in some neighbourhood $U \ni ('w^0, ''w^0)$, $U \subset A^0$ projection $\pi : U \to \mathbb{C}^n$ is biholomorphic. Consequently, restriction $\rho|_A$ of the plurisubharmonic in $\mathbb{C}^N$ function $\rho(w) = \ln \|'w\|$ is plurisubharmonic function in a neighbourhood of $U \ni ('w^0, ''w^0)$. Since point $('w^0, ''w^0) \in A^0$ is arbitrary restriction of $\rho|_A$ is a plurisubharmonic function in A^0. In addition, it is locally bounded from above on A and, therefore, $\rho|_A \in psh(A)$.

It is clear that $\rho|_A$ is special exhaustion function on A and restriction on A of polynomials $p('w, ''w)$ from $\mathbb{C}^N$ are polynomials on A. This implies that set of polynomials $\mathcal{P}_\rho(A)$ is dense in $\mathcal{O}(A)$, i.e., affine-algebraic surface is regular parabolic surface.

Example 2. *Let $A = \{\Phi(z) = 0\} \subset \mathbb{C}^n$ be a pure $(n-1)$ dimensional analytic surface such that*

$$A \subset \{z = ('z, z_n) \in \mathbb{C}^n : |z_n| < \varphi('z)\},$$

where $'z = (z_1, \dots, z_{n-1})$, $\varphi('z)$ is a locally bounded positive function. Then A is S^*-parabolic surface.

Let us consider projection $\pi('z, z_n) = 'z : A \to \mathbb{C}^{n-1}$. For each fixed point $'z^0 \in \mathbb{C}^{n-1}$ intersection $\{'z = 'z^0\} \cap A = \pi^{-1}\{'z^0\}$ consists of a finite number of points $\{'z = 'z^0\} \cap A = (\alpha_1('z^0), \dots, \alpha_m('z^0))$ as a compact analytic set in plane $\mathbb{C}_{'z^0}$. Function $\Phi('z, z_n) \neq 0$ on the boundary of circle $\{|z_n| = \varphi('z)\}$. According to the argument principle, the number of zeros (taking into account multiplicities)

$$N('z) = \frac{1}{2\pi i} \int\limits_{|z_n| = \varphi('z^0)} \frac{\Phi'('z, z_n)}{\Phi('z, z_n)}\, dz_n, \quad 'z \in U,$$

as a continuous integer function is constant, $N('z) \equiv m$ and $\pi^{-1}('z) = (\alpha_1('z), \dots, \alpha_m('z))$, $'z \in \mathbb{C}^{n-1}$. Moreover, function

$$F('z, z_n) = \prod_{k=1}^{m} (z_n - \alpha_k('z)) = z_n^m + f_{m-1}('z) z_n^{m-1} + \dots + f_0('z)$$

is an entire function, where $f_k('z) \in \mathcal{O}(\mathbb{C}^{n-1})$, $k = 0, 1, \dots, m-1$.

If A is not an algebraic set then function $F('z, z_n)$ is not a polynomial, i.e., not all functions $f_k('z) \in \mathcal{O}(\mathbb{C}^{n-1})$, $k = 0, 1, \dots, m-1$ are polynomials. As in Example 1, contraction $\rho|_A$ of plurisubharmonic function $\rho(z) = \ln \|'z\|$ from $\mathbb{C}^N$ is special exhaustion function on A, i.e., surface A is parabolic. However, here restrictions of polynomials $P(z)$ in $\mathbb{C}^n$ on A are not, in general, $\rho|_A$ polynomials.

It was proved ([5], see also [19]) that $X = \mathbb{C}^n \backslash A$ complement of zeros of the Weierstrass polynomial $A = \{z_n^m + f_1('z) z_n^{m-1} + \dots + f_m('z) = 0\}$, where $f_1('z), \dots, f_m('z)$ are entire functions, is S^*-parabolic manifold with special exhaustion function

$$\rho(z) = \frac{1}{2}\ln\left(|'z|^2 + \left|F(z) + \frac{1}{F(z)}\right|^2\right),$$

where $F('z, z_n) = z_n^m + f_{m-1}('z)z_n^{m-1} + \ldots + f_0('z)$. However, $X = \mathbb{C}^n \backslash A$ is not always regular (see [19]). The main result of this section is

Theorem 2.4. *The complement $X = \mathbb{C}^n \backslash A$ of an arbitrary pure $(n-1)$ dimensional algebraic set $A = \{p(z) = 0\} \subset \mathbb{C}^n$ is regular S^*-parabolic manifold. If $p(0) \neq 0$ then function*

$$\rho(z) = -\frac{1}{\deg p}\ln|p(z)| + 2\ln\|z\| \tag{5}$$

is special exhaustion function on X.

The theorem is proved in several steps.

Step 1. Let us show that $\rho(z)$ from (5) is special exhaustion function. In fact, function $-\dfrac{1}{\deg p}\ln|p(z)|$ is pluriharmonic in X and function $2\ln\|z\|$ is maximal in $X\backslash\{0\}$. Therefore, function $\rho(z)$ is the maximal function in $X\backslash\{0\}$. In addition, since $p(0) \neq 0$ then $\{z \in X : \rho(z) < C\} \subset\subset X$ $\forall C > 0$.

Step 2. Using the criterion of W. Rudin [17] and after the corresponding linear transformation of space $\mathbb{C}^n$, A is reduced into special form (see Example 1)

$$A \subset \{z = ('z, z_n) \in \mathbb{C}^n : |z_n| < C(1 + \|'z\|)\}, \quad C - const. \tag{6}$$

Then A has the form

$$A = \left\{p(z) = z_n^m + e_1('z)z_n^{m-1} + \ldots + e_m('z) = 0\right\},$$

where $m = \deg p > 1$, $e_1('z), \ldots, e_m('z)$ are polynomials and $p(0) \neq 0$.

Step 3. The expansion of holomorphic functions in $X = \mathbb{C}^n \backslash A$ into Jacobi–Hartogs series is used. First, let us consider some insights on the theory of Jacobi series ([20], see also [21]). Let $p(z) = z_n^m + e_1 z_n^{m-1} + \ldots + e_m$, $m > 1$ and $e_1, \ldots, e_m$ are constants. Let us denote the lemniscate ring $\{z \in \mathbb{C} : r < |p(z)| < R\}$ by $G_{r,R}$. If function $f(z)$ is holomorphic in some neighbourhood $\overline{G}_{r,R}$ then function of two variables

$$F(z, w) = \frac{1}{2\pi i}\int\limits_{\partial G_{r,R}} \frac{f(\xi)}{p(\xi) - w} \cdot \frac{p(\xi) - p(z)}{\xi - z} d\xi$$

is holomorphic in domain $G_{r,R} \times \{r < |w| < R\}$. According to the Cauchy integral formula, the equality $F(z, p(z)) \equiv f(z)$ $(z \in G_{r,R})$ takes place. The expansion of function $F(z, w)$ into Hartogs–Laurent series (see [11]) with respect to the variable w is

$$F(z, w) = \sum_{k=-\infty}^{\infty} c_k(z) w^k, \tag{7}$$

where

$$c_k(z) = \frac{1}{2\pi i}\int\limits_{|p(\xi)| = r_1} f(\xi)\frac{p(\xi) - p(z)}{p^{k+1}(\xi)(\xi - z)}\, d\xi, \qquad (r < r_1 < R, \;\; z \in G_{r,R}, \;\; k = 0, \pm 1, \pm 2, \ldots). \tag{8}$$

Series (7) converges uniformly inside domain $G_{r,R} \times \{r < |w| < R\}$. If we put $w = p(z)$ then we obtain the series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z) p^k(z), \quad z \in G_{r,R},$$

which is called the Jacobi–Hartogs series of function $f(z)$. It converges uniformly inside domain $G_{r,R}$. One can be see from (8) that coefficients $c_k(z)$ are polynomials of degree $\deg c_k(z) \leqslant m-1$.

It follows that if function $f(z)$ is holomorphic in $G_{0,\infty}$ then it is expanded into the series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z) p^k(z),$$

which converges uniformly inside $G_{0,\infty}$. Here $c_k(z)$ are polynomials of degree $\deg c_k(z) \leqslant m-1$,

$$c_k(z) = \frac{1}{2\pi i} \int\limits_{|p(\xi)|=r} f(\xi) \frac{p(\xi) - p(z)}{p^{k+1}(\xi)\,(\xi - z)} d\xi \quad (0 < r < \infty,\ z \in G_{0,\infty}),$$

and the Cauchy inequality holds:

$$|c_k(z)| \leqslant \frac{\max\{|f(\xi)| : \ |p(\xi)| = r\}}{2\pi r^{k+1}} \int\limits_{|p(\xi)|=r} \left| \frac{p(\xi) - p(z)}{\xi - z} \right| |d\xi|, \ \ k = 0, \pm 1, \pm 2, \ldots \tag{9}$$

Step 4. Let us apply the Jacobi–Hartogs series to the holomorphic function $f('z, z_n) \in \mathcal{O}(X)$ outside the algebraic set $A = \left\{p(z) = z_n^m + e_1('z) z_n^{m-1} + \ldots + e_m('z) = 0\right\}$. We fix $'z \in \mathbb{C}^{n-1}$ and expand function $f('z, z_n)$ in the Jacobi–Hartogs–Laurent series:

$$f('z, z_n) = \sum_{k=-\infty}^{\infty} c_k('z, z_n) \cdot p^k('z, z_n) \tag{10}$$

where coefficients

$$c_k('z, z_n) = \frac{1}{2\pi i} \int\limits_{|p('z,\xi_n)|=r} f('z, \xi_n) \cdot \frac{p('z, \xi_n) - p('z, z_n)}{p^{k+1}('z, \xi_n)(\xi_n - z_n)} d\xi_n$$

are polynomials in variable z_n with holomorphic $\mathbb{C}^{n-1}$ coefficients

$$c_k('z, z_n) = a_{k,m-1}('z)\, z_n^{m-1} + \ldots + a_{k,0}('z), \ a_{k,j}('z) \in \mathcal{O}\left(\mathbb{C}^{n-1}\right), \ j = 0, 1, \ldots, m-1.$$

Series (10) converges uniformly inside domain

$$X = \{('z, z_n) \in \mathbb{C}^n : \quad 0 < |p('z, z_n)| < \infty\}.$$

Step 5. Let us show that rational functions of the form $\dfrac{q(z)}{p^k(z)}$, where $q(z)$ is a polynomial in $\mathbb{C}^n$, $k \geqslant 0$ are integer functions, and only they are ρ-polynomials in X, where $\rho(z) = = -\dfrac{1}{\deg p} \ln|p(z)| + 2\ln\|z\|$. In fact, since

$$\ln\left|\frac{q(z)}{p^k(z)}\right| = -k\ln|p(z)| + \ln|q(z)| \leqslant \max\{k, \deg q\}\, \rho^+(z) + const$$

then function $\dfrac{q\,(z)}{p^k\,(z)}$ is $\rho\,(z)$-polynomial in $X = \mathbb{C}^n \backslash A$.

On the contrary, if $P(z)$ is a $\rho(z)$-polynomial in $X = \mathbb{C}^n \backslash A$, then according to (4)

$$\ln |P(z)| \leqslant d\rho^+(z) + c \ \ \forall\, z \in X, \ \ d, c - const.$$

Let us expand $P(z)$ into the Jacobi–Hartogs–Laurent series (10)

$$f('z, z_n) = \sum_{k=-\infty}^{\infty} c_k('z, z_n) \cdot p^k('z, z_n),$$

with coefficients

$$c_k('z, z_n) = \frac{1}{2\pi i} \int\limits_{|p('z,\xi_n)|=r} P('z, \xi_n) \cdot \frac{p('z, \xi_n) - p('z, z_n)}{p^{k+1}('z, \xi_n)(\xi_n - z_n)} d\xi_n.$$

According to (9), we have the following estimate

$$|c_k('z, z_n)| \leqslant \frac{\max\left\{|P\,('z, \xi_n)| : \ |p\,('z, \xi_n)| = r\right\}}{2\pi r^{k+1}} \int\limits_{|p('z,\xi_n)|=r} \left| \frac{p\,('z, \xi_n) - p\,('z, z_n)}{\xi_n - z_n} \right| |d\xi_n| \leqslant$$
$$\leqslant \frac{\max\left\{\exp\left[c + d\rho^+\,('z, \xi_n)\right] : \ |p\,('z, \xi_n)| = r\right\}}{2\pi r^{k+1}} \int\limits_{|p('z,\xi_n)|=r} \left| \frac{p\,('z, \xi_n) - p\,('z, z_n)}{\xi_n - z_n} \right| |d\xi_n| \,.$$

Substituting $\rho(z) = -\dfrac{1}{\deg p} \ln |p\,(z)| + 2 \ln \|z\|$, we obtain

$$|c_k\,('z, z_n)| \leqslant \frac{\max\left\{\exp\left[c + d\left(-\frac{1}{m}\ln |p\,('z, \xi_n)| + 2\ln \|('z, \xi_n)\|\right)^+\right] : \ |p\,('z, \xi_n)| = r\right\}}{2\pi r^{k+1}} \times$$
$$\times \int\limits_{|p('z,\xi_n)|=r} \left| \frac{p\,('z, \xi_n) - p\,('z, z_n)}{\xi_n - z_n} \right| |d\xi_n| \,. \tag{11}$$

However,

$$\begin{aligned} &\max\left\{\exp\left[c + d\left(-\frac{1}{\deg p}\ln |p\,(z)| + 2\ln \|z\|\right)^+\right] : \ |p\,('z, \xi_n)| = r\right\} \leqslant \\ &\leqslant \exp\left[c + d\left(-\frac{\ln r}{m} + \ln \|('z, \xi_n)\|^2_{|p('z,\xi_n)|=r}\right)^+\right] \leqslant \\ &\leqslant \exp\left[c + d\left(-\frac{\ln r}{m} + \ln\left(r^2 + C_1\left(1 + \|'z\|^2\right)\right)\right)^+\right] \leqslant C_2 \begin{cases} \left(r^2 + \|'z\|^2\right)^d & \text{if } \ r \to \infty \\ \|'z\|^{2d} \cdot r^{-d/m} & \text{if } \ r \to 0 \end{cases} \end{aligned} \tag{12}$$

Here the estimate

$$\|('z, \xi_n)\|^2_{|p('z,\xi_n)|=r} \leqslant \left[\|'z\|^2 + |\xi_n|^2\right]_{|p('z,\xi_n)|=r} \leqslant \|'z\|^2 + r^2 + C^2\left(1 + \|'z\|\right)^2,$$

is used which is easy to obtain by applying relation (6). The integral in (11) is estimated as (see [19] Lemma 4.1)

$$\int\limits_{|p('z,\xi_n)|=r} \left| \frac{p\,('z, \xi_n) - p\,('z, z_n)}{\xi_n - z_n} \right| |d\xi_n| \leqslant C_3 r, \ \ |p\,('z, z_n)| \leqslant r, \ \ C_3 - const. \tag{13}$$

Thus, substituting (12), (13) into (11), we obtain the final estimate

$$|c_k('z, z_n)| \leqslant \frac{C_4}{r^k} \begin{cases} \left(r^2 + \|'z\|^2\right)^d & \text{if } r \to \infty \\ \|'z\|^{2d} \cdot r^{-d/m} & \text{if } r \to 0 \end{cases}, \quad k = 0, \pm1, \pm2, \dots, \ C_4 - const. \tag{14}$$

For indices $k \geqslant 0$ the upper inequality (14)is used:
$|c_k('z, z_n)| \leqslant \frac{C_4}{r^k} \left(r^2 + \|'z\|^2\right)^d \to 0$, for $r \to \infty$ and $k > 2d$. For indices $k < 0$ the lower inequality (14) is taken:
$|c_k('z, z_n)| \leqslant \frac{C_3}{r^k} \|'z\|^{2d} \cdot r^{-d/m} \to 0$, with $r \to 0$ and $k < -\frac{d}{m}$. Consequently, $c_k('z, z_n) \equiv 0$ for all $|k| > 2d$ and

$$f('z, z_n) = \sum_{k=-2d}^{+2d} c_k('z, z_n) \cdot p^k('z, z_n).$$

However, according to (14), each function $c_k('z, z_n)$, $|k| \leqslant 2d$, is a polynomial, i.e.,

$$f('z, z_n) = \sum_{k=-2d}^{+2d} c_k('z, z_n) \cdot p^k('z, z_n) = \frac{q('z, z_n)}{p^{2d}('z, z_n)}.$$

Step 6. It remains to show that space of polynomials $\mathcal{P}_\rho(X)$ is dense in space $\mathcal{O}(X)$, i.e., an arbitrary holomorphic function $f(z)$ is uniformly approximated by ρ-polynomials inside $X = \mathbb{C}^n \backslash A$. This follows from the fact that, as we noted above (step 4), the Jacobi–Hartogs–Laurent series $f('z, z_n) = \sum_{k=-\infty}^{\infty} c_k('z, z_n) \cdot p^k('z, z_n)$ of an arbitrary function $f('z, z_n) \in \mathcal{O}(X)$ converges uniformly inside $X = \mathbb{C}^n \backslash A$. Here coefficients are

$$c_k('z, z_n) = a_{k,m-1}('z) z_n^{m-1} + \ldots + a_{k,0}('z), \ a_{k,j}('z) \in \mathcal{O}(\mathbb{C}^{n-1}), \ j = 0, 1, \dots, m-1.$$

Consequently, the partial sums of $S_M('z, z_n) = \sum_{k=-M}^{M} c_k('z, z_n) \cdot p^k('z, z_n)$ converge uniformly inside $X = \mathbb{C}^n \backslash A$. Approximating coefficients $a_{k,j}('z) \in \mathcal{O}(\mathbb{C}^{n-1})$, $k = 0, \pm1, \dots, \pm M$, $j = 1, 2, \dots, m-1$ by polynomials, we thereby obtain approximation of function $f('z, z_n) \in \mathcal{O}(X)$ by polynomials, i.e., $\overline{\mathcal{P}}_\rho(X) = \mathcal{O}(X)$. Theorem is proved. □

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Функция Грина на параболической аналитической поверхности

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Аннотация. В данной работе рассматривается класс плюрисубгармонических функций на комплексной параболической поверхности, вводятся понятия функции Грина и плюриполярных множеств, изучается ряд их потенциальных свойств.

Ключевые слова: параболическое многообразие, параболические поверхности, регулярные параболические поверхности, функции Грина, плюриполярные множества.