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On Restoring a Local Lie Group by Structural Constants

Vitaly A. Stepanenko*
Siberian Federal University
Krasnoyarsk, Russian Federation

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Abstract. A coordinate system of the 2nd kind is being built, corresponding to canonical coordinates of the 1st kind (terminology A. I. Maltsev), thereby obtaining a parametric solution of the system Lie's equations. The integral representation of the group operations $f(x, y)$ of the local Lie group G in canonical coordinates 1st kind. As the main apparatus is used modified formula of A. P. Yuzhakov for implicit mappings. The operation $f(x, y)$ is also represented as a power series, which is the reduced form of the series Campbell–Hausdorff.

Keywords: local Lie group, series Campbell–Hausdorff, formula of A. P. Yuzhakov.

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There are currently three known recovery methods of a finite-dimensional Lie group with a given Lie algebra: analytic (via solution systems of differential Lie equations [1]), topological (with using cohomologies of groups and Lie algebras [2]), algebraic (using Campbell-Hausdorff series (CH-series) in the form of Dynkin [3]). The CH-series is the most famous, but inconvenient for practical application and is rarely used in applications, because homogeneous terms CH-series contain a lot of uncredited similar elements. These homogeneous terms are found by I.Shur's algorithm [4] (and CH-series is called then SCH-series), or by integrating on the unit segment by Campbell–Baker–Hausdorff formula [5], or Dynkin formula [3].

In this article we introduce a group operation $f(x, y)$ of the local Lie groups G in canonical coordinates of the 1st kind as the power series, which is the reduce form of the CH series.

This formula is obtained from the integral representation of the group operation arising from Yuzhakov's formula [6, 7], which we have a little bit of supplemented (the investigation was also subjected to an obvious generalization).

For the application of the modified Yuzhakov formula and the consequence of it needs a complexification, which in the local case is always possible [1].

1. Let G be Lie Local Group, $\Im$ be its algebra Lie dimensions n, $X_1, \ldots, X_n$ be basis $\Im$, $C^i_{jk}(i, j, k = 1, \ldots, n)$ be specified structural constants.

Definition 1. *Canonical system coordinates of the 1st kind* $(x^1, \ldots, x^n)$ *in some neighborhood of an unit element* e *of the group* G *this is an exponential map* $exp : \Im \longrightarrow G$, *in this case a system of equations* $x^i = a^i t \quad (i = 1, \ldots, n)$ *specifies 1-parametric subgroup for any vector* $a \in \Im$.

Setting coordinates of the first kind does not depend on the group operation $f(x, y) = x{\cdot}y$, but in terms of function $f(x, y)$ Definition 1 will obviously be rewritten so: for any $\lambda, \mu \in \mathbf{R}, x \in G$

$$f(\lambda x, \mu x) = (\lambda + \mu)x. \tag{1}$$

*v-stepanen@mail.ru

Main matrix $v^i_j(x)$ (Jacobi matrix of multiplicative increment group operation $(x+\triangle x)\cdot x^{-1}$) is obtained [1] from system

$$\dot{w}^i_j = \delta^i_j + C^i_{\alpha\beta} a^\alpha w^\beta_j, \; w^i_j(0,a) = 0, \tag{2}$$

where a is an arbitrary vector:

$$v^i_j(x) = w^i_j(1,x). \tag{3}$$

Matrix v^i_j is reversible, because $v^i_j(0) = \delta^i_j$, let

$$\Lambda^i_j(x) = (v^i_j(x))^{-1}. \tag{4}$$

Canonicity criterion coordinates of the first kind can be written using a matrix v :

$$v(x)x = x \quad (v^i_j(x)x^j = x^i), \tag{5}$$

or, obviously, using a matrix Λ :

$$\Lambda(x)x = x \quad (\Lambda^i_j(x)x^j = x^i). \tag{6}$$

Definition 2. *Canonical coordinates of the second kind is called a local card in a unit* e, *specified smooth map* $\boldsymbol{R}^n$ *в* G*:*

$$(t^1,\ldots,t^n) \longrightarrow e^{t^1X_1}\cdot\ldots\cdot e^{t^nX_n} = g(t^1,\ldots,t^n). \tag{7}$$

Each element g from some the neighborhood of the unit e is the uniquely represented in the form (7), jacobian of the map (7) at zero is not zero and converts the standard basis $\mathbf{R}^n$ to the basis $\Im : e_i \longrightarrow X_i, \; (i=1,\ldots,n)$. Set the coordinates of the second kind, unlike the coordinates of the first kind, uses a group operation G. (Coordinates of the first kind are analogue of polar coordinates, coordinates second kind are cartesian.) Using the basis $\Im$ (vector fields)$X_1,\ldots,X_n$, easily write out a Lie group operation G in coordinates of the second kind with vector $(t^1,\ldots,t^n)$: along $X_1,\ldots,X_n$ find 1-parametric $f_1(x;t^1),\ldots,f_n(x;t^n)$ with initial conditions $f_i(x;0) = x \;\; (i=1,\ldots,n)$, and then superpose $f(x;t^1,\ldots,t^n) = f_1(x;t^1)\circ\ldots\circ f_n(x;t^n)$. For example (see [9]), $f_1(x;t^1) = x+t^1, f_2(x;t^2) = xe^{t^2}, f_3(x;t^3) = \dfrac{x}{1-xt^3}$ — 1-parametric along the fields $\partial_x, x\partial_x, x^2\partial_x$ on the straight line $\mathbf{R}^1_x$. Then the general map of the local group G in coordinates of the 2nd kind with vector (t^1,t^2,t^3) this:

$$f(x;t^1,t^2,t^3) = \frac{x}{1-xt^3}e^{t^2} + t^1,$$

herewith $f(x;t^1,0,0) = f(x;t^1)$, $f(x;0,t^2,0) = f_2(x;t^2)$, $f(x;0,0,t^3) = f_3(x;t^3)$. A. I. Maltsev [8] introduces a concept linking these two species coordinate.

Definition 3. *Let vectors* $X_1,\ldots,X_n$ *set in* $\Im$ *system coordinates of the 1st kind, passing through them 1-parametric subgroups* $x_i(t) = tX_i \quad (i=1,\ldots,n)$, *we get a coordinate system of the second kind. This coordinate system will be call the corresponding first.*

Thus Maltsev introduces coordinate system of the 2nd kind within the already existing coordinates first kind.

2. The lines of the main matrix v from (3) are the coefficients of Maurer–Cartan's 1-forms (Pfaff's 1-forms)

$$\omega^i = v^i_j(x)dx^j \;\; (i=1,\ldots,n) \tag{8}$$

with properties

$$d\omega^i = C^i_{jk}\omega^j \wedge \omega^k \quad (j,k,=1,\dots,n), \tag{9}$$

where $\wedge$ is external multiplication. Columns of the inverse matrix Λ from (4) are coefficients of dual vector fields (first-order differential operators)

$$\overline{X}_i = \Lambda^j_i(x)\frac{\partial}{\partial x^j} \quad (i=1,\dots,n) \tag{10}$$

with properties

$$d\omega^j(\overline{X}_i) = \delta^j_i \quad (j,i=1,\dots,n), \tag{11}$$

$$[\overline{X}_i,\overline{X}_j] = -C^k_{ij}\overline{X}_k \quad (i,j,k=1,\dots,n). \tag{12}$$

Rewriting the Li system of equations [1]

$$v^i_k(f)\frac{\partial f^k}{\partial x^j} = v^i_j(x) \quad (i,j,k=1,\dots,n) \tag{13}$$

in the form of

$$\frac{\partial f^k}{\partial x^j}\Lambda^j_i(x) = \Lambda^k_i(f) \quad (i,j,k=1,\dots,n), \tag{14}$$

and considering $x^i(t^1,\dots,t^n)$ $(i=1,\dots,n)$ as solutions of the system

$$(x^i)^{\cdot}_{t^m} = \Lambda^i_m(x) \quad (i,m=1,\dots,n), \tag{15}$$

we get that the components of a group operation $f^i(x^1(t^1,\dots,t^n),\dots,x^n(t^1,\dots,t^n))$ satisfy the same system of equations

$$(f^i)^{\cdot}_{t^m} = \Lambda^i_m(f) \quad (i,m=1,\dots,n), \tag{16}$$

and system solutions (15) and (16) differ only in constants c_k and φ_k:

$$x^i = \chi^i(t^1,\dots,t^n;c_1,\dots,c_n),\ \ f^i = \chi^i(t^1,\dots,t^n;\varphi_1,\dots,\varphi_n) \quad (i=1,\dots,n). \tag{17}$$

The coincidence of the systems equations for x^i and f^i is explained by the right invariance of Pfaff's 1-forms $v^i_j(f)df^j = v^i_j(x)dx^j$, which is obtained from the system of Lie equations (13) after multiplication (13) on dx^j and summation by j ($f(x,y) = x\cdot y$ — right shift of an element x by element y). Let's select the values of constants $c_k = \overline{c}_k, \varphi_k = \overline{\varphi}_k(y)$ $(k=1,\dots,n)$ so, that

$$\chi^i(0,\dots,0;\overline{c}_1,\dots,\overline{c}_n) = 0, \quad \chi^i(0,\dots,0;\overline{\varphi}_1(y),\dots,\overline{\varphi}_n(y)) = y^i \quad (i=1,\dots,n), \tag{18}$$

then $x^i = \chi^i(t^1,\dots t^n;\overline{c}_1,\dots,\overline{c}_n)$ will the desired coordinates of the 2nd kind, corresponding to the available coordinates of the 1st kind, and $f^i = \chi^i(t^1,\dots t^n;\overline{\varphi}_1(y),\dots,\overline{\varphi}_n(y))$ — parametric representation of the group operation component G. Left from the system

$$\left\{\begin{array}{c} x^i = \chi^i(t^1,\dots t^n;\overline{c}_1,\dots,\overline{c}_n \\ f^i = \chi^i(t^1,\dots t^n;\overline{\varphi}_1(y),\dots,\overline{\varphi}_n(y)) \end{array}\right. , \tag{19}$$

$(i=1,\dots,n)$ exclude parameters$(t^1,\dots,t^n)$ and express $f^i(x^1,\dots,x^n;y^1,\dots,y^n)$ obviously. In this we will be helped by Yuzhakov's formula, which gives an integral group operation view $f(x,y)$, and a consequence of the Yuzhakov's theorem will allow you to get the Campbell–Hausdorff series (CH-series) in the reduced form.

3. In 1975, A. P. Yuzhakov [6] (see also [7]) with the help of multidimensional logarithmic residue obtained a generalization of the classical Lagrangian decomposition into arbitrary implicit functions of many complex variables defined by the system of equations. Indexes in this paragraph write lower, as in [6] and as is customary in the theory of functions of many complex variables.

Let $\Phi(w,z)$, $F_j(w,z)$, $j=1,\dots,n$ are holomorphic variable functions $w=(w_1,\dots,w_m)$, $z=(z_1,\dots,z_n)$ in the neighborhood of the point $(0,0)\in C^{m+n}_{w,z}$, $F_j(0,0)=0$, Jacobian

$$\frac{\partial F}{\partial z}\Big|_{(0,0)}=\delta_{jk}\quad (j,k=1,\dots,n). \tag{20}$$

According to the theorem of the existence of implicit functions, the system of equations

$$F_j(w,z)=0\quad (j=1,\dots,n) \tag{21}$$

uniquely defines a system of functions

$$z_j=\psi_j(w)\quad (j=1,\dots,n), \tag{22}$$

holomorphic in the neiborhood of the point $0\in C^m_w$. Then there will be numbers $\epsilon>0,\delta>0$ such that in a closed polycircle $\overline{V}_\delta=\{w\in C^m : |w_j|\leqslant\delta,\, j=1,\dots,m\}$ there is an integral representation

$$\Phi(w,\psi(w))=\frac{1}{(2\pi i)^n}\int_{\Gamma_\epsilon}\frac{\Phi(w,z)}{F^I(w,z)}\cdot\frac{\partial F}{\partial z}dz, \tag{23}$$

where is $F^I=F_1\cdot\ldots\cdot F_n$, $\Gamma_\epsilon=\{z:|z_1|=\ldots=|z_n|=\epsilon\}$.

Asking the purpose to present $\Phi(w,\psi(w))$ as a Taylor series

$$\Phi(w,\psi(w))=\sum_{|\alpha|\geqslant 0}c_\alpha w^\alpha,\quad \alpha=(\alpha_1,\dots,\alpha_m), \tag{24}$$

substitute in the formula for the coefficients of the series (24)

$$c_\alpha=\frac{1}{(2\pi i)^m}\int_{\gamma_\delta}\Phi(w,\psi(w))\frac{dw}{w^{\alpha+I}} \tag{25}$$

expression

$$\Phi(w,\psi(w))=\sum_{|\beta|\geqslant 0}\frac{(-1)^{|\beta|}}{(2\pi i)^n}\int_{\Gamma_\epsilon}\Phi(w,z)g^\beta(w,z)\frac{\partial F(w,z)}{\partial z}\frac{dz}{z^{\beta+I}}, \tag{26}$$

obtained from (23) after the decomposition of the fraction $\dfrac{1}{F^I(w,z)}$ in a multiple geometric progression

$$\frac{1}{F^I(w,z)}=\sum_{|\beta|\geqslant 0}\frac{(-1)^{|\beta|}g^\beta(w,z)}{z^{\beta+I}},$$

where is $g_j(w,z)=F_j(w,z)-z_j$ and at $w\in\overline{V}_\delta,\ z\in\Gamma_\epsilon$ inequalities are met $|g_j(w,z)|<\epsilon$.

Finally from (25) and (26) we get

$$c_\alpha=\sum_{|\beta|\geqslant 0}\frac{(-1)^{|\beta|}}{(2\pi i)^{m+n}}\int_{\gamma_\delta\times\Gamma_\epsilon}\Phi(w,z)g^\beta(w,z)\frac{\partial F(w,z)}{\partial z}\cdot\frac{dw\wedge dz}{w^{\alpha+I}z^{\beta+I}}=$$

$$=\sum_{|\beta|=0}^{2|\alpha|}\frac{(-1)^{|\beta|}}{\alpha!\beta!}D^{(\alpha,\beta)}_{w,z}\left(\Phi(w,z)g^\beta(w,z)\cdot\frac{\partial F(w,z)}{\partial z}\right)\Big|_{w=z=0}. \tag{27}$$

It is easy to see (this is our addition) that if Φ holomorphic depends on more variables $y = (y_1, \dots, y_n)$ from some surroundings point $0 \in C^n_y$, then, similarly (23), fair integral representation then, similarly (23), true integral representation

$$\Phi(w, \psi(w), y) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\epsilon} \frac{\Phi(w, z, y)}{F^I(w, z)} \cdot \frac{\partial F(w, z)}{\partial z} dz, \tag{28}$$

and (generalization consequences of Yuzhakov's theorem) coefficients of the power series

$$\Phi(w, \psi(w), y) = \sum_{|\alpha| \geqslant 0, |\nu| \geqslant 0} c_{\alpha,\nu} w^\alpha y^\nu, \tag{29}$$

are calculated by the formula

$$c_{\alpha,\nu} = \sum_{|\beta|=0}^{2|\alpha|} \frac{(-1)^{|\beta|}}{\alpha!\beta!\nu!} D^{(\alpha,\beta)}_{w,z} \left(\Phi^{(\nu)}_y (w, z, y) \frac{\partial F(w, z)}{\partial z} (F(w, z) - z)^\beta \right) \Big|_{w=z=y=0}. \tag{30}$$

4. Theorem. *Integrated local group operation components Lie G in canonical coordinates of the 1st kind have an integral representation*

$$f^k(x, y) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\epsilon} \frac{\chi^k(t, \overline{\varphi}(y))}{(\chi^1(t, \overline{c}) - x^1) \dots (\chi^n(t, \overline{c}) - x^n)} \cdot \frac{\partial(\chi(t, \overline{c}))}{\partial t} dt, \tag{31}$$

where $\dfrac{\partial(\chi(t, \overline{c}))}{\partial t} = \dfrac{\partial(\chi(t, \overline{c}) - x)}{\partial t}$ *is Jacobian,* $\Gamma_\epsilon = \{t : |t^1| = \ldots = |t^n| = \epsilon\}$ $(k = 1, \dots, n)$.

Proof. We combine constants, parameters, variables in formulas (19) and form functions $F_j(x, t) = \chi^j(t, \overline{c}) - x^j$ $(j = 1, \dots, n)$, which are analogues of holomorphic functions $F_j(w, z)$ from the Yuzhakov's theorem, with the role w play variables x, role z — options t. Holomorphism by x is obvious, and holomorphism on t follows from the local analytics of system solutions (15), (16).

Taking as $\Phi(w, z, y) = \chi^k(t, \overline{\varphi}(y))$ from (19) $(k = 1, \dots, n)$, we get the formula (31). This formula (28) used in the simplified case — function Φ does not depend on w (in our case $\chi^k(t, \overline{\varphi}(y))$ $(k = 1, \dots, n)$ does not depend on x). Jacobian $\dfrac{\partial(\chi(t, \overline{c}) - x)}{\partial t} = \dfrac{\partial(\chi(t, \overline{c}))}{\partial t}$ at $t = 0$ turns into a single matrix, because $\chi(t, \overline{c})$ is the solution of the system (15) with zero initial data, and $\Lambda^i_j(0) = \delta^i_j$.

Calculating multidimensional residue (31) is often convenient use the transformation formula from the monograph of A. K. Tsikh [10]: let holomorphic mapping $f = (f_1, \dots, f_n)$ has an isolated zero in point $a \in C^n$ and $g = Af$, where the elements of the matrix A are holomorphic in some surroundings point a. Then for anyone $h \in O_a$ $res_{f,a}(h) = res_{g,a}(h \cdot detA)$, In here O_a -ring of germs of holomorphic functions at a point $a \in C^n$.

From (29), (30) follows what interests us

Investigation. Components $f^k(x, y)$ group operation local Lie groups G allow decomposition into power series

$$f^k(x, y) = \sum_{|\alpha| \geqslant 0, |\nu| \geqslant 0} c^{(k)}_{\alpha,\nu} x^\alpha y^\nu \tag{32}$$

with coefficients

$$c^{(k)}_{\alpha,\nu} = \sum_{|\beta|=0}^{2|\alpha|} \frac{(-1)^{|\beta|}}{\alpha!\beta!\nu!} D^{(\alpha,\beta)}_{x,t} \left((\chi^k(t, y))^{(\nu)}_y \frac{\partial(\chi(t))}{\partial t} (\chi(t) - x - t)^\beta \right) \Big|_{x=t=y=0}, \tag{33}$$

which, because of uniqueness in canonical coordinates of the 1st kind, coincides with CH-series in the reduced form.

Recall (see [3]) that if the bracket $[a_1\, a_2\, \dots\, a_n]$ means multiple cjmmutator $[a_1[a_2 \dots [a_{n-1}, a_n] \dots]]$, that CH-series has the form:

$$f(x,y) = x + y + \frac{1}{2}[x,y] + \sum_{p_i,q_i} \frac{(-1)^{m-1}}{m} \cdot \frac{[x^{p_1} y^{q_1} \dots x^{p_m} y^{q_m}]}{\sum_i (p_i + q_i) \prod_i p_i! q_i!}, \tag{34}$$

where summation is done for all non-negative integer values p_i, q_i $(i = 1, 2, \dots, m;\ m = 2, 3, \dots)$, $p_i + q_i \neq 0$, except in the simplest cases when $m = 0, 1$, which are written separately in the form of the first three terms.

A. D. Mednykh aptly put it that CH-series expresses algebraic essence group operation, and power series (32), (33), which is a consequence of the Yuzhakov's formula — her analytical essence.

5. Let's illustrate the theory with two examples from [11].

Example 1. Consider a group G affine transformations of the straight line $\mathbf{R}^1$ matrices of the species

$$\begin{pmatrix} 1 + x^1 & x^2 \\ 0 & 1 \end{pmatrix}, \qquad x^1, x^2 \in \mathbf{R}^1, \tag{35}$$

with group operation components in non-canonical coordinates

$$\begin{cases} f^1(x,y) = x^1 + y^1 + x^1 y^1, \\ f^2(x,y) = x^2 + y^2 + x^1 y^2. \end{cases} \tag{36}$$

From the decompositions (36) it follows that structural constants of this group $C^1_{jk} = 0$ $(j, k = 1, 2)$,

$$C^2_{12} = 1, C^2_{21} = -1. \tag{37}$$

Let's restore the group G by its structural constants (37) in the canonical coordinates of the 1st kind. The system (2) from Pontryagin's theorem [1] is divided into two:

$$\begin{cases} \dot{w}^1_1 = 1, \\ a^2 w^1_1 + \dot{w}^2_1 - a^1 w^2_1 = 0, \end{cases} \quad \begin{cases} \dot{w}^1_2 = 0, \\ a^2 w^1_2 + \dot{w}^2_2 - a^1 w^2_2 = 1, \end{cases} \tag{38}$$

with initial conditions $w^i_j(0, a) = 0$. Using Laplace transforms we get a solution

$$\begin{pmatrix} w^1_1 & w^1_2 \\ w^2_1 & w^2_2 \end{pmatrix} = \begin{pmatrix} t & 0 \\ \dfrac{a^2}{a^1}\left(t - \dfrac{e^{a^1 t} - 1}{a^1}\right) & \dfrac{e^{a^1 t} - 1}{a^1} \end{pmatrix}. \tag{39}$$

Formula (3) $v(x) = w(1, x)$ we get the main auxiliary matrix $v^i_j(x)$

$$\begin{pmatrix} v^1_1 & v^1_2 \\ v^2_1 & v^2_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{x^2}{x^1}\left(1 - \dfrac{e^{x^1} - 1}{x^1}\right) & \dfrac{e^{x^1} - 1}{x^1} \end{pmatrix} \tag{40}$$

and inverse to it

$$\Lambda^i_j(x) = \begin{pmatrix} 1 & 0 \\ \dfrac{x^2}{x^1}\left(1 - \dfrac{x^1}{e^{x^1} - 1}\right) & \dfrac{x^1}{e^{x^1} - 1} \end{pmatrix}. \tag{41}$$

Obviously, the criterion of canonicity is met. Coordinates of the 1st kind: $v(x)x = x$ and $\Lambda(x)x = x$. Using the formula (8) we get 1-forms

$$\begin{cases} \omega^1 = dx^1, \\ \omega^2 = \dfrac{x^2}{x^1}\left(1 - \dfrac{e^{x^1}-1}{x^1}\right)dx^1 + \dfrac{e^{x^1}-1}{x^1}dx^2, \end{cases}$$

with properties (9) $d\omega^1 = 0$, $d\omega^2 = \dfrac{e^{x^1}-1}{x^1}dx^1 \wedge dx^2 = 1 \cdot \omega^1 \wedge \omega^2$. According to the formula (10) we get dual vector fields

$$\begin{cases} \overline{X}_1 = \partial_{x^1} + \dfrac{x^2}{x^1}\left(1 - \dfrac{x^1}{e^{x^1}-1}\right)\partial_{x^2}, \\ \overline{X}_2 = \dfrac{x^1}{e^{x^1}-1}\partial_{x^2}, \end{cases}$$

with property (12): $[\overline{X_1}, \overline{X_2}] = -\dfrac{x^1}{e^{x^1}-1}\partial_{x^2} = -1 \cdot \overline{X_2}$.

Also running (11): $\omega^1(\overline{X_1}) = 1$, $\omega^2(\overline{X_1}) = 0$, $\omega^1(\overline{X_2}) = 0$, $\omega^2(\overline{X_2}) = 1$.

Next, entering parameters t^1, t^2 and using matrix columns $\wedge^i_j(x)$ (41), we will make systems (15), (16):

$$\begin{cases} (x^1)^{\cdot}_{t^1} = 1, \\ (x^2)^{\cdot}_{t^1} = \dfrac{x^2}{x^1}\left(1 - \dfrac{x^1}{e^{x^1}-1}\right), \end{cases} \quad \begin{cases} (x^1)^{\cdot}_{t^2} = 0, \\ (x^2)^{\cdot}_{t^2} = \dfrac{x^1}{e^{x^1}-1}, \end{cases} \tag{42}$$

$$\begin{cases} (f^1)^{\cdot}_{t^1} = 1, \\ (f^2)^{\cdot}_{t^1} = \dfrac{f^2}{f^1}\left(1 - \frac{f^1}{e^{f^1}-1}\right), \end{cases} \quad \begin{cases} (f^1)^{\cdot}_{t^2} = 0, \\ (f^2)^{\cdot}_{t^2} = \dfrac{f^1}{e^{f^1}-1}. \end{cases} \tag{43}$$

Separately 1-parametric solutions of both systems (42) and (43) have the form

$$\begin{cases} \theta^1(t^1; c_1, c_2) = t^1 + c_1, \\ \theta^2(t^1; c_1, c_2) = c_2\dfrac{(t^1+c_1)e^{t^1+c_1}}{e^{t^1+c_1}-1}, \end{cases} \quad \begin{cases} \theta^1(t^2; c_1, c_2) = c_1, \\ \theta^2(t^2; c_1, c_2) = \dfrac{c_1}{e^{c_1}-1}(t^2 + c_2e^{c_1}), \end{cases},$$

a desired functions (17) χ^i :

$$\begin{cases} x^1 = \chi^1(t^1, t^2; c_1, c_2) = t^1 + c_1, \\ x^2 = \chi^2(t^1, t^2; c_1, c_2) = \dfrac{t^1+c_1}{e^{t^1+c_1}-1}(t^2 + c_2e^{t^1+c_1}), \end{cases}$$
$$\begin{cases} f^1 = \chi^1(t^1, t^2; \varphi_1, \varphi_2) = t^1 + \varphi_1, \\ f^2 = \chi^2(t^1, t^2; \varphi_1, \varphi_2) = \dfrac{t^1+\varphi_1}{e^{t^1+\varphi_1}-1}(t^2 + \varphi_2e^{t^1+\varphi_1}). \end{cases} \tag{44}$$

Note that $\chi^1(t^1, 0, ; c_1, c_2) = \theta^1(t^1; c_1, c_2)$, $\chi^2(0, t^2; c_1, c_2) = \theta^2(t^2; c_1, c_2)$. Further, counting c, φ, t, x, y, f complex and, believing $\overline{c}_1 = \overline{c}_2 = 0$, $\overline{\varphi}_1(y) = y^1$, $\overline{\varphi}_2(y) = \dfrac{y^2}{y^1}(e^{y^1}-1)e^{-y^1}$, we get (18), i.e. $\chi^i(0, 0; \overline{c_1}, \overline{c_2}) = 0$, $\chi^i(0, 0; \overline{\varphi}_1(y)), \overline{\varphi}_2(y)) = y^i$ $(i = 1, 2)$, besides

$$x_1 = \chi^1(t, \overline{c}) = t^1, \quad x_2 = \chi^2(t, \overline{c}) = \frac{t^1}{e^{t^1}-1}t^2 \tag{45}$$

are coordinates of the second kind, corresponding to the coordinates 1st kind. Constructing functions in Yuzhakov's notations $F_j(x,t) = \chi^j(t,\overline{c}) - x^j$ $(j = 1,2)$:

$$F_1 = t^1 - x^1, \quad F_2 = \frac{t^1}{e^{t^1}-1} \cdot t^2 - x_2;$$

$$\Phi^1 = t^1 + y^1, \quad \Phi^2 = \frac{t^1+y^1}{e^{t^1+y^1}-1}\left(t^2 + \frac{y^2}{y^1}(e^{y^1}-1)e^{t^1}\right),$$

jacobian

$$\frac{\partial(\chi(t))}{\partial t} = \frac{t^1}{e^{t^1}-1}. \tag{46}$$

According to formula (31), calculating components f^i of a group operation is reduced to finding double deductions

$$f^1(x,y) = \frac{1}{(2\pi i)^2}\int_{\Gamma_\epsilon} \frac{t^1+y^1}{(t^1-x^1)(\frac{t^1}{e^{t^1}-1}t^2 - x^2)} \cdot \frac{t^1}{e^{t^1}-1} dt^1 \wedge dt^2, \tag{47}$$

$$f^2(x,y) = \frac{1}{(2\pi i)^2}\int_{\Gamma_\epsilon} \frac{(t^1+y^1)(t^2 + \frac{y^2}{y^1}(e^{y^1}-1)e^{t^1})}{(e^{t^1+y^1}-1)(t^1-x^1)(\frac{t^1}{e^{t^1}-1}t^2 - x^2)} \cdot \frac{t^1}{e^{t^1}-1} dt^1 \wedge dt^2. \tag{48}$$

Formulas (47), (48) give integral representations of components group operation G (36) in canonical coordinates of the 1st kind. Calculating residues, we get

$$f^1(x,y) = x^1 + y^1, \tag{49}$$

$$f^2(x,y) = \frac{x^1+y^1}{e^{x^1+y^1}-1}\left(\frac{e^{x^1}-1}{x^1}x^2 + e^{x^1}\frac{e^{y^1}-1}{y^1}y^2\right), \tag{50}$$

(which in this model example can be obtained directly, Substituting t^1, t^2 from (45) in Φ_1, Φ_2 from (46)).

It's easy verify that for (49),(50) associativity is performed, unit is the origin of the coordinates $(0,0)$, back to x is $-x$, the structural constants coincide with (37), and for any $\lambda, \mu \in \mathbf{R}^1$ $f(\lambda x, \mu x) = (\lambda+\mu)x$, which indicates canonical coordinates of the 1st kind. Finally, we will show how formulas (32), (33) decomposition of components work group operations of the group G (35), (36) in the form of power series and compare with the Campbell-Hausdorff formula of [3]:

$$f(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}\big([x,[x,y]] + [[x,y],y]\big) - \frac{1}{24}[x,[y,[x,y]]]\ldots \tag{51}$$

Because $[x,y]^i = C^i_{jk}x^jy^k$ and $C^1_{jk} = 0$ (see.(37) (cm(37)), then $f^1(x,y) = x^1 + y^1$, and

$$\begin{aligned} f^2(x,y) = x^2 + y^2 + \frac{1}{2}C^2_{12}x^1y^2 + \frac{1}{2}C^2_{21}x^2y^1 + \frac{1}{12}\big(C^2_{12}x^1(C^2_{12}x^1y^2 + C^2_{21}x^2y^1) + \\ + C^2_{21}(C^2_{12}x^1y^2 + C^2_{21}x^2y^1)y^1\big) - \frac{1}{24}C^2_{12}x^1\big(C^2_{12}y^1(C^2_{12}x^1y^2 + C^2_{21}x^2y^1)\big) + \ldots \end{aligned} \tag{52}$$

Substitute in (52) $C^2_{12} = 1, C^2_{21} = -1$ from (37), we get

$$\begin{aligned} f^2(x,y) = x^2 + y^2 + \frac{1}{2}(x^1y^2 - x^2y^1) + \frac{1}{12}\big((x^1)^2y^2 - x^1x^2y^1 - x^1y^2y^1 + x^2(y^1)^2\big) - \\ - \frac{1}{24}\big((x^1)^2y^1y^2 - x^1x^2(y^1)^2\big) + \ldots, \end{aligned} \tag{53}$$

here we have brought the commuting factors. Consider, for example, monom coefficient $(x^1)^2y^2 = (x^1)^2(x^2)^0(y^1)^0(y^2)^1$ equal $\frac{1}{12}$, and calculate it by the formula (33) from the investigation modified Yuzhakov's formula:

$$c^{(2)}_{2,0;0,1} = \sum_{|\beta|=0}^{4} \frac{(-1)^{|\beta|}}{2!0!0!1!} D^{(2,0;\beta_1,\beta_2)}_{x^1,x^2;t^1,t^2} \left(\left(\frac{t^1+y^1}{e^{t^1+y^1}-1} \left(t^2 + \frac{y^2}{y^1}(e^{y^1}-1)e^{t^1} \right) \right)^{(0,1)}_{y^1,y^2} \times \right.$$

$$\left. \times \frac{t^1}{e^{t^1}-1}(-x^1)^{\beta_1} \left(\frac{t^1}{e^{t^1}-1}t^2 - x^2 - t^2 \right)^{\beta_2} \right) \Big|_{x=t=y=0} =$$

(it is immediately apparent that $\beta_1 = 2$, a x^2 should be taken by zero)

$$= \sum_{\beta_2=0}^{2} \frac{(-1)^{\beta_2}}{2 \cdot \beta_2!} D^{(2,\beta_2)}_{t^1,t^2} \left(\left(\frac{t^1}{e^{t^1}-1} \right)^2 e^{t^1} \left(\frac{t^1}{e^{t^1}-1} - 1 \right)^{\beta_2} (t^2)^{\beta_2} \right) \Big|_{t=0} =$$

$$= \frac{1}{2} \left(\left(\frac{t^1}{e^{t^1}-1} \right)^2 e^{t^1} \left(1 - \left(\frac{t^1}{e^{t^1}-1} - 1 \right) + \left(\frac{t^1}{e^{t^1}-1} - 1 \right)^2 \right) \right)^{\cdot\cdot}_{t^1} \Big|_{t^1=0} =$$

(producing function of Bernoulli numbers $\frac{t^1}{e^{t^1}-1} = 1 - \frac{1}{2}t^1 + \frac{1}{12}(t^1)^2 + \dots$, $\frac{t^1}{e^{t^1}-1} - 1 = = -\frac{1}{2}t^1 + \frac{1}{12}(t^1)^2 + \dots$)

$$= \frac{1}{2}(1 + \frac{1}{2}t^1 + \frac{1}{12}(t^1)^2 + \dots)^{\cdot\cdot}_{t^1} \Big|_{t^1=0} = \frac{1}{12},$$

which coincides with the corresponding factor CH-series (51).

Finally, without giving detailed calculations (unfortunately, cumbersome), get a group $ISO(2)$ movements of the Euclidean plane [11] in canonical coordinates of the 1st kind.

Example 1. Let's imagine the product of matrices from $ISO(2)$

$$\begin{pmatrix} \cos x^3 & -\sin x^3 & x^1 \\ \sin x^3 & \cos x^3 & x^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos y^3 & -\sin y^3 & y^1 \\ \sin y^3 & \cos y^3 & y^2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the form of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} \cos x^3 & -\sin x^3 & 0 \\ \sin x^3 & \cos x^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}, \tag{54}$$

group operation components

$$\begin{cases} f^1 = x^1 + \cos x^3 \cdot y^1 - \sin x^3 \cdot y^2, \\ f^2 = x^2 + \sin x^3 \cdot y^1 + \cos x^3 \cdot y^2, \\ f^3 = x^3 + y^3, \end{cases}$$

allow you to calculate the structural constants of this group :

$$C^1_{23} = 1, \ \ C^1_{32} = -1, \ \ C^2_{31} = 1, \ \ C^2_{13} = -1 \tag{55}$$

(the other constants are zero). Systems of equations (2):

$$\begin{cases} \dot{w}^1_1 + a^3 w^2_1 - a^2 w^3_1 = 1, \\ -a^3 w^1_1 + \dot{w}^2_1 + a^1 w^3_1 = 0, \\ \dot{w}^3_1 = 0, \end{cases} \quad \begin{cases} \dot{w}^1_2 + a^3 w^2_2 - a^2 w^3_2 = 0, \\ -a^3 w^1_2 + \dot{w}^2_2 + a^1 w^3_2 = 1, \\ \dot{w}^3_2 = 0, \end{cases}$$

$$\begin{cases} \dot{w}_3^1 + a^3 w_3^2 - a^2 w_3^3 = 0, \\ -a^3 w_3^1 + \dot{w}_2^2 + a^1 w_3^3 = 0, \\ \dot{w}_3^3 = 1, \end{cases} \tag{56}$$

with initial conditions $w_j^i(0,a) = 0$, have decision

$$\begin{pmatrix} w_1^1 & w_2^1 & w_3^1 \\ w_1^2 & w_2^2 & w_3^3 \\ w_1^3 & w_2^3 & w_3^3 \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{\sin a^3 t}{a^3} & -\dfrac{1-\cos a^3 t}{a^3} & \dfrac{a^2}{a^3}\cdot\dfrac{1-\cos a^3 t}{a^3} + \dfrac{a^1}{a^3}\left(t - \dfrac{\sin a^3 t}{a^3}\right) \\ \dfrac{1-\cos a^3 t}{a^3} & \dfrac{\sin a^3 t}{a^3} & -\dfrac{a^1}{a^3}\cdot\dfrac{1-\cos a^3 t}{a^3} + \dfrac{a^2}{a^3}\left(t - \dfrac{\sin a^3 t}{a^3}\right) \\ 0 & 0 & t \end{pmatrix}. \tag{57}$$

According to the formula (3) we get the basic auxiliary matrix

$$v_j^i(x) = \begin{pmatrix} \dfrac{\sin x^3}{x^3} & -\dfrac{1-\cos x^3}{x^3} & \dfrac{x^2}{x^3}\cdot\dfrac{1-\cos x^3}{x^3} + \dfrac{x^1}{x^3}\left(1 - \dfrac{\sin x^3}{x^3}\right) \\ \dfrac{1-\cos x^3}{x^3} & \dfrac{\sin x^3}{x^3} & -\dfrac{x^1}{x^3}\cdot\dfrac{1-\cos x^3}{x^3} + \dfrac{x^2}{x^3}\left(1 - \dfrac{\sin x^3}{x^3}\right) \\ 0 & 0 & 1 \end{pmatrix} \tag{58}$$

and inverse to it

$$\Lambda_j^i(x) = \begin{pmatrix} \dfrac{x^3 \sin x^3}{2(1-\cos x^3)} & \dfrac{x^3}{2} & -\dfrac{x^2}{2} + \dfrac{x^1}{x^3} - \dfrac{x^1 \sin x^3}{2(1-\cos x^3)} \\ -\dfrac{x^3}{2} & \dfrac{x^3 \sin x^3}{2(1-\cos x^3)} & \dfrac{x^1}{2} + \dfrac{x^2}{x^3} - \dfrac{x^2 \sin x^3}{2(1-\cos x^3)} \\ 0 & 0 & 1 \end{pmatrix}. \tag{59}$$

Canonicity criterion of coordinates of the 1st kind done because. $v(x)x = x$ и $\Lambda(x)x = x$. Entering parameters t^1, t^2, t^3 and using the formula $\dfrac{\sin x^3}{1-\cos x^3} = \operatorname{ctg}\dfrac{x^3}{2}$, compose systems (15):

$$\begin{cases} (x^1)^{\cdot}_{t^1} = \dfrac{x^3}{2} ctg \dfrac{x^3}{2}, \\ (x^2)^{\cdot}_{t^1} = -\dfrac{x^3}{2}, \\ (x^3)^{\cdot}_{t^1} = 0, \end{cases} \quad \begin{cases} (x^1)^{\cdot}_{t^2} = \dfrac{x^3}{2}, \\ (x^2)^{\cdot}_{t^2} = \dfrac{x^3}{2} \operatorname{ctg} \dfrac{x^3}{2}, \\ (x^3)^{\cdot}_{t^2} = 0, \end{cases} \quad \begin{cases} (x^1)^{\cdot}_{t^3} = -\dfrac{x^2}{2} + \dfrac{x^1}{x^3} - \dfrac{x^1}{2} \operatorname{ctg} \dfrac{x^3}{2}, \\ (x^2)^{\cdot}_{t^3} = \dfrac{x^1}{2} + \dfrac{x^2}{x^3} - \dfrac{x^2}{2} \operatorname{ctg} \dfrac{x^3}{2}, \\ (x^3)^{\cdot}_{t^3} = 1. \end{cases} \tag{60}$$

Equations for f do not discharge, because they coincide with the equations for x. We get the desired functions (44):

$$\begin{cases} x^1 = \dfrac{t^3 + c_3}{2}\left(\dfrac{\cos\frac{t^3+c_3}{2}}{\sin\frac{t^3+c_3}{2}} \cdot t^1 + t^2 + c_2 \dfrac{\cos\frac{t^3+c_1}{2}}{\sin\frac{t^3+c_3}{2}}\right), \\ x^2 = \dfrac{t^3 + c_3}{2}\left(-t^1 + \dfrac{\cos\frac{t^3+c_3}{2}}{\sin\frac{t^3+c_3}{2}} \cdot t^2 + c_2 \dfrac{\sin\frac{t^3+c_1}{2}}{\sin\frac{t^3+c_3}{2}}\right), \\ x^3 = t^3 + c_3, \end{cases}$$

$$\begin{cases} f^1 = \dfrac{t^3+\varphi_3}{2}\left(\dfrac{\cos\frac{t^3+\varphi_3}{2}}{\sin\frac{t^3+\varphi_3}{2}}\cdot t^1 + t^2 + \varphi_2\dfrac{\cos\frac{t^3+\varphi_1}{2}}{\sin\frac{t^3+\varphi_3}{2}}\right), \\ f^2 = \dfrac{t^3+\varphi_3}{2}\left(-t^1 + \dfrac{\cos\frac{t^3+\varphi_3}{2}}{\sin\frac{t^3+\varphi_3}{2}}\cdot t^2 + \varphi_2\dfrac{\sin\frac{t^3+\varphi_1}{2}}{\sin\frac{t^3+\varphi_3}{2}}\right). \\ f^3 = t^3+\varphi_3. \end{cases} \tag{61}$$

We complexify constants, parameters, variables, and find constants $\overline{c}$ и $\overline{\varphi}(y)$ so that it is executed $x^i(0,\overline{c}) = 0, f^i(0,\overline{\varphi}(y)) = y^i$, Get

$$\begin{cases} x^1 = \dfrac{t^3}{2}\left(\operatorname{ctg}\frac{t^3}{2}\cdot t^1 + t^2\right), \\ x^2 = \dfrac{t^3}{2}\left(-t^1 + \operatorname{ctg}\frac{t^3}{2}\cdot t^2\right), \\ x^3 = t^3, \end{cases}$$

$$\begin{cases} f^1 = \dfrac{t^3+y^3}{2}\left(\operatorname{ctg}\dfrac{t^3+y^3}{2}\cdot t^1 + t^2 + \dfrac{\sin\frac{y^3}{2}}{\frac{y^3}{2}}\cdot\dfrac{\cos\frac{t^3}{2}\cdot y^1 - \sin\frac{t^3}{2}\cdot y^2}{\sin\frac{t^3+y^3}{2}}\right), \\ f^2 = \dfrac{t^3+y^3}{2}\left(-t^1 + \operatorname{ctg}\dfrac{t^3+y^3}{2}\cdot t^2 + \dfrac{\sin\frac{y^3}{2}}{\frac{y^3}{2}}\cdot\dfrac{\sin\frac{t^3}{2}\cdot y^1 + \cos\frac{t^3}{2}\cdot y^2}{\sin\frac{t^3+y^3}{2}}\right), \\ f^3 = t^3+y^3, \end{cases} \tag{62}$$

in the canonical coordinates of the seco2nd kind, corresponding to the coordinates of the 1st kind because $\overline{\varphi}_1(y) = 2\operatorname{arctg}\dfrac{y^2}{y^1}$, $\overline{\varphi}_2(y) = \sqrt{(y^1)^2+(y^2)^2}\dfrac{\sin\frac{y^3}{2}}{\frac{y^3}{2}}$, $\overline{\varphi}_3(y) = y^3$. Excluding from (62) parameters t, we get

$$\begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix} = \frac{\frac{x^3+y^3}{2}}{\sin\frac{x^3+y^3}{2}}\left(\frac{\sin\frac{x^3}{2}}{\frac{x^3}{2}}\begin{pmatrix} \cos\frac{y^3}{2} & \sin\frac{y^3}{2} & 0 \\ -\sin\frac{y^3}{2} & \cos\frac{y^3}{2} & 0 \\ 0 & 0 & \dfrac{\sin\frac{x^3+y^3}{2}}{\frac{x^3+y^3}{2}}\cdot\dfrac{\frac{x^3}{2}}{\sin\frac{x^3}{2}} \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \right.$$

$$\left. + \frac{\sin\frac{y^3}{2}}{\frac{y^3}{2}}\begin{pmatrix} \cos\frac{x^3}{2} & -\sin\frac{x^3}{2} & 0 \\ \sin\frac{x^3}{2} & \cos\frac{x^3}{2} & 0 \\ 0 & 0 & \dfrac{\sin\frac{x^3+y^3}{2}}{\frac{x^3+y^3}{2}}\cdot\dfrac{\frac{y^3}{2}}{\sin\frac{y^3}{2}} \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}\right), \tag{63}$$

group operation components $ISO(2)$ in canonical coordinates of the 1st kind. In (63) $f(\lambda x, \mu x) = (\lambda+\mu)x$, associativity is performed, inverse to x is $-x$, the structural constants coincide with (55). In non-canonical coordinates (54) vector (y^1, y^2) turns to corner x^3 counterclockwise, then shifted to a vector (x^1, x^2), argument y^3 grows on x^3. In canonical coordinates (60) vector (y^1, y^2) turns against the clock arrows on half the corner $\dfrac{x^3}{2}$, at this time, the shear vector (x^1, x^2) turns halfway to the argument $\dfrac{y^3}{2}$, then they are shortened, respectively, in $\dfrac{\sin\frac{y^3}{2}}{\frac{y^3}{2}}$ and in $\dfrac{\sin\frac{x^3}{2}}{\frac{x^3}{2}}$ times, and stored. The result then experiences a general, "compensating" stretch in $\dfrac{\frac{x^3+y^3}{2}}{\sin\frac{x^3+y^3}{2}}$ times. Argument y^3 also grows on x^3. Example 2 is considered to demonstrate how "actually" (63) turns work, and shifts on the plane (54).

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О восстановлении локальной группы Ли по структурным константам

Виталий А. Степаненко
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Строится система координат 2-го рода, соответствующих каноническим координатам 1-го рода (терминология А. И. Мальцева), тем самым получается параметрическое решение системы уравнений Ли. Приводится интегральное представление групповой операции $f(x, y)$ локальной группы Ли G в канонических координатах 1-го рода. В качестве основного аппарата используется модифицированная формула А. П. Южакова для неявных отображений. Операция $f(x, y)$ также представлена в виде степенного ряда, являющегося приведенной формой ряда Кэмпбелла–Хаусдорфа.

Ключевые слова: локальная группа Ли, ряд Кэмпбелла–Хаусдорфа, формула А. П. Южакова.