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The Cauchy Problem for Equation of Elasticity Theory

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Abstract. A problem on the analytic continuation of the solution of equation of elasticity theory in a spatial domain is considered. Continuation is based on the values of the solution and stresses on a part of the boundary of this domain. Hence the problem presents the Cauchy problem.

Keywords: Cauchy problem,Lame equation, elliptic system,ill-posed problem, Carleman matrix, regularization.

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1. Introduction and preliminaries

It is well known that Cauchy problem for an elliptic equation is ill-posed. The solution of the problem is unique but unstable (Hadamard‘s example). For ill-posed problems the existence of a solution and it belonging to the correctness class is usually assumed a priori. Moreover, the solution is assumed to belong to some given subset of the function space, that is usually a compact subset [1]. The uniqueness of the solution follows from the general Holmgren theorem [2].

The Cauchy problem for elliptic equations was the subject of study for mathematicians throughout the twentieth century and it continues to attract the attention of researchers to this day.

The development of special methods that allows one to deal with ill-posed Cauchy problems was stimulated by practical demands. Such problems can be found in hydrodynamics, signal transmission theory, tomography, geological exploration, geophysics, elasticity theory, and so on.

A solution of the Cauchy problem for the one-dimensional system of Cauchy–Riemann equations was first obtained in 1926 by Carleman [3]. He proposed the idea of introducing an additional function into the Cauchy integral formula which allows one to take the limit in order to damp the influence of integrals over that part of the boundary where the values of the function to be continued are not given. The idea of Carleman was developed in 1933 by Goluzin and Krylov [4]. They found a general way to obtain Carleman formulas for the one-dimensional system of Cauchy–Riemann equations.

Resting on the results of Carleman and Goluzin–Krylov, Lavrent'ev introduced the concept of the Carleman function for the one-dimensional system of Cauchy–Riemann equations. The

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method proposed by Lavrent'ev [5] consists in approximation of the Cauchy kernel on the additional part of the domain boundary outside the support of the data of the Cauchy problem.

The Carleman function of the Cauchy problem for the Laplace equation is a fundamental solution that depends on a positive parameter. It tends to zero together with its normal derivative on the part of the domain boundary outside the Cauchy data support as the parameter tends to infinity. Using the Carleman function and Green's integral formula, a Carleman formula is produced. It gives an exact solution of the Cauchy problem when the data are specified exactly. Construction of the Carleman function allows one to construct a regularization if the Cauchy data are given approximately. The existence of the Carleman function follows from the Mergelyan approximation theorem [6].

Fock and Kuni [7] found in 1959 an application of the Carleman formula to the one-dimensional system of Cauchy–Riemann equations. When part of the domain boundary is a segment of the real axis they used the Carleman formula to establish a criterion for the solvability of the Cauchy problem for the system of Cauchy–Riemann equations on the plane. An analog of the Carleman formula and criteria for the solvability of the Cauchy problem were obtained for analytic functions of several variables [8, 9], for harmonic functions [10–12] and also [13–16].

A fairly complete survey on Carleman formulas can be found in [5, 11, 17, 18].

In the present paper, a regularized solution of the Cauchy problem for the system of elasticity equations is constructed on the basis of the Carleman function method.

Let us assume that $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ are points in R^m, D_ρ is a bounded simple connected domain in R^m. Its boundary is a cone surface:

$$\Sigma: \quad \alpha_1 = \tau y_m, \;\; \alpha_1^2 = y_1^2 + \ldots + y_{m-1}^2, \;\; \tau = tg\frac{\pi}{2\rho}, \;\; y_m > 0, \;\; \rho > 1.$$

Let us also consider a smooth surface S that lies inside the cone.

Let us consider in domain D_ρ the system of equations of elasticity theory

$$\mu \Delta U(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} U(x) = 0;$$

here $U = (U_1, \dots, U_m)$ is the displacement vector, Δ is the Laplace operator, λ and μ are the Lame constants. For brevity, it is convenient to use matrix notation. Let us introduce the matrix differential operator

$$A(\partial_x) = \|A_{ij}(\partial_x)\|_{m \times m},$$

where

$$A_{ij}(\partial_x) = \delta_{ij} \mu \Delta + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then the elliptic system of equations can be written in matrix form

$$A(\partial_x) U(x) = 0. \tag{1}$$

Statement of the problem. Let us assume that Cauchy data of a solution U are given on S,

$$U(y) = f(y), \quad y \in S,$$

$$T(\partial_y, n(y)) U(y) = g(y), \quad y \in S, \tag{2}$$

where $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_m)$ are prescribed continuous vector functions on S, $T(\partial_y, n(y))$ is the strain operator, i.e.,

$$T(\partial_y, n(y)) = \|T_{ij}(\partial_y, n(y))\|_{m \times m} = \left\| \lambda n_i \frac{\partial}{\partial y_j} + \mu n_j \frac{\partial}{\partial y_i} + \mu \delta_{ij} \frac{\partial}{\partial n} \right\|_{m \times m},$$

δ_{ij} is the Kronecker delta, and $n(y) = (n_1(y), \dots, n_m(y))$ is the unit normal vector to the surface S at the point y.

It is required to determine function $U(y)$ in D, i.e., to find an analytic continuation of the solution of the system of equations in the domain from the values of f and g on a smooth part of S of the boundary.

In this paper, the Cauchy problem for system of static equations of elasticity theory is solved for cone type domains by the method of regularization of the solution according to Lavrentiev.

In earlier works [14–16], this problem was considered either in two or three-dimensional spaces or for other special domains for which it is required to construct special matrices of fundamental solutions in explicit form that depends on the domain and dimension of the space.

Similar problems were considered for an arbitrary domain, by expanding the fundamental solution into a series in terms of spherical functions [12, 19].

Let us suppose that instead of $f(y)$ and $g(y)$ their approximations $f_\delta(y)$ and $g_\delta(y)$ with accuracy δ, $0 < \delta < 1$ (in the metric of C) are given. They do not necessarily belong to the class of solutions. In this paper, a family of functions $U(x, f_\delta, g_\delta) = U_{\sigma\delta}(x)$ that depends on parameter σ is constructed. It is also proved that under certain conditions and special choice of parameter $\sigma(\delta)$ the family $U_{\sigma\delta}(x)$ converges in the usual sense to the solution $U(x)$ of problem (1), (2) as $\delta \to 0$.

Following A. N. Tikhonov, $U_{\sigma\delta}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem [1].

2. Construction of the matrix of fundamental solution for the system of equations of elasticity

Definition 2.1. *Matrix $\Gamma(y,x) = ||\Gamma_{ij}(y,x)||_{m\times m}$, is called the matrix of fundamental solutions of system (1), where*

$$\Gamma_{ij}(y,x) = \frac{1}{2\mu(\lambda+2\mu)}((\lambda+3\mu)\delta_{ij}q(y,x) - (\lambda+\mu)(y_j - x_j)\frac{\partial}{\partial x_i}q(y,x)), \quad i,j = 2,\dots,m,$$

$$q(y,x) = \begin{cases} \dfrac{1}{(2-m)\omega_m}\cdot\dfrac{1}{|y-x|^{m-2}}, & m > 2 \\ \dfrac{1}{2\pi}\ln|y-x|, & m = 2, \end{cases}$$

and ω_m is the area of unit sphere in R^m.

Matrix $\Gamma(y,x)$ is symmetric and its columns and rows satisfy equation (1) at an arbitrary point $x \in R^m$, except $y = x$. Thus, we have

$$A(\partial_x)\Gamma(y,x) = 0, \quad y \neq x.$$

Developing idea of Lavrent‘ev on the notion of Carleman function of the Cauchy problem for the Laplace equation [5], the following notion is introduced.

Definition 2.2. *The Carleman matrix of problem (1), (2) is $(m\times m)$ matrix $\Pi(y,x,\sigma)$ that satisfies the following two conditions*

1) $\Pi(y,x,\sigma) = \Gamma(y,x) + G(y,x,\sigma)$, *where σ is a positive parameter, and matrix $G(y,x,\sigma)$ satisfies system (1) everywhere in domain D with respect to the variable y.*

2) The following relation holds

$$\int_{\partial D\setminus S} (|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|)ds_y \leqslant \varepsilon(\sigma),$$

where $\varepsilon(\sigma)\to 0$ *as* $\sigma\to\infty$ *uniformly in* x *on compact subsets of* D*. Here and elsewhere* $|\Pi|$ *denotes the Euclidean norm of matrix* $\Pi = ||\Pi_{ij}||$*, i.e.,* $|\Pi| = \Big(\sum\limits_{i,j=1}^{m}\Pi_{ij}^2\Big)^{\frac{1}{2}}$*. In particular,* $|U| = \Big(\sum\limits_{i=1}^{m}U_i^2\Big)^{\frac{1}{2}}$ *for a vector* $U=(U_1,\ldots,U_m)$*.*

Definition 2.3. *A vector function* $U(y) = (U_1(y),\ldots,U_m(y))$ *is said to be regular in* D *if it is continuous together with its partial derivatives of second order in* D *and partial derivatives of first order in* $\overline{D} = D\bigcup\partial D$*.*

In the theory of partial differential equations solution functions of potential type play an important role. As an example of such representation, the formula of Somilian–Bettis is considered below [20].

Theorem 2.1. *Any regular solution* $U(x)$ *of equation (1) in the domain* D *is represented as*

$$U(x) = \int_{\partial D}(\Gamma(y,x)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Gamma(y,x)\}^*U(y))ds_y,\ \ x\in D, \tag{3}$$

here A^* *is conjugate to* A*.*

Suppose that Carleman matrix $\Pi(y,x,\sigma)$ of problem (1), (2) exists. Then for the regular functions $v(y)$ and $u(y)$ the following relation holds

$$\int_{\partial D_\rho}[v(y)\{A(\partial_y)U(y)\} - \{A(\partial_y)v(y)\}^*U(y)]dy =$$

$$= \int_{\partial D_\rho}[v(y)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)v(y)\}^*U(y)]ds_y.$$

Substituting $v(y) = G(y,x,\sigma)$ and $u(y) = U(y)$ into the above relation, we obtain

$$\int_{\partial D_\rho}[G(y,x,\sigma)\{A(\partial_y)U(y)\} - \{A(\partial_y)G(y,x,\sigma)\}^*U(y)]dy = 0. \tag{4}$$

The theorem follows from (3) and (4).

Theorem 2.2. *Any regular solution* $U(x)$ *of equation (1) in domain* D_ρ *is represented as*

$$U(x) = \int_{\partial D_\rho}(\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y))ds_y,\ \ x\in D_\rho, \tag{5}$$

where $\Pi(y,x,\sigma)$ *is the Carleman matrix.*

Suppose that $K(\omega)$, $\omega = u+iv$ (u and v are real) is an entire function that takes real values on the real axis. It satisfies the following conditions

$$K(u)\neq 0,\ \sup_{v\geqslant 1}|v^pK^{(p)}(\omega)| = M(p,u)<\infty,\ \ p=0,\ldots,m,\ \ u\in R^1.$$

Let

$$s = \alpha^2 = (y_1 - x_1)^2 + \dots + (y_{m-1} - x_{m-1})^2.$$

For $\alpha > 0$ function $\Phi(y, x)$ is defined by the following relations. If $m = 2$ then

$$-2\pi K(x_2)\Phi(y,x) = \int_0^\infty Im\left[\frac{K(i\sqrt{u^2+\alpha^2}+y_2)}{i\sqrt{u^2+\alpha^2}+y_2-x_2}\right]\frac{udu}{\sqrt{u^2+\alpha^2}}. \tag{6}$$

If $m = 2n + 1,\ \ n \geqslant 1$ then

$$C_m K(x_m)\Phi(y,x) = \frac{\partial^{n-1}}{\partial s^{n-1}}\int_0^\infty Im\left[\frac{K(i\sqrt{u^2+\alpha^2}+y_m)}{i\sqrt{u^2+\alpha^2}+y_m-x_m}\right]\frac{du}{\sqrt{u^2+\alpha^2}}, \tag{7}$$

where $C_m = (-1)^{n-1}\cdot 2^{-n}(m-2)\pi\omega_m(2n-1)!$. If $m = 2n,\ \ n \geqslant 2$ then

$$C_m K(x_m)\Phi(y,x) = \frac{\partial^{n-2}}{\partial s^{n-2}} Im\frac{K(\alpha i + y_m)}{\alpha(\alpha + y_m - x_m)}, \tag{8}$$

where $C_m = (-1)^{n-1}(n-1)!(m-2)\omega_m$.

The following theorem is valid [10]

Theorem 2.3. *Function $\Phi(y, x)$ can be expressed as*

$$\Phi(y,x) = \frac{1}{2\pi}ln\frac{1}{r} + g_2(y,x),\ \ m = 2,\ \ r = |y-x|,$$

$$\Phi(y,x) = \frac{r^{2-m}}{\omega_m(m-2)} + g_m(y,x),\ \ m \geqslant 3,\ \ r = |y-x|,$$

where $g_m(y,x),\ \ m \geqslant 2$ is a functions defined for all values of y, x and it is harmonic with respect to variable y in R^m.

Using function $\Phi(y, x)$, the following matrix is constructed

$$\Pi(y,x) = \|\Pi_{ij}(y,x)\|_{m\times m} = \Big\| \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\delta_{ij}\Phi(y,x) - \\ - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j - x_j)\frac{\partial}{\partial y_i}\Phi(y,x)\Big\|_{m\times m}, i,j = 1,2,\dots,m. \tag{9}$$

3. The solution of problems (1), (2) in domain D_ρ

I. Let $x_0 = (0, \dots, 0, x_m) \in D_\rho$. Let us introduce the following designations

$$\beta = \tau y_m - \alpha_0,\ \ \gamma = \tau x_m - \alpha_0,\ \ \alpha_0^2 = x_1^2 + \ldots + x_{m-1}^2,\ \ r = |x-y|,$$

$$s = \alpha^2 = (y_1 - x_1)^2 + \ldots + (y_{m-1} - x_{m-1})^2,\ \ w = i\tau\sqrt{u^2+\alpha^2} + \beta,\ \ w_0 = i\tau\alpha + \beta.$$

Let us construct a Carleman matrix for problem (1), (2) for domain D_ρ. The Carleman matrix is explicitly expressed in terms of the Mittag–Löffler entire function. It is defined by series [21]

$$E_\rho(w) = \sum_{n=0}^{\infty}\frac{w^n}{\Gamma\left(1+\frac{n}{\rho}\right)}, \quad \rho > 0, \quad E_1(w) = \exp w,$$

where $\Gamma(\cdot)$ is the Euler function.

Let us denote the contour in complex plane w by $\gamma = \gamma(1,\theta)$, $0 < \theta < \dfrac{\pi}{\rho}$, $\rho > 1$. It is in the direction of nondecreasing $argw$ and it consists of the following parts.
1) ray $argw = -\theta$, $|w| \geqslant 1$,
2) arc $-\theta \leqslant argw \leqslant \theta$ circle $|w| = 1$,
3) ray $argw = \theta$, $|w| \geqslant 1$.

Contour γ divides complex plane on two parts: D^- and D^+. They are on the left and the right sides of γ, respectively. Suppose that $\dfrac{\pi}{2\rho} < \theta < \dfrac{\pi}{\rho}$, $\rho > 1$. Then the following relation holds

$$E_\rho(w) = \exp w^\rho + \Psi_\rho(w), \quad w \in D^+$$

$$E_\rho(w) = \Psi_\rho(w), \quad E'_\rho(w) = \Psi'_\rho(w), \quad w \in D^-, \tag{10}$$

where

$$\Psi_\rho(w) = \frac{\rho}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho}{\zeta - w}d\zeta, \qquad \Psi'_\rho(w) = \frac{\rho}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho}{(\zeta - w)^2}d\zeta. \tag{11}$$

$$\begin{aligned} Re\Psi_\rho(w) &= \frac{\Psi_\rho(w) + \Psi_\rho(\overline{w})}{2} = \frac{\rho}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho(\zeta - Rew)}{(\zeta - w)(\zeta - \overline{w})}d\zeta,\\ Im\Psi_\rho(w) &= \frac{\Psi_\rho(w) - \Psi_\rho(\overline{w})}{2i} = \frac{\rho Imw}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho}{(\zeta - w)(\zeta - \overline{w})}d\zeta,\\ \frac{Im\Psi'_\rho(w)}{Imw} &= \frac{\rho}{2\pi i}\int_\gamma \frac{2\exp\zeta^\rho(\zeta - Rew)}{(\zeta - w)^2(\zeta - \overline{w})^2}d\zeta. \end{aligned} \tag{12}$$

In what follows, $\theta = \dfrac{\pi}{2\rho} + \dfrac{\varepsilon_2}{2}$, $\rho > 1, \varepsilon_2 > 0$. It is clear that if $\dfrac{\pi}{2\rho} + \varepsilon_2 \leqslant |argw| \leqslant \pi$ then $w \in D^-$ and $E_\rho(w) = \Psi_\rho(w)$.

Let us set

$$E_{k,q}(w) = \frac{\rho}{2\pi i}\int_\gamma \frac{\zeta^q \exp\zeta^\rho}{(\zeta - w)^k(\zeta - \overline{w})^k}d\zeta, \quad k = 1, 2, \ldots, \quad q = 0, 1, 2, \ldots.$$

If $\dfrac{\pi}{2\rho} + \dfrac{\varepsilon_2}{2} \leqslant |argw| \leqslant \pi$ then the following inequalities are valid

$$|E_\rho(w)| \leqslant \frac{M_1}{1 + |w|}, \quad |E'_\rho(w)| \leqslant \frac{M_2}{1 + |w|^2},$$

$$|E_{k,q}(w)| \leqslant \frac{M_3}{1 + |w|^{2k}}, \quad k = 1, 2, \ldots, \tag{13}$$

where M_1, M_2, M_3 are constants.

Suppose that $\theta = \dfrac{\pi}{2\rho} + \dfrac{\varepsilon_2}{2} < \dfrac{\pi}{\rho}$, $\rho > 1$ in (10). Then $E_\rho(w) = \Psi_\rho(w)$, $\cos\rho\theta < 0$ and

$$\int_\gamma |\zeta|^q \exp(\cos\rho\theta|\zeta|^q)|d\zeta| < \infty, \quad q = 0, 1, 2, \ldots. \tag{14}$$

In this case for sufficiently large $|w|$ $(w \in D^+, \overline{w} \in D^-)$ we have

$$\min_{\zeta\in\gamma}|\zeta - w| = |w|\sin\frac{\varepsilon_2}{2}, \quad \min_{\zeta\in\gamma}|\zeta - \overline{w}| = |w|\sin\frac{\varepsilon_2}{2}. \tag{15}$$

Now from (10) and

$$
\begin{aligned}
\frac{1}{\zeta - w} &= -\frac{1}{w} + \frac{\zeta}{w(\zeta - \overline{w})}, \\
\frac{1}{\zeta - \overline{w}} &= -\frac{1}{\overline{w}} + \frac{\zeta}{\overline{w}(\zeta - \overline{w})},
\end{aligned} \tag{16}
$$

for large $|w|$ we obtain

$$\left| E_\rho(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{w}\right| \leqslant \frac{\rho}{2\pi \sin\frac{\varepsilon_2}{2}}\frac{1}{|w|^2}.$$

$$\int_\gamma |\zeta| \exp\left[\cos\rho\theta|\zeta|^\rho\right]|d\zeta| \leqslant \frac{const}{|w|^2},$$

$$\Gamma^{-1}\left(1 - \frac{1}{\rho}\right) = \frac{\rho}{2\pi i}\int_\gamma \exp\left(\zeta^\rho\right) d\zeta.$$

It follows from this that

$$|E_\rho(w)| \leqslant \frac{M_1}{1 + |w|}.$$

From (11), (15) and

$$\frac{1}{(\zeta - w)^2} = \frac{1}{w^2} - \frac{2\zeta}{w^2(\zeta - w)} + \frac{\zeta^2}{w^2(\zeta - w)^2}$$

for large $|w|$ we obtain

$$\left| E'_\rho(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{w^2}\right| \leqslant \frac{const}{|w|^3}$$

or

$$|E'_\rho(w)| = \frac{M_2}{1 + |w|^2}.$$

Considering (16), for $k = 1, 2, \ldots$ we have

$$
\begin{aligned}
\frac{1}{(\zeta - w)^k(\zeta - \overline{w})^k} &= \left[\frac{(-1)^k}{w^k} + \ldots + \frac{\zeta^k}{w^k(\zeta - w)^k}\right]\left[\frac{(-1)^k}{\overline{w}^k} + \ldots + \frac{\zeta^k}{\overline{w}^k(\zeta - \overline{w})^k}\right] = \\
&= \frac{1}{|w|^{2k}} - \frac{k}{|w|^{2k+1}|\zeta - w|} + \ldots.
\end{aligned}
$$

Taking into account previous relations and (14), (15), for large $|w|$ we obtain

$$\left| E_{k,q}(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{|w|^{2k}}\right| \leqslant \frac{const}{|w|^{2k+1}}$$

or

$$|E'_{k,q}(w)| = \frac{M_3}{1 + |w|^{2k}}, \quad k = 1, 2, \ldots.$$

Therefore, since

$$(\zeta - w)(\zeta - \overline{w}) = \zeta^2 - 2\zeta(y_m - x_m) + u^2 + \alpha^2 + (y_m - x_m)^2, \quad \alpha^2 = s,$$

then

$$\frac{\partial^{n-1}}{\partial s^{n-1}}\frac{1}{(\zeta - w)(\zeta - \overline{w})} = \frac{(-1)^{n-1}(n-1)!}{(\zeta - w)^n(\zeta - \overline{w})^n}.$$

Now we obtain from (11) that

$$\frac{d^{n-1}}{ds^{n-1}} ReE_\rho(w) = \frac{(-1)^{n-1}(n-1)!\rho}{2\pi i} \int_\gamma \frac{(\zeta - (y_m - x_m)) \exp \zeta^\rho}{(\zeta - w)^n (\zeta - \overline{w})^n} d\zeta,$$

$$\frac{d^{n-1}}{ds^{n-1}} \frac{ImE_\rho(w)}{\sqrt{u^2+\alpha^2}} = \frac{(-1)^{n-1}(n-1)!\rho}{\pi i} \int_\gamma \frac{\exp \zeta^\rho}{(\zeta - w)^n (\zeta - \overline{w})^n} d\zeta,$$

Then from (3.) we have

$$\left| \frac{d^{n-1}}{ds^{n-1}} ReE_\rho(w) \right| \leqslant \frac{const \cdot r}{1+|w|^2}$$

$$\left| \frac{d^{n-1}}{ds^{n-1}} \frac{ImE_\rho(w)}{\sqrt{u^2+\alpha^2}} \right| \leqslant \frac{const \cdot r}{1+|w|^2}.$$

For $\sigma > 0$ we set in formulas (6)-(9)

$$K(w) = E_\rho(\sigma^{\frac{1}{\rho}} w), \quad K(x_m) = E_\rho(\sigma^{\frac{1}{\rho}} \gamma). \tag{17}$$

Then, for $\rho > 1$ we obtain

$$\Phi(y,x) = \Phi_\sigma(y,x) = \frac{\varphi_\sigma(y,x)}{c_m E_\rho(\sigma^{\frac{1}{\rho}}\gamma)}, \quad y \neq x,$$

where $\varphi_\sigma(y,x)$ is defined as follows:
if $\frac{1}{\rho}$ $m=2$ then

$$\varphi_\sigma(y,x) = \int_0^\infty Im \frac{E_\rho(\sigma w)}{i\sqrt{u^2+\alpha^2} + y_2 - x_2} \frac{udu}{\sqrt{u^2+\alpha^2}},$$

if $m = 2n+1$, $n \geqslant 1$ then

$$\varphi_\sigma(y,x) = \frac{d^{n-1}}{ds^{n-1}} \int_0^\infty Im \frac{E_\rho(\sigma^{\frac{1}{\rho}} w)}{i\sqrt{u^2+\alpha^2} + y_m - x_m} \frac{udu}{\sqrt{u^2+\alpha^2}}, y \neq x,$$

if $m = 2n$, $n \geqslant 2$ then

$$\varphi_\sigma(y,x) = \frac{d^{n-2}}{ds^{n-2}} Im \frac{E_\rho\left(\sigma^{\frac{1}{\rho}} w\right)}{\alpha(i\alpha + y_m - x_m}, y \neq x.$$

Let us define matrix $\Pi(y,x,\sigma)$ using (9) for $\Phi(y,x) = \Phi_\sigma(y,x)$.
It was proved [10]

Theorem 3.1. *Function $\Phi_\sigma(y,x)$ can be expressed as*

$$\Phi_\sigma(y,x) = \frac{1}{2\pi} ln\frac{1}{r} + g_2(y,x,\sigma), \ \ m=2, \ \ r = |y-x|,$$

$$\Phi_\sigma(y,x) = \frac{r^{2-m}}{\omega_m(m-2)} + g_m(y,x,\sigma), \ \ m \geqslant 3, \ \ r = |y-x|,$$

where $g_m(y,x,\sigma)$, $m \geqslant 2$ is a function defined for all y,x and it is harmonic with spect to variable y in R^m.

We obtain from this theorem

Theorem 3.2. *Matrix $\Pi(y,x,\sigma)$ defined in (7)–(9) and (17) is a Carleman matrix for problem (1), (2).*

Let us first consider some properties of function $\Phi_\sigma(y,x)$.

I. Let $m=2n+1,\ n\geqslant 1,\ x\in D_\rho,\ y\neq x, \sigma\geqslant\sigma_0>0$ then
1) for $\beta\leqslant\alpha$ the following inequalites hold:

$$|\Phi_\sigma(y,x)|\leqslant C_1(\rho)\frac{\sigma^{m-2}}{r^{m-2}}\exp(-\sigma\gamma^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_2(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_3(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho),\ \ i=1,\dots,m, \tag{18}$$

2) for $\beta>\alpha$ the following inequalities hold:

$$|\Phi_\sigma(y,x)|\leqslant C_4(\rho)\frac{\sigma^{m-2}}{r^{m-2}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_5(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_6(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ i=1,\dots,m. \tag{19}$$

II. Let $m=2n,\ n\geqslant 2,\ x\in D_\rho,\ x\neq y,\ \sigma\geqslant\sigma_0>0$ then
1) for $\beta\leqslant\alpha$ the following inequalities hold:

$$|\Phi_\sigma(y,x)|\leqslant \widetilde{C}_1(\rho)\frac{\sigma^{m-3}}{r^{m-2}}\exp(-\sigma\gamma^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_2(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_3(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho),\ \ y\in\partial D_\rho,\ \ i=1,\dots,m, \tag{20}$$

2) for $\beta>\alpha$ the following inequalities hold:

$$|\Phi_\sigma(y,x)|\leqslant \widetilde{C}_4(\rho)\frac{\sigma^{m-2}}{r^{m-2}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_5(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_6(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ y\in\partial D_\rho,\ \ i=1,\dots,m. \tag{21}$$

III. Let $m=2,\ x\in D_\rho,\ x\neq y,\ \sigma\geqslant\sigma_0>0$ then
1) if $\beta\leqslant\alpha$ then

$$|\Phi_\sigma(y,x)|\leqslant C_7(\rho)E^{-1}(\sigma^{\frac{1}{\rho}}\gamma)\ln\frac{1+r^2}{r^2},$$

$$\left|\frac{\partial\Phi_\sigma}{\partial y_i}(y,x)\right|\leqslant C_8(\rho)\frac{E_\rho^{-1}(\sigma^{\frac{1}{\rho}}\gamma)}{r}, \tag{22}$$

2) if $\beta > \alpha$ then

$$|\Phi_\sigma(y,x)| \leqslant \widetilde{C}_7(\rho) E^{-1}(\sigma^{\frac{1}{\rho}}\gamma)(\ln \frac{1+r^2}{r^2}) \exp(\sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial \Phi_\sigma}{\partial y_i}(y,x)\right| \leqslant \widetilde{C}_8(\rho) E_\rho^{-1}(\sigma^{\frac{1}{\rho}}\gamma)\frac{1}{2}\exp(\sigma Re\omega_0^\rho). \tag{23}$$

Here all coefficients $C_i(\rho)$ and $\widetilde{C}_i(\rho)$, $\quad i = 1, \dots 8$ depend on ρ.

Proof of Theorem 3.2. From the definition of $\Pi(y,x,\sigma)$ and Lemma 1 we have

$$\Pi(y,x,\sigma) = \Gamma(y,x) + G(y,x,\sigma),$$

where

$$G(y,x,\sigma) = ||G_{kj}(y,x,\sigma)||_{m\times m} = \\ = \left|\left|\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\delta_{kj} g_m(y,x,\sigma) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j - x_j)\frac{\partial}{\partial y_i} g_m(y,x,\sigma)\right|\right|_{m\times m}.$$

Let us prove that $A(\partial_y)G(y,x,\sigma) = 0$. Since $\Delta_y g_m(y,x,\sigma) = 0$, $\Delta_y = \sum\limits_{k=1}^{m} \frac{\partial^2}{\partial y_k^2}$ and taking into account relation for the jth column $G^j(y,x,\sigma)$

$$\div G^j(y,x,\sigma) = \frac{1}{2\mu(\lambda+2\mu)} \cdot \frac{\partial}{\partial y_j} g_m(y,x,\sigma),$$

we obtain relation for the kth components of $A(\partial_y)G^j(y,x,\sigma)$

$$\sum_{i=1}^{m} A(\partial_y)_{ki} G_{ij}(y,x,\sigma) = \mu\Delta_y\left[\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} \cdot \delta_{kj} g_m(y,x,\sigma) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j - x_j)\frac{\partial}{\partial y_k} g_m(y,x,\sigma)\right] + \\ + (\lambda+\mu)\frac{\partial}{\partial y_k} div G^j(y,x,\sigma) = \\ = -\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2} g_m(y,x,\sigma) + \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2} g_m(y,x,\sigma) = 0$$

Therefore, each column of matrix $G(y,x,\sigma)$ satisfies system (1) with respect to the variable y everywhere in R^m.

The second condition on the Carleman matrix follows from inequalities (18)–(23). The proof of the theorem is complete. □

For fixed $x \in D_\rho$ we denote the part of S, where $\beta \geqslant \alpha$ by S^*. If $x = x_0 = (0, \dots, 0, x_m) \in D_\rho$ then $S = S^*$. Consider the point $(0, \dots, 0) \in D_\rho$. Suppose that

$$\frac{\partial U}{\partial n}(0) = \frac{\partial U}{\partial y_m}(0), \quad \frac{\partial \Phi_\sigma(0,x)}{\partial n} = \frac{\partial \Phi_\sigma(0,x)}{\partial y_m}.$$

Let

$$U_\sigma(y) = \int_{S^*} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^* U(y)] ds_y, \;\; x \in D_\rho. \tag{24}$$

Theorem 3.3. *Let $U(x)$ be a regular solution of system (1) in D_ρ, such that*

$$|U(y)| + |T(\partial_y, n)U(y)| \leqslant M, \ \ y \in \Sigma. \tag{25}$$

Then
1) if $m = 2n+1$, $n \geqslant 1$ and for $x \in D_\rho$, $\sigma \geqslant \sigma_0 > 0$ the following estimate is valid:

$$|U(x) - U_\sigma(x)| \leqslant MC_1(x)\sigma^{m+1}\exp(-\sigma\gamma^\rho),$$

2) if $m = 2n$, $n \geqslant 1$, $x \in D_\rho$, $\sigma \geqslant \sigma_0 > 0$ the following estimate is valid

$$|U(x) - U_\sigma(x)| \leqslant MC_2(x)\sigma^{m}\exp(-\sigma\gamma^\rho),$$

where

$$C_k(x) = C_k(\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}, \ \ k = 1, 2,$$

$C_k(\rho)$ is a constant that depends on ρ.

Proof. It follows from (5) that

$$U(x) = \int_{S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y +$$
$$+ \int\limits_{\partial D_\rho \setminus S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y, \ \ x \in D_\rho.$$

Therefore, we have from (24) that

$$|U(x) - U_\sigma(x)| \leqslant \int\limits_{\partial D_\rho \setminus S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y \leqslant$$
$$\leqslant \int\limits_{\partial D_\rho \setminus S^*}[|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]\,[|T(\partial_y,n)\Pi(y,x,\sigma)| + |U(y)|]\,ds_y.$$

Taking into accoun inequalities (18)–(23) and condition (25), we obtain for $\beta \leqslant \alpha$ and $m = 2n+1$, $n \geqslant 1$

$$|U(x) - U_\sigma(x)| \leqslant MC_1(\rho)\sigma^{m+1}\exp(-\sigma\gamma^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}.$$

For $m = 2n$, $n \geqslant 1$ we obtain

$$|U(x) - U_\sigma(x)| \leqslant MC_2(\rho)\sigma^{m}\exp(-\sigma\gamma^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}.$$

The proof of the theorem is complete. □

One can determine $U(x)$ approximately if, instead of $U(y)$ and $T(\partial_y, n)U(y)$, their continuous approximations $f_\delta(y)$ and $g_\delta(y)$ are given on surface S:

$$\max_S|U(y) - f_\delta(y)| + \max_S|T(\partial_y,n)U(y) - g_\delta(y)| \leqslant \delta, \ \ 0 < \delta < 1. \tag{26}$$

Function $U_{\sigma\delta}(x)$ is defined as follows

$$U_{\sigma\delta}(x) = \int_{s^*}[\Pi(y,x,\sigma)g_\delta(y) - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^* f_\delta(y)]ds_y, \ \ x \in D_\rho, \tag{27}$$

where
$$\sigma = \frac{1}{R^\rho} \ln \frac{M}{\delta}, R^\rho = \max_{y \in S} Re\omega_0^\rho.$$
Then the following theorem holds.

Theorem 3.4. *Let $U(x)$ be a regular solution of system (1) in D_ρ, such that*
$$|U(y)| + |T(\partial_y, n)U(y)| \leqslant M, \;\; y \in \partial D_\rho.$$
Then,
1) if $m = 2n+1, \;\; n \geqslant 1$ then the following estimate is valid
$$|U(x) - U_{\sigma\delta}(x)| \leqslant C_1(x)\delta^{(\frac{\gamma}{R})^\rho} \left(ln \frac{M}{\delta} \right)^{m+1},$$
2) if $m = 2n, \;\; n \geqslant 1$ then the following estimate is valid:
$$|U(x) - U_{\sigma\delta}(x)| \leqslant C_2(x)\delta^{(\frac{\gamma}{R})^\rho} \left(\ln \frac{M}{\delta} \right)^{m},$$
where
$$C_k(x) = C_k(\rho) \int_{\partial D_\rho} \frac{ds_y}{r^m}, \;\; k = 1, 2.$$

Proof. It follows from (5) and (27) that
$$U(x) - U_{\sigma\delta}(x) = \int\limits_{\partial D_\rho \setminus S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* U(y)] ds_y +$$
$$+ \int_{S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y) - g_\delta(y)\} + \{T(\partial_y, n)\Pi(y, x, \sigma)\}^*(U(y) - f_\delta(y))] ds_y =$$
$$= I_1 + I_2.$$
Taking into account Theorem 3.3, we obtain for $m = 2n+1, \;\; n \geqslant 1$,
$$|I_1| = MC_1(\rho)\sigma^{m+1} \exp(-\sigma\gamma^\rho) \int_{\partial D_\rho} \frac{ds_y}{r^m},$$
and for $m = 2n, \;\; n \geqslant 1$
$$|I_1| = MC_2(\rho)\sigma^{m} \exp(-\sigma\gamma^\rho) \int_{\partial D_\rho} \frac{ds_y}{r^m}.$$
Let us consider $|I_2|$:
$$|I_2| = \int_{S^*} \left(|\Pi(y, x, \sigma)| + |T(\partial_y, n)\Pi(y, x, \sigma)|\right) \left(|T(\partial_y, n)U(y) - g_\delta(y)| + |U(y) - f_\delta(y)|\right) ds_y.$$
Taking into account Theorem 3.1 and condition (26), we obtain for $m = 2n+1, \;\; n \geqslant 1$
$$|I_2| = \widetilde{C}_1(\rho)\sigma^{m+1}\delta \exp(-\sigma\gamma^\rho + \sigma Re\omega_0^\rho) \int_{\partial D_\rho} \frac{ds_y}{r^m}$$
and for $m = 2n, \;\; n \geqslant 1$,
$$|I_2| = \widetilde{C}_2(\rho)\sigma^{m}\delta \exp(-\sigma\gamma^\rho + \sigma Re\omega_0^\rho) \int_{\partial D_\rho} \frac{ds_y}{r^m}.$$

Therefore, from

$$\sigma = \frac{1}{R^\rho} \ln \frac{M}{\delta}, \quad R^\rho = \max_{y \in S} Re\omega_0^\rho.$$

The theorem is proved.

Corollary 1. The limits

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x), \quad \lim_{\delta \to 0} U_{\sigma\delta}(x) = U(x)$$

hold uniformly on any compact set from D_ρ.

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Задача Коши для уравнения теории упругости

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Аннотация. Рассматривается задача об аналитическом продолжении решения системы теории упругости в область по значениям решения и его напряжений на части границы этой области, т. е. задача Коши.

Ключевые слова: задача Коши, теория упругости, эллиптическая система, некорректно поставленная задача, матрица Карлемана, регуляризация.