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On Multiple Zeros of Entire Functions of Finite Order of Growth

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Abstract. The article is devoted to determination of the number of multiple zeros of entire functions of finite order of growth.

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Our work is devoted to the problem of multiple zeros of entire functions. For polynomials, this question is a classical problem, and its solution is included in algebra textbooks (see, for example, [1]).

Recall the statement. Consider a polynomial $P(z)$ of degree n. Denote by S_j the power sums of the roots of a polynomial of degree j.

Theorem 1. *In order for the polynomial $P(z)$ to have multiple roots, it is necessary and sufficient that*

$$D(P) = a_0^{2n-2} \begin{vmatrix} n & S_1 & S_2 & \dots & S_{n-1} \\ S_1 & S_2 & S_3 & \dots & S_n \\ \dots & \dots & \dots & \dots & \dots \\ S_{n-1} & S_n & S_{n+1} & \dots & S_{2n-2} \end{vmatrix} = 0.$$

Here a_0 is the leading coefficient of the polynomial $P(z)$.

The determinant of $D(P)$ is called the *discriminant* of the polynomial $P(z)$.

For entire functions, the question of multiple zeros needs to be clarified. An entire function may have no zeros at all, like, for example, the function e^z, or an infinite number of zeros like $\sin z$. Therefore, we have to consider various options here.

1. Let an entire function have the form

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k, \quad f(0) = a_0 = 1. \tag{1}$$

The following statement is true ([2], corollary 1.4.1).

Theorem 2. *In order for the function $f(z)$ to be an entire function of finite order k_0 that has no zeros, it is necessary and sufficient that the determinant*

$$D(P) = a_0^{2n-2} \begin{vmatrix} a_1 & a_0 & 0 & \dots & 0 \\ 2a_2 & a_1 & a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ka_k & a_{k-1} & a_{k-2} & \dots & a_1 \end{vmatrix} = 0 \quad \text{for all} \quad k > k_0, \tag{2}$$

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where k_0 is the minimal number with this property.

2. Consider an entire function of finite order of growth of the form (1). Find the order ρ of the function f. To do this, we apply the formula ([3], ch. 7)

$$\varliminf_{n\to\infty} \frac{\ln(1/|a_n|)}{n\ln n} = \frac{1}{\rho}.$$

If ρ is a fractional number, then the function $f(z)$ is known to have an infinite number of zeros (see [3]). First we will assume that ρ is an integer.

Let us take a sequence of complex numbers $s_0,\ s_1,\ s_2,\ \dots$. It defines an infinite Hankel matrix

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & \dots \\ s_1 & s_2 & s_3 & \dots \\ s_2 & s_3 & s_4 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3}$$

The consecutive main minors of the matrix S are denoted by $D_0,\ D_1,\ D_2,\ \dots$. In addition, we set $D_{-1} = 1$.

If for every $p \in \mathbb{N}$ there exists a minor of the matrix S of order p that is not equal to zero, then the matrix has infinite rank. If, starting from some p, all minors of larger orders are zero, then the matrix S has finite rank. The smallest such p is called the *rank* of the matrix.

We recall a statement regarding matrices S of finite rank p ([4], ch. 16, Sec. 10).

Theorem 3. *If an infinite Hankel matrix has finite rank p, then the minor $D_{p-1} \neq 0$.*

Thanks to the properties of entire functions, the power sums of σ_k

$$\sigma_k = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N}$$

are absolutely convergent series for $k > \rho$. Here, the zeros of the entire function $f(z)$ are denoted by α_n. We will arrange them in ascending order of modules $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \dots \leqslant |\alpha_n| \leqslant \dots$.

The smallest such k is denoted by k_0 and we denote $s_j = \sigma_{2k_0+j}$, $j = 0, 1, \dots$. Consider an infinite Hankel matrix S of the form (3). In the monograph ([2], Theorem 1.4.5) the following statement is proved

Theorem 4. *In order for the function f to have a finite number of zeros, it is necessary and sufficient that the rank of the matrix S is finite, while the number of different zeros of f is equal to the rank of S.*

In this case, we can write the function f as follows:

$$f(z) = e^{-Q(z)}\, P(z), \tag{4}$$

where $Q(z)$ is a polynomial of degree $p = \rho$, and $P(z)$ is a polynomial of some degree m

$$P(z) = \sum_{k=0}^{m} b_k\, z^k = 1 + b_1\, z + \dots + b_m\, z^m.$$

The number m is the number of roots of the function $f(z)$ together with their multiplicities. To find the polynomial $P(z)$, one needs to factorize the function $f(z)$ (see Sec. 1.6.5 from [2]).

Take the logarithm of both parts in the formula (4). Let

$$\ln f(z) = \sum_{k=1}^{\infty} \tilde{a}_k\, z^k = \tilde{a}_1\, z + \ldots + \tilde{a}_n\, z^n + \ldots,$$

$$\ln P(z) = \sum_{k=1}^{\infty} \tilde{b}_k\, z^k = \tilde{b}_1\, z + \ldots + \tilde{b}_n\, z^n + \ldots.$$

The coefficients of $\tilde{a}_n$ can be found by the following formula (see [2], Lemma 1.2.1)

$$\tilde{a}_n = \frac{(-1)^{n-1}}{n} \begin{vmatrix} a_1 & 1 & 0 & \ldots & 0 \\ 2a_2 & a_1 & 1 & \ldots & 0 \\ 3a_3 & a_2 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ na_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{vmatrix} = 0 \quad \text{for all} \quad n \geqslant 1,$$

The coefficients b_k are found from the theorem ([2], Theorem 1.6.4, [6]).

Theorem 5. *The formulas are valid*

$$b_k = \frac{-\begin{vmatrix} (m+p)\,\tilde{a}_{m+p} & \ldots & (m+p+1)\,\tilde{a}_{m+p+1} & \ldots & (p+1)\,\tilde{a}_{p+1} \\ (m+p+1)\,\tilde{a}_{m+p+1} & \ldots & (m+p+2)\,\tilde{a}_{m+p+2} & \ldots & (p+2)\,\tilde{a}_{p+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (2m+p-1)\,\tilde{a}_{2m+p-1} & \ldots & (2m+p)\,\tilde{a}_{2m+p} & \ldots & (p+m)\,\tilde{a}_{p+m} \end{vmatrix}}{\begin{vmatrix} (m+p)\,\tilde{a}_{m+p} & \ldots & (p+1)\,\tilde{a}_{p+1} \\ (m+p+1)\,\tilde{a}_{m+p+1} & \ldots & (p+2)\,\tilde{a}_{p+2} \\ \ldots & \ldots & \ldots \\ (2m+p-1)\,\tilde{a}_{2m+p-1} & \ldots & (p+m)\,\tilde{a}_{p+m} \end{vmatrix}},$$

$k = 1, \ldots, m$. *In the numerator k, the th column is replaced by the column*

$$\begin{pmatrix} -(m+p+1)\,\tilde{a}_{m+p+1} \\ -(m+p+2)\,\tilde{a}_{m+p+2} \\ \ldots \\ -(2m+p)\,\tilde{a}_{2m+p} \end{pmatrix}$$

Here m this is the smallest k for which b_k is different from zero.

Corollary 1. *A function $f(z)$ has multiple roots if and only if the polynomial $P(z)$ has multiple roots.*

Let us give an example.

Consider the function

$$f(z) = 1 + 2z + \sum_{k=2}^{\infty} \left(\frac{2^k}{k!} - \frac{2^{k-2}}{(k-2)!} \right) z^k = 1 + 2z + z^2 - \frac{2z^3}{3} - \frac{4z^4}{3} - \frac{16z^5}{15} + \ldots.$$

It is not difficult to calculate that the order of growth of this function is $\rho = 1$.

By Lemma 1.2.1 of [2], the power sums of S_j with even numbers are 2, with odd numbers are 0. Therefore, the rank of the Hankel matrix is S

$$S = \begin{pmatrix} 2 & 0 & 2 & \ldots \\ 0 & 2 & 0 & \ldots \\ 2 & 0 & 2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

is equal to 2.

From Theorem 5 and Lemma 1.2.1 from [2] we get

$$\tilde a_2 = \frac{-1}{2}\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -1,$$

$$\tilde a_3 = \frac{1}{3}\begin{vmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{vmatrix} = 0,$$

$$\tilde a_4 = \frac{-1}{4}\begin{vmatrix} 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ -\frac{16}{3} & -\frac{2}{3} & 1 & 2 \end{vmatrix} = -\frac{1}{2},$$

$$\tilde a_5 = \frac{1}{5}\begin{vmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ -2 & 1 & 2 & 1 & 0 \\ -\frac{16}{3} & -\frac{2}{3} & 1 & 2 & 1 \\ -\frac{16}{3} & -\frac{4}{3} & -\frac{2}{3} & 1 & 2 \end{vmatrix} = 0.$$

From here we find

$$b_1 = -\frac{\begin{vmatrix} 4\tilde a_4 & 2\tilde a_2 \\ 5\tilde a_5 & 3\tilde a_3 \end{vmatrix}}{\begin{vmatrix} 3\tilde a_3 & 2\tilde a_2 \\ 4\tilde a_4 & 3\tilde a_3 \end{vmatrix}} = -\frac{\begin{vmatrix} -2 & -2 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}} = 0,$$

$$b_2 = -\frac{\begin{vmatrix} 3\tilde a_3 & 4\tilde a_4 \\ 4\tilde a_4 & 5\tilde a_5 \end{vmatrix}}{\begin{vmatrix} 3\tilde a_3 & 2\tilde a_2 \\ 4\tilde a_4 & 3\tilde a_3 \end{vmatrix}} = -\frac{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}} = -1.$$

The remaining b_k is zero. Therefore, the polynomial $P(z)$ is equal to

$$P(z) = 1 - z^2.$$

It has two roots ± 1 and has no multiple roots. Therefore, the function $f(z)$ has no multiple roots.

3. Let an function $f(z)$ of the form (1) have an infinite number of zeros, then the rank of the matrix S (3) is infinite. Multiple zeros can only have finite multiplicities. Therefore, if $f(z)$ has an infinite number of zeros, then it has an infinite number of distinct zeros.

Multiple zeros are the common zeros of the function and its derivative, i.e., the zeros of the resultant. So the question is whether the function and its derivative have common zeros.

The approach to determining the resultant of two integer functions is considered in a number of papers [5–7], but for arbitrary entire functions of finite growth order it is not yet known how to find the common zeros of the function and its derivative.

Let an entire function $f(z)$ have the order ρ. Due to the properties of entire functions, power sums σ_k

$$\sigma_k = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N},$$

are absolutely convergent series for $k > \rho$. Here, as before, α_n are zeros of the entire function $f(z)$. We will arrange them in ascending order of modules $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots \leqslant |\alpha_n| \leqslant \ldots$. The smallest such k is denoted by k_0. We assume that k_0 is an integer.

We will introduce, as in the previous section, power sums $s_j = \sigma_{2k_0+j}$, $j = 0, 1, \dots$ and an infinite Hankel matrix S of the form (3). Its rank is infinite.

Consider its submatrices of the order m:

$$S^m = \begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_m \\ s_1 & s_2 & s_3 & \dots & s_{m+1} \\ s_2 & s_3 & s_4 & \dots & s_{m+2} \\ \dots & \dots & \dots & \dots & \dots \\ s_m & s_{m+1} & s_{m+2} & \dots & s_{2m+1} \end{pmatrix}. \tag{5}$$

We introduce finite power sums

$$\sigma_k^m = \sum_{n=1}^{m} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N},$$

$s_j^m = \sigma_{2k_0+j}^m$ and matrices

$$S_m^m = \begin{pmatrix} s_0^m & s_1^m & s_2^m & \dots & s_m^m \\ s_1^m & s_2^m & s_3^m & \dots & s_{m+1}^m \\ s_2^m & s_3^m & s_4^m & \dots & s_{m+2}^m \\ \dots & \dots & \dots & \dots & \dots \\ s_m^m & s_{m+1}^m & s_{m+2}^m & \dots & s_{2m+1}^m \end{pmatrix}. \tag{6}$$

Consider an infinite matrix

$$A = \begin{pmatrix} \frac{1}{\alpha_1^{k_0}} & \frac{1}{\alpha_2^{k_0}} & \frac{1}{\alpha_3^{k_0}} & \dots \\ \frac{1}{\alpha_1^{k_0+1}} & \frac{1}{\alpha_2^{k_0+1}} & \frac{1}{\alpha_3^{k_0+1}} & \dots \\ \frac{1}{\alpha_1^{k_0+2}} & \frac{1}{\alpha_2^{k_0+2}} & \frac{1}{\alpha_3^{k_0+3}} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{7}$$

Then we have

$$S = A \cdot A',$$

where A' is the transpose of the matrix A. If the function f has multiple zeros, then the matrix A has the same columns.

Denote by A_m the main submatrix of the matrix A of order m. Then $S_m^m = A_m \cdot A_m'$. If the function $f(z)$ has multiple zeros, then $\det A_m = 0$, starting from some m. Since the matrix A_m is a Vandermonde matrix up to a nonzero multiplier, the opposite is also true: if its determinant is 0, at least two of its columns coincide.

Thus, the next statement is true.

Lemma 1. *In order for the function $f(z)$ to have multiple zeros, it is necessary and sufficient that* $\det A_m = 0$ *starting from some m.*

Since $S_m^m = A_m \cdot A_m'$, the following statement is true

Proposition 1. *In order for the function $f(z)$ to have multiple zeros, it is necessary and sufficient that* $\det S_m^m = 0$ *starting from some m.*

In order to find $\det S_m^m$, we first need to factorize the function f (see point 2). Suppose that after factorization, the function $f(z)$ takes the form

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\alpha_j}\right).$$

Thus, the function $f(z)$ is a function of genus zero (or an entire function of the first order of growth of minimal type ([3], Chapter 7). In this case, the series

$$\sum_{j=0}^{\infty} \frac{1}{\alpha_j}$$

absolutely converges. Then the coefficients a_k of the function $f(z)$ take the form

$$a_k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\alpha_{j_1} \cdots \alpha_{j_k}}.$$

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Кратные нули целых функций конечного порядка роста

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Аннотация. Статья посвящена определению числа кратных нулей целой функции конечного порядка роста.

Ключевые слова: целая функция, кратный нуль.