EDN: BBJNGZ
УДК 517.55

Summation of Functions and Polynomial Solutions to a Multidimensional Difference Equation

Andrey A. Grigoriev*
Evgeniy K. Leinartas†
Siberian Federal University
Krasnoyarsk, Russian Federation
Alexander P. Lyapin‡
Siberian Federal University
Krasnoyarsk, Russian Federation
Fairmont State University
Fairmont, WV, USA

Received 10.10.2022, received in revised form 09.11.2022, accepted 08.12.2022

Abstract. We define a set of polynomial difference operators which allows us to solve the summation problem and describe the space of polynomial solutions for these operators in equations with the polynomial right-hand side. The criterion describing these polynomial difference operators was obtained. The theorem describing the space of polynomial solutions for the operators was proved.

Keywords: Bernoulli numbers, Bernoulli polynomials, summation problem, multidimensional difference equation, Euler–Maclaurin formula, Todd operator.

Citation: A. A. Grigoriev, E. K. Leinartas, A. P. Lyapin, Summation of Functions and Polynomial Solutions to a Multidimensional Difference Equation, J. Sib. Fed. Univ. Math. Phys., 2023, 16(2), 153–161.

1. Introduction and preliminaries

The summation of functions is one of the main problems of the theory of finite differences, and the answer was given in the famous Euler–Maclaurin formula obtained by Euler in 1733 and independently by Maclaurin in 1738 (see [6, 7, 21]).

In [1, 2, 13] the problem of rational summation was studied, that is, finding sums of the form

$$S(x) = \sum_{t=0}^{x} \varphi(t), \tag{1}$$

where the function $\varphi(t)$ is a rational function. The solution to the problem consists in finding a solution in symbolic form, that is, explicitly in the form of a mathematical function (formula) and is called the *indefined summation problem* (see also [8, 9]).

In the *definite summation problem*, the function φ can depend not only on the summation index, but also on the summation boundary x, that is, $S(x) = \sum\limits_{t=0}^{x} \varphi(t, x)$ (see, for example, [11, 20]).

*grigrow@yandex.ru
†lein@mail.ru
‡aplyapin@sfu-kras.ru

The problem of indefinite summation is reduced to solving the so-called (see [8,9]) telescopic equation — the inhomogeneous difference equation

$$(\delta - 1)f(x) = \varphi(x), \tag{2}$$

where δ is a shift operator: $\delta f(x) := f(x+1)$.

By analogy with the problem of integrating functions, the solution $f(x)$ to equation (2) is called *the discrete antiderivative of the function* $\varphi(x)$. If $f(x)$ is the discrete antiderivative function $\varphi(x)$, then the required sum is

$$S(x) = f(x+1) - f(0). \tag{3}$$

Formula (3) is called the *discrete analogue of the Newton–Leibniz formula.*

Euler's approach to the problem of finding a discrete antiderivative is based on the operator equality $\delta = e^D$, which allows us to write (2) in the form

$$Df(x) = \left[\frac{D}{e^D - 1}\right]\varphi(x),$$

where D is a differentiation operator.

The expression in square brackets on the right-hand side of the last equality is called the *Todd operator* and is understood as follows: $\left[\frac{D}{e^D - 1}\right] = \sum\limits_{m=0}^{\infty} \frac{B_m}{m!} D^m$, where b_m are Bernoulli numbers (see, for example, [3,6,10,17,19]). Thus, we obtain the Euler–Maclaurin formula

$$\sum_{t=0}^{x} \varphi(t) = \int_0^{x+1} \varphi(t)dt + \sum_{m=1}^{\infty} \frac{B_m}{m!} \left[\varphi^{(m-1)}(x+1) - \varphi^{(m-1)}(0)\right],$$

in which the required sum is expressed in terms of the derivatives and the integral of the function $\varphi(t)$.

Remark 1. *In the summation problem we can use other operators instead of $\delta - 1$. For example, we can consider the operator $(\delta - 1)(\delta - 2)$ and solve the difference equation*

$$f(x+2) - 3f(x+1) + 2f(x) = \varphi(x), \quad x = 0, 1, 2, \ldots.$$

If a solution to this equation is found then the sum $S(x)$ can be written as $S(x) = f(x+2) - 2f(x+1) - [f(1) - 2f(0)]$. For $n = 1$ polynomial difference operators $P(\delta) = c_0 + c_1\delta + \cdots + c_m\delta^m$, where $c_0 + \cdots + c_m = 0$, has a similar property (effect), see Theorem 2.3.

Euler's approach to the problem of indefinite summation of a function $\varphi(t) = \varphi(t_1, \ldots, t_n)$ of several variables suggests that you need to find a multidimensional analogue of (2), and compute a discrete antiderivative to obtain an analogue of the Newton-Leibniz formula (3). In Section 2 we implemented it to sum a function over the integer points in an n-dimensional parallelepiped (Lemma 2.2 and Theorem 2.3).

Bernoulli numbers and polynomials play an important role in classical one-dimensional summation theory and various branches of combinatorial analysis. Bernoulli polynomials are solutions of difference equation (2) with polynomial right-hand side $\varphi(t) = t^\mu$:

$$B_\mu(t+1) - B_\mu(t) = \mu t^{\mu-1}.$$

In the third section of this paper, we use spaces of polynomial solutions (generalized Bernoulli polynomials) to sum functions of several discrete arguments.

2. Operators with a summing effect and a discrete analogue of the Newton–Leibniz formula

To formulate the main result of the paper (Theorem 2.3), we need the following definitions and notations. For a given function of several discrete arguments $\varphi(t) = \varphi(t_1, \ldots, t_n)$, we consider the problem of finding the sum of its values over all integer points of an n-dimensional parallelepiped with a "variable" vertex $x \in \mathbb{Z}^n_{\geqslant}$:

$$\Pi(x) = \{t \in \mathbb{R}^n_{\geqslant} : 0 \leqslant t_j \leqslant x_j, j = 1, \ldots, n\}. \tag{4}$$

This sum can be written as follows:

$$S(x) = \sum_{t_1=0}^{x_1} \cdots \sum_{t_n=0}^{x_n} \varphi(t_1, \ldots, t_n) = \sum_{t \in \Pi(x)} \varphi(t). \tag{5}$$

To solve the summation problem means to find a formula expressing (5) in terms of a (finite) number of terms independent of x.

Operating on the complex-valued functions $f(x)$ of integer arguments $x = (x_1, \ldots, x_n)$, we define the shift operator δ_j with respect to the j-th variable

$$\delta_j f(x) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n), \quad \delta_j^{\alpha_j} = \underbrace{\delta_j \circ \cdots \circ \delta_j}_{\alpha_j \ times},$$

where δ_j^0 is the identity operator. Some properties of the shift operator were studied in [12]. Denote $P(\delta) = \sum\limits_{0 \leqslant \alpha \leqslant l} c_\alpha \delta^\alpha$ — polynomial difference operator with constant coefficients c_α, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n_{\geqslant}$, and the inequality $l \geqslant \alpha$ means $l_j \geqslant \alpha_j, j = 1, \ldots, n$. We will also use the notation $l \ngeqslant \alpha$, if there is at least one j_0 for which $l_{j_0} < \alpha_{j_0}$.

The difference equation for the unknown function $f(x)$ is written as follows:

$$P(\delta) f(x) = \varphi(x), \ x \in \mathbb{Z}^n_{\geqslant}. \tag{6}$$

Definition 2.1. *A polynomial difference operator $P(\delta)$ of the difference equation* (6) *is called an operator with a summing effect if the sum* (5) *can be represented through solutions $f(x)$ to this equation at finite set of points regardlessly of the numbers of summands in $S(x)$.*

In this case, naturally, $f(x)$ can be called *the discrete antiderivative of the function $\varphi(x)$*, and the corresponding expression solving the summation problem (5) is *a discrete analogue of the Newton–Leibniz formula.*

For any point x, we define the projection operator π_j along the x_j axis:

$$\pi_j x := (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$$

and define its action: $\pi_j f(x) := f(\pi_j x)$.

Let $\mathcal{P}(A)$ be the power set of A and $V := \mathcal{P}(\{1, \ldots, n\}), J = \{j_1, \ldots, j_k\} \in V$. If we denote $\pi_J = \pi_{j_1} \circ \ldots \circ \pi_{j_k}$, then the set of vertices of the parallelepiped $\Pi(x)$ can be written as $\{\pi_J x, J \in V\}$. Note that $\pi_\varnothing x = x$.

Lemma 2.2. *In (6) let $P(\delta) = R(\delta)(\delta - I)$, where $R(\delta)$ is a polynomial operator. Then for any solution f of (6), the discrete analogue of the Newton–Leibniz formula is*

$$\sum_{t \in \Pi(x)} \varphi(t) = R(\delta) \sum_{J \in V} (-1)^{\#J} f(\pi_J(x + I)),$$

where $\#J$ is a number of elements of the set J.

Proof. Since

$$\sum_{t_j=0}^{x_j}(\delta_j-1)f(t)=\sum_{t_j=0}^{x_j}(\delta_j-1)\delta_j^{t_j}\pi_j f(t)=(\delta_j-1)\Big(\sum_{t_j=0}^{x_j}\delta_j^{t_j}\Big)\pi_j f(t)=$$
$$=(\delta_j-1)\frac{\delta_j^{x_j+1}-1}{\delta_j-1}\pi_j f(t)=(\delta_j^{x_j+1}-1)\pi_j f(t),$$

we get

$$\sum_{0\leqslant t\leqslant x}\varphi(t)=R(\delta)\prod_{j=1}^{n}(\delta_j^{x_j+1}-1)\pi_j f(t)=R(\delta)\prod_{j=1}^{n}(\delta_j^{x_j+1}\pi_j-\pi_j)f(t),$$

hense, since π_j and δ_k permute for $j\neq k$, we have

$$\prod_{j=1}^{n}(\delta_j^{x_j+1}\pi_j-\pi_j)=\sum_{J\in V}(-1)^{\#J}\delta_{\overline{J}}^{x_{\overline{J}}+I}\pi_{\overline{J}}\pi_J,$$

where $\overline{J}=\{1,\dots,n\}\backslash J$, $\delta=(\delta_1,\cdots,\delta_n)$.

Thus we conclude that

$$\sum_{t\in\Pi(x)}\varphi(t)=R(\delta)\prod_{j=1}^{n}(\delta_j^{x_j+1}\pi_j-\pi_j)f(t)=R(\delta)\sum_{J\in V}(-1)^{\#J}f(\pi_J(x+I)).$$

□

Note that the case $R(\delta)\equiv 1$ was proved in [18].

We see that in Lemma 2.2 finding the value of (5) is reduced to calculating the values of the function $f(x)$ at the vertices of the parallelepiped $\Pi(x+I)$, the number of which is 2^n and does not depend on x. Thus, the operator $P(\delta)=R(\delta)(\delta-I)$ has a summing effect.

We denote $\partial=(\partial_1,\dots,\partial_n)$, where ∂_j is the differenctiation operators with respect to the j-th variable, $j=1,\dots,n$, and $\partial^\mu=\partial_1^{\mu_1}\dots\partial_n^{\mu_n}$.

Theorem 2.3. *In the summation problem (5), the polynomial difference operators*

$$P(\delta)=R(\delta)\prod_{j=1}^{n}(\delta_j-1)=R(\delta)(\delta-I)$$

and only they have a summing effect, where $R(\delta)$ is a polynomial.

Proof. We transform (5), assuming that $f(t)$ is a solution to the difference equation (6) and using the equality $f(t)=\delta^t f(0)$ yields

$$S(x)=\sum_{t\in\Pi(x)}\varphi(t)=\sum_{t\in\Pi(x)}P(\delta)f(t)=\sum_{t\in\Pi(x)}\delta^t P(\delta)f(0). \tag{7}$$

Next, we use the multiple geometric progression formula $\sum\limits_{t\in\Pi(x)}\delta^t=\dfrac{\delta^{x+I}-I}{\delta-I}$ and expand the characteristic polynomial in a Taylor series at the point $I=(1,1,\dots,1)$: $P(z)=\sum\limits_{0\leqslant\alpha\leqslant l}\frac{\partial^\alpha P(I)}{\alpha!}(z-I)^\alpha$. Then we transform the resulting expression

$$P(z)=\sum_{\substack{\alpha\geqslant 0\\ \alpha\ngeqslant I}}\frac{\partial^\alpha P(I)}{\alpha!}(z-I)^\alpha+(z-I)\sum_{I\leqslant\alpha\leqslant l}\frac{\partial^\alpha P(I)}{\alpha!}(z-I)^{\alpha-I}$$

and to express (5) as

$$S(x) = \left(\sum_{\substack{\alpha \geqslant 0 \\ \alpha \ngeqslant I}} \frac{\partial^\alpha P(I)}{\alpha!} (\delta - I)^\alpha \right) \sum_{t \in \Pi(x)} f(t) +$$

$$+ \left(\sum_{I \leqslant \alpha \leqslant l} \frac{\partial^\alpha P(I)}{\alpha!} (\delta - I)^{\alpha - I} \right) \left(\delta^{x+I} - I\right) f(0), \quad (8)$$

where $\delta^{x+I} - I = (\delta_1^{x_1+1} - 1) \cdots (\delta_n^{x_n+1} - 1)$.

Note that the number of summands in the second sum of the right-hand side of (8) does not depend on numbers of summands in $S(x)$, but in the first sum it does. If $P(\delta) = R(\delta)(\delta - I)$, then the first term is absent and $P(\delta)$ has a summing effect.

On the other hand, if $P(\delta)$ has a summing effect, then $\sum\limits_{\substack{\alpha \geqslant 0 \\ \alpha \ngeqslant I}} \frac{\partial^\alpha P(I)}{\alpha!} (\delta - I)^\alpha \equiv 0$, but then

$$P(\delta) = \sum_{I \leqslant \alpha \leqslant l} \frac{\partial^\alpha P(I)}{\alpha!} (\delta - I)^\alpha = (\delta - I) R(\delta),$$

where $R(\delta) = \sum\limits_{I \leqslant \alpha \leqslant l} \frac{\partial^\alpha P(I)}{\alpha!} (\delta - I)^{\alpha - I}$. □

Example. Find the sum

$$S(x_1, x_2) = \sum_{t_1=0}^{x_1} \sum_{t_2=0}^{x_2} \varphi(t_1, t_2)$$

for the function

$$\varphi(t_1, t_2) = \frac{1}{(t_1 + t_2 + 1)(t_1 + t_2 + 2)(t_1 + t_2 + 3)}.$$

We note that the function

$$f(t_1, t_2) = \frac{1}{2} \frac{1}{t_1 + t_2 + 1}$$

is a solution to the difference equation $(\delta_1 - 1)(\delta_2 - 1) f(t) = \varphi(t)$. Since $P(\delta) = (\delta_1 - 1)(\delta_2 - 1)$, $R \equiv 1$, the sum is

$$S(x) = f(x_1 + 1, x_2 + 1) - f(x_1 + 1, 0) - f(0, x_2 + 1) + f(0, 0) =$$
$$= \frac{1}{2} \left(\frac{1}{x_1 + x_2 + 3} - \frac{1}{x_1 + 2} - \frac{1}{x_2 + 2} + 1 \right).$$

3. Polynomial solutions to a multidimensional difference equation

Bernoulli numbers and polynomials play an important role in the classical one-dimensional summation theory. Bernoulli polynomials are solutions of the difference equation (2) with the polynomial right-hand side $\varphi(t) = t^{\mu - 1}$:

$$\frac{1}{\mu} \left(B_\mu(t + 1) - B_\mu(t) \right) = t^{\mu - 1}. \quad (9)$$

Bernoulli numbers and polynomials are well studied (see, for example, [6, 19]) and have numerous applications in various branches of mathematics (see [5, 15, 16]).

One of the options for finding the Bernoulli polynomials is to use the operator equality $\delta = e^D$. From (9) we find the formula for the Bernoulli polynomials

$$B_\mu(t) = \frac{\mu}{\delta - 1} t^{\mu-1} = \frac{\mu}{e^D - 1} t^{\mu-1},$$

whence we get

$$B_\mu(t) = \frac{D}{e^D - 1} t^\mu, \tag{10}$$

where $\dfrac{D}{e^D - 1} = \sum\limits_{\nu=0}^{\infty} B_\nu \dfrac{D^\nu}{\nu!}$ is a differential operator of infinite order, $B_\nu = B_\nu(0)$ are Bernoulli numbers.

The action of the operator $\dfrac{D}{e^D - 1}$ on polynomials is well defined. We obtain a formula for finding the Bernoulli polynomials

$$B_\mu(t) = \sum_{\nu=0}^{\mu} \frac{B_\nu}{\nu!} D^\nu t^\mu.$$

Remark. The above scheme for finding the Bernoulli polynomials can be viewed as a method for finding a particular solution of the equation (2) in the case when the right-hand side of $\varphi(t)$ is a polynomial.

We are interested in computing polynomial solutions to difference equation (6) with polynomial right-hand sides. In this case, without loss of generality, we can consider the case $\varphi(t) = t^\mu = t_1^{\mu_1} \dots t_n^{\mu_n}$. In addition, we are interested in polynomial difference operators $P(\delta)$ with a summing effect, which, by virtue of Theorem 2.3, can be written in the form

$$P(\delta) = R(\delta) \prod_{j=1}^{n} (\delta_j - 1)^{k_j}, \tag{11}$$

where $R(\delta)$ is a polynomial difference operator with constant coefficients, $R(I) \neq 0$.

We consider the difference equation

$$R(\delta) \prod_{j=1}^{n} (\delta_j - 1)^{k_j} f(t) = t^\mu, \quad t \in \mathbb{Z}^n_{\geqslant}, \tag{12}$$

and find its particular polynomial solutions by analogy with the one-dimensional case, that is, we use the operator equalities $\delta_j = e^{D_j}$, $j = 1, 2, \dots, n$.

The function $\mathrm{Td}(\xi) = \dfrac{1}{R(e^\xi)} \prod\limits_{j=1}^{n} \dfrac{\xi_j^{k_j}}{(e^{\xi_j} - 1)^{k_j}}$ is holomorphic at the point $\xi = 0$ and therefore admits its expansion in some neighborhood of zero as a power series

$$\mathrm{Td}(\xi) = \sum_{m \geqslant 0} \frac{\tilde{b}_{k,m}}{m!} \xi^m. \tag{13}$$

Substituting the differentiation operator D_j into (13) in place of the variable ξ_j, we define the *differential operator of infinite order*:

$$\mathrm{Td}(D) = \sum_{m \geqslant 0} \frac{\tilde{b}_{k,m}}{m!} D^m. \tag{14}$$

For $k_1 = \ldots = k_n = 1$ and $R(\delta) \equiv 1$, the operator defined in (14) is called the *Todd operator* (see, for example, [4, 14]). In the general case, it is natural to call it *the generalized Todd operator*, and the numbers $\tilde{b}_{k,m}$ — *generalized Bernoulli numbers*. Any polynomial solution to equation (12) is called the Bernoulli polynomial associated with the polynomial difference operator (11).

The case $R(\delta) \equiv 1$ was considered in [18].

We set $\mu^{(m)} = \mu(\mu-1)(\mu-2)\cdots(\mu-(m-1))$.

Theorem 3.1. *Let $P(\delta)$ be an operator with summing effect of the form (11). Then the set of Bernoulli polynomials associated with this operator is described by the formula*

$$f(x) = \sum_{0 \leqslant m \leqslant \mu} \frac{\tilde{b}_{k,m}}{m!} \frac{\mu^{(m)} x^{\mu+k-m}}{(\mu+k-m)^{(k)}} + \sum_{i=1}^{n} \sum_{m_i=1}^{k_i} x_i^{k_i - m_i} q_{m_i}(x_1, \ldots [i] \ldots, x_n), \tag{15}$$

where q_{m_i} are arbitrary polynomials in $(n-1)$-th variables $x_1, \ldots, [i], \ldots, x_n$.

Proof. From the difference equation (12), using $\delta_j = e^{D_j}$, $j = 1, 2, \ldots, n$, and the definition of the Todd operator, we obtain

$$D^k f(x) = \mathrm{Td}(D) x^\mu = \sum_{0 \leqslant m \leqslant \mu} \frac{\tilde{b}_{k,m}}{m!} D^m x^\mu = \sum_{0 \leqslant m \leqslant \mu} \frac{\tilde{b}_{k,m}}{m!} \mu^{(m)} x^{\mu-m}. \tag{16}$$

Integrating (16) k_j times over the variable x_j for all $j = 1, \ldots, n$, we get (15). □

Example. As an illustration of the application of (15), we present the solution of the difference equation

$$(\delta_1 - 1)(\delta_2 - 1) f(x, y) = xy. \tag{17}$$

We have $P(\delta) = (\delta_1 - 1)(\delta_2 - 1), R \equiv 1, (\mu_1, \mu_2) = (1, 1), (k_1, k_2) = (1, 1)$, and $f(x, y) = \tilde{B}_{11,11}(x) + Q(x) + S(y)$, where $\tilde{B}_{11,11}(x) = \frac{\tilde{b}_{11,00}}{2 \cdot 2} x^2 y^2 + \frac{\tilde{b}_{11.01}}{2 \cdot 1} x^2 y + \frac{\tilde{b}_{11.10}}{1 \cdot 2} x y^2 + \frac{\tilde{b}_{11,11}}{1 \cdot 1} xy$ is the generalized Bernoulli polynomial, $\tilde{b}_{11,m}$ are the expansion coefficients of the generating function

$$\frac{D_1 D_2}{(e^{D_1} - 1)(e^{D_2} - 1)}$$

into the Taylor series at the point $D = 0$; $Q(x), S(y)$ are arbitrary polynomials in one variable.

Calculations give: $\tilde{b}_{11.00} = 1$, $\tilde{b}_{11.01} = -\frac{1}{2}$, $\tilde{b}_{11.10} = -\frac{1}{2}$, $\tilde{b}_{11,11} = \frac{1}{4}$.

Thus, any polynomial solution to (17) has the form

$$f(x, y) = \frac{1}{4}(x^2 y^2 - x^2 y - x y^2 + xy) + Q(x) + S(y).$$

The second author is supported by the Russian Science Foundation no. 20-11-20117.

References

[1] S.A.Abramov, Indefinite sums of rational functions, Proceedings of ISSAC'95, 1995, 303–308.

[2] S.A.Abramov, On the summation of rational functions, *USSR Computational Mathematics and Mathematical Physics*, **11**(1971), 324–330. DOI: 10.1016/0041-5553(71)90028-0

[3] T.Arakawa, T.Ibukiyama, M.Kaneko, Bernoulli Numbers and Zeta Functions, Springer, 2014.

[4] M.Brion, Lattice points in simple polytopes, *Journal of the American Mathematical Society*, **10**(1997), no. 2, 371–392.

[5] C.E.Froberg, Introduction to Numerical Analysis, Addison–Wesley, Reading, Mass., 1965.

[6] A.O.Gelfond, Calculus of finite differences, Moscow, Nauka, 1977 (in Russian).

[7] G.Hardy, Divergent series, Oxford University Press, London, 1949.

[8] M.Kauers, Algorithms for Nonlinear Higher Order Difference Equations, Ph.D. Thesis, RISC-Linz, Johannes Kepler University, 2005.

[9] M.Kauers, The Concrete Tetrahedron, Springer-Verlag Wien, 2011.

[10] E.K.Leinartas, O.A.Shishkina, The Discrete Analog of the Newton-Leibniz Formula in the Problem of Summation over Simplex Lattice Points, *Journal of Siberian Federal University. Mathematics & Physics*, **12**(2019), no. 4, 503–508.
DOI: 10.17516/1997-1397-2019-12-4-503-508

[11] A.P.Lyapin, S.Chandragiri, Generating functions for vector partition functions and a basic recurrence relation, *Journal of Difference Equations and Applications*, **25**(2019), no. 7, 1052–1061.

[12] A.P.Lyapin, T.Cuchta, Sections of the Generating Series of a Solution to a Difference Equation in a Simplicial Cone, *The Bulletin of Irkutsk State University. Series Mathematics*, **42**(2022), 75–89. DOI: 10.26516/1997-7670.2022.42.75

[13] S.A.Polyakov, Indefinite summation of rational functions with factorization of denominators, *Programming and Computer Software*, **37**(2011), no. 6, 322–325.
DOI: 10.1134/S0361768811060077

[14] A.V.Pukhlikov, A.G.Khovanskii, The Riemann–Roch theorem for integrals and sums of quasipolynomials on virtual polytopes, *St. Petersburg Math. J.*, **4**(1993), (4), 789—812.

[15] J.Riordan, Combinatorial identities, Huntington, N.Y., 1979.

[16] G.C.Rota, D.Kahaner, A.Odlyzko, On the foundations of combinatorial theory. VIII. Finite operator calculus, *Journal of Mathematical Analysis and Applications*, **42**(1973), no. 3, 684–760.

[17] O.A.Shishkina, Multidimensional Analog of the Bernoulli Polynomials and its Properties, *Journal of Siberian Federal University. Mathematics & Physics*, **9**(2016), no. 3, 376–384.

[18] O.A.Shishkina, The Euler-Maclaurin Formula for Rational Parallelotope, The Bulletin of Irkutsk State University. Series Mathematics, **13**(2015), 56–71.

[19] N.M.Temme, Bernoulli polynomials old and new: Generalization and asymptotics, *CWI Quarterly*, **8**(1995), no. 1, 47–66.

[20] S.P.Tsarev, On rational definite summation, *Programming & Computer Software*, **31**(2005), no. 2, 56–59. DOI: 10.1007/s11086-005-0013-9

[21] A.V.Ustinov, A Discrete Analog of Euler's Summation Formula, *Math. Notes*, **71**(2002), no. 6, 851–856. DOI: 10.4213/mzm397

Сумма функций и полиномиальные решения многомерного разностного уравнения

Андрей А. Григорьев
Евгений К. Лейнартас
Сибирский федеральный университет
Красноярск, Российская Федерация
Александр П. Ляпин
Сибирский федеральный университет
Красноярск, Российская Федерация
Государственный университет Фэрмонт
Фэрмонт, Западная Вирджиния, США

Аннотация. Определен набор полиномиальных разностных операторов, позволяющий решить задачу суммирования, и описано пространство полиномиальных решений этих операторов в уравнениях с полиномиальной правой частью. Получен критерий, описывающий эти полиномиальные разностные операторы. Доказана теорема, описывающая пространство полиномиальных решений для операторов.

Ключевые слова: числа Бернулли, многочлены Бернулли, задача суммирования, многомерное разностное уравнение, формула Эйлера–Маклорена, оператор Тодда.