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Spectrum of One-dimensional Eigenoscillations of Two-phase Layered Composites

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Abstract. The spectrum of one-dimensional eigenoscillations of two-phase composites with a periodic structure is studied. Their phases are isotropic elastic or viscoelastic materials, and the period consists of $2M$ alternating plane layers of the first and second phases. Equations whose roots form the spectrum are derived and their asymptotic behaviour is investigated. In particular, it is established that all finite limits of sequences of the spectrum points depend on the volume fractions of the phases and do not depend on the number M and distances between the layers boundaries inside the period.

Keywords: spectrum, eigenoscillations, layered composite.

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Studying the spectra of eigenoscillations of composite materials is one of the most relevant problems in heterogeneous media mechanics. Information on their structure and an accurate description of the spectra points make it possible to find such important dynamic characteristics of composites as eigenfrequencies and damping coefficients.

Modern composites often have a periodic geometry with a small period $\varepsilon > 0$. In this case, the mathematical study of their spectra of eigenoscillations can be reduced to the spectral analysis of boundary value problems for homogeneous systems of differential equations with ε-periodic coefficients. In the numerical search for the discrete part of the spectrum, one has to choose fairly good initial approximations of the eigenvalues of the corresponding boundary value problem. However, the direct searching of such approximations turns out to be an extremely difficult problem, especially in the case of complex eigenvalues with nonzero real and imaginary parts. In this situation, it is natural to find finite limits of sequences of eigenvalues as $\varepsilon \to 0$ and propose to use them as the initial approximations.

The asymptotic behaviour of the spectra of various boundary value problems arising in the study of ε-periodic heterogeneous media was investigated in [1–8]. In particular, the spectrum S_ε of one-dimensional eigenoscillations of two-phase layered composites was studied in [6] and [7] under the assumption that the period formed by one elastic and one viscoelastic layer. Simultaneously, it was considered the spectrum S of one-dimensional eigenoscillations of the corresponding homogenized material constructed as $\varepsilon \to 0$. The main result of the above two papers consisted in proving that the spectrum S_ε converges in the sense of Hausdorff to the union of the spectrum S and some finite set V consisting of real negative points. This convergence means the fulfillment of the following two conditions [2]: 1) for any $s \in S \cup V$ there exists a sequence $s_\varepsilon \in S_\varepsilon$ such that $s_\varepsilon \to s$ as $\varepsilon \to 0$; 2) if $s_\varepsilon \in S_\varepsilon$ and $s_\varepsilon \to s < \infty$ as $\varepsilon \to 0$ then $s \in S \cup V$.

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In the present work, we consider layered composites consisting of two isotropic solid (elastic or viscoelastic) materials. We suppose that the layers are parallel to the plane Ox_2x_3 and the composites have ε-periodic structure with the period formed by $2M$ alternating layers of the first and second phases. Using a matrix approach, we derive transcendental equations for finding points of the spectrum of eigenoscillations along the Ox_1 axis and then find that finite limits of sequences of the spectrum points as $\varepsilon \to 0$ are the roots of rational equations and are independent of the number M and distances between the layers boundaries inside the period. Finally, to clarify the meaning of the limits, we also describe the spectrum of eigenoscillations along the Ox_1 axis for the corresponding homogenized materials.

1. Mathematical description of two-phase layered composites

Consider a cube $\Omega = (0, L)^3$ occupied by a two-phase composite with a periodic microstructure. Let its period be an elementary stripe $Y_\varepsilon = (0, \varepsilon) \times (0, L)^2$ consisting of $2M$ alternating layers of the first and second phases. Assume additionally that the layers are parallel to the plane Ox_2x_3 (see Fig. 1) and the cube Ω contains a whole number N of periods. Note that by our construction, $\varepsilon = L/N$ and $2MN$ is the total number of layers in Ω.

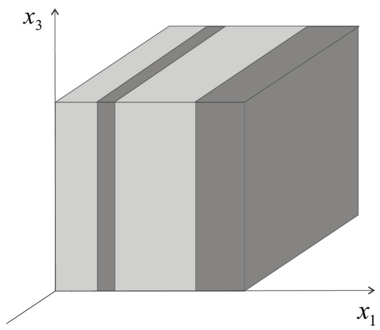

Fig. 1. The set $Y_\varepsilon \cap (0, \varepsilon)^3$ for $M = 2$

In order to clarify the geometric position of the phases inside Ω, we denote

$$I_{1\varepsilon} = \bigcup_{n=0}^{N-1} \bigcup_{m=0}^{M-1} \left(\varepsilon(n + h_{2m}), \varepsilon(n + h_{2m+1})\right),$$

$$I_{2\varepsilon} = \bigcup_{n=0}^{N-1} \bigcup_{m=0}^{M-1} \left(\varepsilon(n + h_{2m+1}), \varepsilon(n + h_{2m+2})\right),$$

$$0 = h_0 < h_1 < h_2 < \cdots < h_{2M} = 1$$

and assume that the set $\Omega_{s\varepsilon} = I_{s\varepsilon} \times (0, L)^2$ is occupied by the s-th phase of the composite, $s = 1, 2$. The total volume fraction of the s-th phase inside Ω is denoted by d_s, $s = 1, 2$. This means that

$$d_1 = \frac{|\Omega_{1\varepsilon}|}{L^3} = \frac{N\varepsilon}{L} \sum_{m=0}^{M-1} (h_{2m+1} - h_{2m}) = \sum_{m=0}^{M-1} (h_{2m+1} - h_{2m}),$$

$$d_2 = \frac{|\Omega_{2\varepsilon}|}{L^3} = \frac{N\varepsilon}{L}\sum_{m=0}^{M-1}(h_{2m+2} - h_{2m+1}) = \sum_{m=0}^{M-1}(h_{2m+2} - h_{2m+1}).$$

The constitutive relations between the stress and small strain tensor components have the form

$$\sigma_{ij}^{\varepsilon} = a_{ijkh}^{(s)} e_{kh}(u^{\varepsilon}) + b_{ijkh}^{(s)} e_{kh}\left(\frac{\partial u^{\varepsilon}}{\partial t}\right) - d_{ijkh}^{(s)}(t) * e_{kh}(u^{\varepsilon}), \quad x \in \Omega_{s\varepsilon}, \tag{1}$$
$$1 \leqslant i, j, k, h \leqslant 3, \quad s = 1, 2,$$

where $u^{\varepsilon}(x,t)$ is the displacement vector, σ^{ε} is the stress tensor, $e(u^{\varepsilon})$ is the small strain tensor, $e_{kh}(u^{\varepsilon}) = e^{x}_{kh}(u^{\varepsilon}) = (\partial u_k^{\varepsilon}/\partial x_h + \partial u_h^{\varepsilon}/\partial x_k)/2$, the symbol $*$ denotes the convolution operation with respect to t, $a^{(s)}$ and $b^{(s)}$ are the tensors of elasticity and viscosity coefficients, respectively, and $d^{(s)}(t)$ are the tensors of the regular parts of relaxation kernels. Note that in (1) and everywhere below the summation convention over repeated subscripts is employed.

Both tensors $a^{(1)}$ and $a^{(2)}$ are positive definite and their components are given by

$$a_{ijkh}^{(s)} = \lambda^{(s)}\delta_{ij}\delta_{kh} + \mu^{(s)}(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \quad s = 1, 2,$$

where $\lambda^{(s)}$ и $\mu^{(s)}$ are the Lamé parameters in $\Omega_{s\varepsilon}$, and δ_{ij} is the Kronecker symbol.

In what follows, we assume that if $b^{(s)} \neq 0$, then the tensor $b^{(s)}$ is positive definite and

$$b_{ijkh}^{(s)} = \zeta^{(s)}\delta_{ij}\delta_{kh} + \eta^{(s)}(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}),$$

where $\zeta^{(s)}$ and $\eta^{(s)}$ are the viscosity coefficients [9], and if $d^{(s)}(t) \neq 0$, then

$$d_{ijkh}^{(s)}(t) = \left(G_{1s}(t) - \frac{1}{3}G_s(t)\right)\delta_{ij}\delta_{kh} + \frac{1}{2}G_s(t)(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}),$$

where $G_{1s}(t)$ and $G_s(t)$ are the regular parts of the bulk and shear relaxation kernels [10] and satisfy the following conditions:

$$G_{1s}(t) = k_s G_s(t), \quad G_s(t) = \sum_{n=1}^{N_s} v_n^{(s)} e^{-\gamma_n^{(s)} t}, \quad \sum_{n=1}^{N_s} \frac{v_n^{(s)}}{\gamma_n^{(s)}} < K_s,$$

where $k_s \in \mathbb{R}^+ \cup \{0\}$, $v_n^{(s)} \in \mathbb{R}^+$, $\gamma_n^{(s)} \in \mathbb{R}^+$, $\gamma_i^{(s)} \neq \gamma_j^{(s)}$ for $i \neq j$, $K_s = 2\mu^{(s)}$ for $k_s = 0$, and $K_s = \min\{2\mu^{(s)}, (\lambda^{(s)} + 2\mu^{(s)}/3)/k_s\}$ for $k_s > 0$. Moreover, for the sake of definiteness, we will assume that if $d^{(1)}(t) \neq 0$ and $d^{(2)}(t) \neq 0$ then $\gamma_{n_1}^{(1)} \neq \gamma_{n_2}^{(2)}$ for all $n_1 = 1, \ldots, N_1$ and $n_2 = 1, \ldots, N_2$.

Note that the case $b^{(s)} = 0$, $d^{(s)}(t) = 0$ corresponds to the elastic s-th phase while the case $b^{(s)} \neq 0$, $d^{(s)}(t) = 0$ corresponds to the viscoelastic Kelvin-Voigt s-th phase.

The mathematical model describing oscillations of the layered composite in Ω is written in the form

$$\begin{gathered} \rho_s \frac{\partial^2 u_i^{\varepsilon}}{\partial t^2} = \frac{\partial \sigma_{ij}^{\varepsilon}}{\partial x_j} + f_i(x,t), \quad x \in \Omega_{s\varepsilon}, \quad t > 0, \quad s = 1, 2, \\ [u^{\varepsilon}]|_{S_\varepsilon} = 0, \quad [\sigma_{i1}^{\varepsilon}]|_{S_\varepsilon} = 0, \quad S_\varepsilon = \partial\Omega_{1\varepsilon} \cap \partial\Omega_{2\varepsilon}, \\ u^{\varepsilon}(x,t)|_{\partial\Omega} = 0, \quad u^{\varepsilon}(x,0) = \frac{\partial u^{\varepsilon}}{\partial t}(x,0) = 0, \end{gathered} \tag{2}$$

where $\rho_s = \text{const} > 0$ is the density in $\Omega_{s\varepsilon}$, $f(x,t)$ is the external force vector, and $[\cdot]|_{S_\varepsilon}$ means the jump in the enclosed quantity across the boundary S_ε.

2. Spectrum of one-dimensional eigenoscillations

From now on, we assume that the layered composite occupies an unbounded strip $0 < x_1 < L$ and consider one-dimensional oscillations propagating along the normal to the layers, i.e., along the Ox_1 axis. In this case $u^\varepsilon(x,t) = (u_1^\varepsilon(x_1,t), 0, 0)$ and $f(x,t) = (f_1(x_1,t), 0, 0)$. Therefore, it follows from (2) that such oscillations are described by the following initial-boundary value problem:

$$\rho_s \frac{\partial^2 u_1^\varepsilon}{\partial t^2} = \frac{\partial \sigma_1^\varepsilon}{\partial x_1} + f_1(x_1,t), \quad x_1 \in I_{s\varepsilon}, \quad t > 0, \quad s = 1, 2,$$

$$[u_1^\varepsilon]|_{\varepsilon(n+h_m)} = 0, \quad [\sigma_1^\varepsilon]|_{\varepsilon(n+h_m)} = 0,$$

$$u_1^\varepsilon(0,t) = u_1^\varepsilon(L,t) = 0, \quad t > 0; \quad u_1^\varepsilon(x_1,0) = \frac{\partial u_1^\varepsilon}{\partial t}(x_1,0) = 0, \quad x_1 \in (0,L),$$

$$n = 0, \ldots, N-1, \quad m = 0, \ldots, 2M-1, \quad n+m \neq 0$$

with

$$\sigma_1^\varepsilon = a_s \frac{\partial u_1^\varepsilon}{\partial x_1} + b_s \frac{\partial^2 u_1^\varepsilon}{\partial x_1 \partial t} - R_s \left(k_s + \frac{2}{3}\right) \sum_{n=1}^{N_s} v_n^{(s)} e^{-\gamma_n^{(s)} t} * \frac{\partial u_1^\varepsilon}{\partial x_1}, \quad x_1 \in I_{s\varepsilon},$$

$$a_s = a_{1111}^{(s)} = \lambda^{(s)} + 2\mu^{(s)}, \quad b_s = b_{1111}^{(s)} = \zeta^{(s)} + 2\eta^{(s)},$$

where we set $b_s = 0$ for $b^{(s)} = 0$, $R_s = 0$ for $d^{(s)}(t) = 0$, and $R_s = 1$ for $d^{(s)}(t) \neq 0$.

To define the spectrum of eigenoscillations of the composite along the Ox_1 axis, we apply the Laplace transform in time to the last problem with $f_1(x_1,t) \equiv 0$. As a result, we obtain the boundary value problem

$$\lambda^2 \rho_s u_{1\lambda}^\varepsilon = A_{s\lambda} \frac{d^2 u_{1\lambda}^\varepsilon}{dx_1^2}, \quad x_1 \in I_{s\varepsilon}, \quad s = 1, 2, \tag{3}$$

$$[u_{1\lambda}^\varepsilon]|_{\varepsilon(n+h_m)} = 0, \quad [\sigma_{1\lambda}^\varepsilon]|_{\varepsilon(n+h_m)} = 0, \quad u_{1\lambda}^\varepsilon(0) = u_{1\lambda}^\varepsilon(L) = 0, \tag{4}$$

$$n = 0, \ldots, N-1, \quad m = 0, \ldots, 2M-1, \quad n+m \neq 0,$$

where $u_{1\lambda}^\varepsilon(x_1)$ is the Laplace image of $u_1^\varepsilon(x_1,t)$, λ is the Laplace transform parameter, and

$$A_{s\lambda} = a_s + b_s\lambda - R_s \left(k_s + \frac{2}{3}\right) \sum_{n=1}^{N_s} \frac{v_n^{(s)}}{\lambda + \gamma_n^{(s)}}, \quad \sigma_{1\lambda}^\varepsilon = A_{s\lambda} \frac{du_{1\lambda}^\varepsilon}{dx_1}, \quad x_1 \in I_{s\varepsilon}.$$

Below we will consider λ as a spectral parameter. Then, by the spectrum of one-dimensional eigenoscillations of the composite along the Ox_1 axis we mean the set of eigenvalues of the spectral problem (3), (4), i.e, the set S_ε of all complex values of $\lambda = \lambda(\varepsilon)$, for which this problem has nontrivial solutions $u_{1\lambda}^\varepsilon(x_1)$.

Our aim now is to find the elements of S_ε. For this purpose, we firstly derive relations connecting $u_{1\lambda}^\varepsilon((n+1)\varepsilon - 0)$ and $\sigma_{1\lambda}^\varepsilon((n+1)\varepsilon - 0)$ with $u_{1\lambda}^\varepsilon(n\varepsilon + 0)$ and $\sigma_{1\lambda}^\varepsilon(n\varepsilon + 0)$ for all $n = 0, \ldots, N-1$. In other words, we seek a square matrix $P_{\lambda n}^\varepsilon$ such that

$$\begin{pmatrix} u_{1\lambda}^\varepsilon((n+1)\varepsilon - 0) \\ A_{2\lambda} \dfrac{du_{1\lambda}^\varepsilon}{dx_1}((n+1)\varepsilon - 0) \end{pmatrix} = P_{\lambda n}^\varepsilon \begin{pmatrix} u_{1\lambda}^\varepsilon(n\varepsilon + 0) \\ A_{1\lambda} \dfrac{du_{1\lambda}^\varepsilon}{dx_1}(n\varepsilon + 0) \end{pmatrix}, \quad n = 0, \ldots, N-1.$$

Note that in view of the periodicity in x_1, the matrix $P^{\varepsilon}_{\lambda n}$ does not depend on n, i.e., $P^{\varepsilon}_{\lambda n} = P^{\varepsilon}_{\lambda}$; therefore, it suffices to find it for $n = 0$. To do this, we write general solutions of equations (3) in the interval $(0, \varepsilon)$:

$$u^{\varepsilon}_{1\lambda}(x_1) = C^{(1)}_{\lambda m} e^{-B_{1\lambda}x_1} + C^{(2)}_{\lambda m} e^{B_{1\lambda}x_1}, \quad x_1 \in (\varepsilon h_{2m}, \varepsilon h_{2m+1}), \quad m = 0, \ldots, M-1,$$

$$u^{\varepsilon}_{1\lambda}(x_1) = C^{(3)}_{\lambda m} e^{-B_{2\lambda}x_1} + C^{(4)}_{\lambda m} e^{B_{2\lambda}x_1}, \quad x_1 \in (\varepsilon h_{2m+1}, \varepsilon h_{2m+2}), \quad m = 0, \ldots, M-1,$$

where

$$B_{s\lambda} = \lambda\sqrt{\frac{\rho_s}{A_{s\lambda}}}, \quad s = 1, 2.$$

It is easy to check that

$$C^{(1)}_{\lambda m} = \frac{\exp(\varepsilon B_{1\lambda} h_{2m})}{2B_{1\lambda}} \left(B_{1\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m} + 0) - \frac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m} + 0) \right),$$

$$C^{(2)}_{\lambda m} = \frac{\exp(-\varepsilon B_{1\lambda} h_{2m})}{2B_{1\lambda}} \left(B_{1\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m} + 0) + \frac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m} + 0) \right),$$

$$C^{(3)}_{\lambda m} = \frac{\exp(\varepsilon B_{2\lambda} h_{2m+1})}{2B_{2\lambda}} \left(B_{2\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} + 0) - \frac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m+1} + 0) \right),$$

$$C^{(4)}_{\lambda m} = \frac{\exp(-\varepsilon B_{2\lambda} h_{2m+1})}{2B_{2\lambda}} \left(B_{2\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} + 0) + \frac{du^{\varepsilon}_{\lambda}}{dx_1}(\varepsilon h_{2m+1} + 0) \right).$$

Thus

$$u^{\varepsilon}_{1\lambda}(x_1) = \frac{\exp(B_{1\lambda}(-x_1 + \varepsilon h_{2m}))}{2B_{1\lambda}} \left(B_{1\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m} + 0) - \frac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m} + 0) \right) +$$

$$+ \frac{\exp(B_{1\lambda}(x_1 - \varepsilon h_{2m}))}{2B_{1\lambda}} \left(B_{1\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m} + 0) + \frac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m} + 0) \right)$$

for $x_1 \in (\varepsilon h_{2m}, \varepsilon h_{2m+1})$ and

$$u^{\varepsilon}_{1\lambda}(x_1) = \frac{\exp(B_{2\lambda}(-x_1 + \varepsilon h_{2m+1}))}{2B_{2\lambda}} \left(B_{2\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} + 0) - \frac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m+1} + 0) \right) +$$

$$+ \frac{\exp(B_{2\lambda}(x_1 - \varepsilon h_{2m+1}))}{2B_{2\lambda}} \left(B_{2\lambda} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} + 0) + \frac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m+1} + 0) \right)$$

for $x_1 \in (\varepsilon h_{2m+1}, \varepsilon h_{2m+2})$. This yields

$$\begin{pmatrix} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} - 0) \\ A_{1\lambda} \dfrac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m+1} - 0) \end{pmatrix} = P^{1\varepsilon}_{\lambda m} \begin{pmatrix} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m} + 0) \\ A_{1\lambda} \dfrac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m} + 0) \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+2} - 0) \\ A_{2\lambda} \dfrac{du^{\varepsilon}_{1\lambda}}{dx}(\varepsilon h_{2m+2} - 0) \end{pmatrix} = P^{2\varepsilon}_{\lambda m} \begin{pmatrix} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} + 0) \\ A_{2\lambda} \dfrac{du^{\varepsilon}_{1\lambda}}{dx}(\varepsilon h_{2m+1} + 0) \end{pmatrix}, \tag{6}$$

where the matrices $P^{1\varepsilon}_{\lambda m}$ and $P^{2\varepsilon}_{\lambda m}$ have the following elements:

$$(P^{s\varepsilon}_{\lambda m})_{11} = (P^{s\varepsilon}_{\lambda m})_{22} = \cosh(\varepsilon B_{s\lambda} d_{sm}), \quad (P^{s\varepsilon}_{\lambda m})_{12} = \frac{\sinh(\varepsilon B_{s\lambda} d_{sm})}{\lambda\sqrt{\rho_s A_{s\lambda}}},$$

$$(P^{s\varepsilon}_{\lambda m})_{21} = \lambda\sqrt{\rho_s A_{s\lambda}}\sinh(\varepsilon B_{s\lambda} d_{sm}), \quad s = 1, 2,$$

$$d_{1m} = h_{2m+1} - h_{2m}, \quad d_{2m} = h_{2m+2} - h_{2m+1}, \quad m = 0, 1, \dots, M-1.$$

Since by condition (4)

$$u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} - 0) = u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+1} + 0)$$

and

$$A_{1\lambda}\frac{du^{\varepsilon}_{1\lambda}}{dx}(\varepsilon h_{2m+1} - 0) = A_{2\lambda}\frac{du^{\varepsilon}_{1\lambda}}{dx}(\varepsilon h_{2m+1} + 0),$$

combining (5) and (6) gives

$$\begin{pmatrix} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m+2} - 0) \\ A_{2\lambda}\dfrac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m+2} - 0) \end{pmatrix} = P^{\varepsilon}_{\lambda m}\begin{pmatrix} u^{\varepsilon}_{1\lambda}(\varepsilon h_{2m} + 0) \\ A_{1\lambda}\dfrac{du^{\varepsilon}_{1\lambda}}{dx_1}(\varepsilon h_{2m} + 0) \end{pmatrix}, \tag{7}$$

$$P^{\varepsilon}_{\lambda m} = P^{2\varepsilon}_{\lambda m}\cdot P^{1\varepsilon}_{\lambda m}, \quad m = 0, 1, \dots, M-1,$$

where

$$(P^{\varepsilon}_{\lambda m})_{11} = \frac{1}{2}\sum_{s=0}^{1}\Big(1 + (-1)^s R_\lambda\Big)\cosh\Big(\varepsilon(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})\Big), \quad R_\lambda = \sqrt{\frac{\rho_1 A_{1\lambda}}{\rho_2 A_{2\lambda}}},$$

$$(P^{\varepsilon}_{\lambda m})_{22} = \frac{1}{2}\sum_{s=0}^{1}\left(1 + \frac{(-1)^s}{R_\lambda}\right)\cosh\Big(\varepsilon(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})\Big),$$

$$(P^{\varepsilon}_{\lambda m})_{12} = \frac{1}{2\lambda}\sum_{s=0}^{1}\left(\frac{1}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{(-1)^s}{\sqrt{\rho_2 A_{2\lambda}}}\right)\sinh\Big(\varepsilon(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})\Big),$$

$$(P^{\varepsilon}_{\lambda m})_{21} = \frac{\lambda}{2}\sum_{s=1}^{1}\left(\sqrt{\rho_1 A_{1\lambda}} + (-1)^s\sqrt{\rho_2 A_{2\lambda}}\right)\sinh\Big(\varepsilon(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})\Big).$$

Using the matrix relations (7) for $m = M-1, \dots, 1, 0$ we obtain

$$\begin{pmatrix} u^{\varepsilon}_{1\lambda}(\varepsilon - 0) \\ A_{2\lambda}\dfrac{du^{\varepsilon}_{1\lambda}}{dx}(\varepsilon - 0) \end{pmatrix} = P^{\varepsilon}_{\lambda(M-1)}\cdot \ldots \cdot P^{\varepsilon}_{\lambda 1}\cdot P^{\varepsilon}_{\lambda 0}\begin{pmatrix} u^{\varepsilon}_{1\lambda}(+0) \\ A_{1\lambda}\dfrac{du^{\varepsilon}_{1\lambda}}{dx}(+0) \end{pmatrix},$$

which means that the matrix $P^{\varepsilon}_{\lambda}$ mentioned above is the product of M matrices:

$$P^{\varepsilon}_{\lambda} = P^{\varepsilon}_{\lambda(M-1)}\cdot \ldots \cdot P^{\varepsilon}_{\lambda 1}\cdot P^{\varepsilon}_{\lambda 0}.$$

It can easily be checked that

$$(P^{\varepsilon}_{\lambda})_{ij} = \sum_{j_1=1}^{2}(P^{\varepsilon}_{\lambda 1})_{ij_1}(P^{\varepsilon}_{\lambda 0})_{j_1 j} \tag{8}$$

for $M = 2$ and

$$(P^{\varepsilon}_{\lambda})_{ij} = \sum_{j_{M-1}=1}^{2}\cdots\sum_{j_2=1}^{2}\sum_{j_1=1}^{2}(P^{\varepsilon}_{\lambda(M-1)})_{ij_{M-1}}(P^{\varepsilon}_{\lambda(M-2)})_{j_{M-1}j_{M-2}}\ldots(P^{\varepsilon}_{\lambda 1})_{j_2 j_1}(P^{\varepsilon}_{\lambda 0})_{j_1 j} \tag{9}$$

for $M \geqslant 3$.

Since the strip $0 < x_1 < L$ contains exactly N periods and

$$u^{\varepsilon}_{1\lambda}(n\varepsilon - 0) = u^{\varepsilon}_{1\lambda}(n\varepsilon + 0), \quad A_{1\lambda}\frac{du^{\varepsilon}_{1\lambda}}{dx_1}(n\varepsilon + 0) = A_{2\lambda}\frac{du^{\varepsilon}_{1\lambda}}{dx_1}(n\varepsilon - 0) = \sigma^{\varepsilon}_{1\lambda}(n\varepsilon)$$

for all $n = 1, \dots, N-1$, we arrive at the matrix equality

$$\begin{pmatrix} u^{\varepsilon}_{1\lambda}(L) \\ \sigma^{\varepsilon}_{1\lambda}(L) \end{pmatrix} = (P^{\varepsilon}_{\lambda})^N \begin{pmatrix} u^{\varepsilon}_{1\lambda}(0) \\ \sigma^{\varepsilon}_{1\lambda}(0) \end{pmatrix},$$

which, due to the boundary conditions (4), can be written in the form

$$\begin{pmatrix} 0 \\ \sigma^{\varepsilon}_{1\lambda}(L) \end{pmatrix} = (P^{\varepsilon}_{\lambda})^N \begin{pmatrix} 0 \\ \sigma^{\varepsilon}_{1\lambda}(0) \end{pmatrix}.$$

This implies that λ belongs to the spectrum S_ε if λ is the root of the equation

$$(P^{\varepsilon}_{\lambda})^N_{12} = 0. \tag{10}$$

Taking into account that

$$\det P^{s\varepsilon}_{\lambda m} = \det P^{\varepsilon}_{\lambda m} = \det P^{\varepsilon}_{\lambda} = 1, \quad m = 0, \dots, M-1, \quad s = 1, 2$$

and using the same arguments as in [6-8], we can prove that equation (10) is split into N equations

$$(P^{\varepsilon}_{\lambda})_{12} = 0, \tag{11}$$

$$(P^{\varepsilon}_{\lambda})_{11} + (P^{\varepsilon}_{\lambda})_{22} = 2\cos\frac{\pi k}{N}, \quad k = 1, \dots, N-1. \tag{12}$$

Note that in the simplest case when the period $0 < x_1 < \varepsilon$ consists of only two layers, i.e., when $M = 1$, equations (11) and (12) can be written explicitly as follows [6-8]:

$$\frac{1}{\lambda}\Big((1 + R_\lambda)\sinh\Big(\varepsilon(B_{1\lambda}d_1 + B_{2\lambda}d_2)\Big) + (1 - R_\lambda)\sinh\Big(\varepsilon(B_{1\lambda}d_1 - B_{2\lambda}d_2)\Big)\Big) = 0,$$

$$\left(1 + R_\lambda + \frac{1}{R_\lambda}\right)\cosh\Big(\varepsilon(B_{1\lambda}d_1 + B_{2\lambda}d_2)\Big) +$$
$$+ \left(1 - R_\lambda - \frac{1}{R_\lambda}\right)\cosh\Big(\varepsilon(B_{1\lambda}d_1 - B_{2\lambda}d_2)\Big) = 4\cos\frac{\pi k}{L}, \quad k = 1, \dots, N-1.$$

In all other cases ($N \geqslant 2$) the explicit form of equations (11) and (12) is very cumbersome and is not given here.

3. Asymptotic behavior of the spectrum as $\varepsilon \to 0$

In the previous section we have proved that the search of the spectrum S_ε reduces to solving the transcendental equations (11) and (12). These equations cannot be solved analytically even in the case $M = N = 1$ [6–8]. For this reason, our next aim is to study the asymptotic behaviour of the roots of equations (11) and (12) as $\varepsilon \to 0$ and thereby find the initial approximations for numerical solving of these equations. To do this, we use the series expansions

$$\cosh\Big(\varepsilon(B_{1\lambda}d_{1m} \pm B_{2\lambda}d_{2m})\Big) = \sum_{n=0}^{\infty}\frac{\varepsilon^{2n}(B_{1\lambda}d_{1m} \pm B_{2\lambda}d_{2m})^{2n}}{(2n)!}, \tag{13}$$

$$\sinh\Big(\varepsilon(B_{1\lambda}d_{1m} \pm B_{2\lambda}d_{2m})\Big) = \sum_{n=0}^{\infty} \frac{\varepsilon^{2n+1}(B_{1\lambda}d_{1m} \pm B_{2\lambda}d_{2m})^{2n+1}}{(2n+1)!}, \tag{14}$$

$$\cos\frac{\pi k}{N} = \cos\frac{\pi k\varepsilon}{L} = \sum_{n=0}^{\infty} \frac{(-1)^n\varepsilon^{2n}}{(2n)!}\left(\frac{\pi k}{L}\right)^{2n}. \tag{15}$$

It follows from (13) and (14) that the elements of the matrices $P^{\varepsilon}_{\lambda m}$ can be represented as

$$(P^{\varepsilon}_{\lambda m})_{11} = 1 + \frac{\varepsilon^2}{2}\left(B^2_{1\lambda}d^2_{1m} + B^2_{2\lambda}d^2_{2m} + 2R_\lambda B_{1\lambda}B_{2\lambda}d_{1m}d_{2m}\right) + \frac{\varepsilon^4}{2}Q^{1\varepsilon}_{\lambda m}, \tag{16}$$

$$(P^{\varepsilon}_{\lambda m})_{22} = 1 + \frac{\varepsilon^2}{2}\left(B^2_{1\lambda}d^2_{1m} + B^2_{2\lambda}d^2_{2m} + \frac{2}{R_\lambda}B_{1\lambda}B_{2\lambda}d_{1m}d_{2m}\right) + \frac{\varepsilon^4}{2}Q^{2\varepsilon}_{\lambda m}, \tag{17}$$

$$(P^{\varepsilon}_{\lambda m})_{12} = \frac{\varepsilon}{\lambda}\left(\frac{B_{1\lambda}d_{1m}}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{B_{2\lambda}d_{2m}}{\sqrt{\rho_2 A_{2\lambda}}}\right) + \frac{\varepsilon^3}{2\lambda}Q^{3\varepsilon}_{\lambda m}, \tag{18}$$

$$(P^{\varepsilon}_{\lambda m})_{21} = \varepsilon\lambda\left(B_{1\lambda}d_{1m}\sqrt{\rho_1 A_{1\lambda}} + B_{2\lambda}d_{2m}\sqrt{\rho_2 A_{2\lambda}}\right) + \frac{\varepsilon^3\lambda}{2}Q^{4\varepsilon}_{\lambda m}, \tag{19}$$

where we have used the notation

$$Q^{1\varepsilon}_{\lambda m} = \sum_{s=0}^{1}(1+(-1)^s R_\lambda)\sum_{n=2}^{\infty}\frac{\varepsilon^{2n-4}}{(2n)!}(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})^{2n},$$

$$Q^{2\varepsilon}_{\lambda m} = \sum_{s=0}^{1}\left(1+\frac{(-1)^s}{R_\lambda}\right)\sum_{n=2}^{\infty}\frac{\varepsilon^{2n-4}}{(2n)!}(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})^{2n},$$

$$Q^{3\varepsilon}_{\lambda m} = \sum_{s=0}^{1}\left(\frac{1}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{(-1)^s}{\sqrt{\rho_2 A_{2\lambda}}}\right)\sum_{n=1}^{\infty}\frac{\varepsilon^{2n-2}}{(2n+1)!}(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})^{2n+1},$$

$$Q^{4\varepsilon}_{\lambda m} = \sum_{s=0}^{1}\left(\sqrt{\rho_1 A_{1\lambda}} + (-1)^s\sqrt{\rho_2 A_{2\lambda}}\right)\sum_{n=1}^{\infty}\frac{\varepsilon^{2n-2}}{(2n+1)!}(B_{1\lambda}d_{1m} + (-1)^s B_{2\lambda}d_{2m})^{2n+1}.$$

Substituting (16)-(19) into (8) or (9) and setting $i = 1$, $j = 2$, it can be easily shown that

$$(P^{\varepsilon}_{\lambda})_{12} = \frac{\varepsilon}{\lambda}\sum_{m=0}^{M-1}\left(\frac{B_{1\lambda}d_{1m}}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{B_{2\lambda}d_{2m}}{\sqrt{\rho_2 A_{2\lambda}}}\right) + \varepsilon^2 D^{\varepsilon}_{1\lambda},$$

where $\lim_{\varepsilon\to 0} D^{\varepsilon}_{1\lambda} < \infty$ for every finite value of λ. Therefore, after dividing by ε, equation (11) takes the form

$$\frac{1}{\lambda}\sum_{m=0}^{M-1}\left(\frac{B_{1\lambda}d_{1m}}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{B_{2\lambda}d_{2m}}{\sqrt{\rho_2 A_{2\lambda}}}\right) + \varepsilon D^{\varepsilon}_{1\lambda} = 0.$$

Let us pass to the limit in the last equation as $\varepsilon \to 0$, assuming that the sequence of roots $\lambda(\varepsilon)$ is bounded and $\lambda(\varepsilon) \to \theta_0 < \infty$. Since

$$\sum_{m=0}^{M-1}\left(\frac{B_{1\lambda}d_{1m}}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{B_{2\lambda}d_{2m}}{\sqrt{\rho_2 A_{2\lambda}}}\right) = \frac{B_{1\lambda}d_1}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{B_{2\lambda}d_2}{\sqrt{\rho_2 A_{2\lambda}}} = \lambda\left(\frac{d_1}{A_{1\lambda}} + \frac{d_2}{A_{2\lambda}}\right),$$

it follows that the limit point θ_0 is a root of the equation

$$A_{1\theta}d_2 + A_{2\theta}d_1 = 0,$$

which can be rewritten as

$$b_{12}\theta + a_{12} = \sum_{s=1}^{2} R_s d_{3-s}\left(k_s + \frac{2}{3}\right)\sum_{n=1}^{N_s}\frac{v_n^{(s)}}{\theta+\gamma_n^{(s)}}, \tag{20}$$

where $b_{12} = b_1 d_2 + b_2 d_1$ and $a_{12} = a_1 d_2 + a_2 d_1$. Obviously, if both phases are elastic, then equation (20) loses its meaning. In this case, there are no finite limits of sequences of roots for equation (11) as $\varepsilon \to 0$.

Further, substituting (16)–(19) into (8) or (9) and setting $i = j = 1$ and then $i = j = 2$, we get

$$(P_\lambda^\varepsilon)_{11} = 1 + \frac{\varepsilon^2}{2}\sum_{m=0}^{M-1}\left(B_{1\lambda}^2 d_{1m}^2 + B_{2\lambda}^2 d_{2m}^2 + 2R_\lambda B_{1\lambda}B_{2\lambda}d_{1m}d_{2m}\right) +$$
$$+\varepsilon^2\sum_{m=0}^{M-2}\sum_{n=m+1}^{M-1}\left(B_{1\lambda}d_{1m}\sqrt{\rho_1 A_{1\lambda}} + B_{2\lambda}d_{2m}\sqrt{\rho_2 A_{2\lambda}}\right)\left(\frac{B_{1\lambda}d_{1n}}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{B_{2\lambda}d_{2n}}{\sqrt{\rho_2 A_{2\lambda}}}\right) + \varepsilon^4 P_{2\lambda}^\varepsilon,$$

$$(P_\lambda^\varepsilon)_{22} = 1 + \frac{\varepsilon^2}{2}\sum_{m=0}^{M-1}\left(B_{1\lambda}^2 d_{1m}^2 + B_{2\lambda}^2 d_{2m}^2 + \frac{2}{R_\lambda}B_{1\lambda}B_{2\lambda}d_{1m}d_{2m}\right) +$$
$$+\varepsilon^2\sum_{m=0}^{M-2}\sum_{n=m+1}^{M-1}\left(\frac{B_{1\lambda}d_{1m}}{\sqrt{\rho_1 A_{1\lambda}}} + \frac{B_{2\lambda}d_{2m}}{\sqrt{\rho_2 A_{2\lambda}}}\right)\left(B_{1\lambda}d_{1n}\sqrt{\rho_1 A_{1\lambda}} + B_{2\lambda}d_{2n}\sqrt{\rho_2 A_{2\lambda}}\right) + \varepsilon^4 P_{3\lambda}^\varepsilon,$$

where $\lim_{\varepsilon\to 0} P_{2\lambda}^\varepsilon < \infty$ and $\lim_{\varepsilon\to 0} P_{3\lambda}^\varepsilon < \infty$ for every finite value of λ. Therefore,

$$(P_\lambda^\varepsilon)_{11} + (P_\lambda^\varepsilon)_{22} = 2 + \varepsilon^2 B_{1\lambda}^2\left(\sum_{m=0}^{M-1} d_{1m}^2 + 2\sum_{m=0}^{M-2}\sum_{n=m+1}^{M-1} d_{1m}d_{1n}\right) +$$
$$+\varepsilon^2 B_{2\lambda}^2\left(\sum_{m=0}^{M-1} d_{2m}^2 + 2\sum_{m=0}^{M-2}\sum_{n=m+1}^{M-1} d_{2m}d_{2n}\right) +$$
$$+\varepsilon^2 B_{1\lambda}B_{2\lambda}\left(R_\lambda + \frac{1}{R_\lambda}\right)\left(\sum_{m=0}^{M-1} d_{1m}d_{2m} + \sum_{m=0}^{M-2}\sum_{n=m+1}^{M-1}(d_{1m}d_{2n} + d_{1n}d_{2m})\right) + \varepsilon^4 P_{4\lambda}^\varepsilon =$$
$$= 2 + \varepsilon^2\left(B_{1\lambda}^2 d_1^2 + B_{2\lambda}^2 d_2^2 + \left(R_\lambda + \frac{1}{R_\lambda}\right)B_{1\lambda}B_{2\lambda}d_1 d_2\right) + \varepsilon^4 P_{4\lambda}^\varepsilon, \quad P_{4\lambda}^\varepsilon = P_{2\lambda}^\varepsilon + P_{3\lambda}^\varepsilon,$$

where we have used the relation

$$\left(\sum_{m=0}^{M-1} d_{1m}\right)\left(\sum_{m=0}^{M-1} d_{2m}\right) = \sum_{m=0}^{M-1} d_{1m}d_{2m} + \sum_{m=0}^{M-2}\sum_{n=m+1}^{M-1}(d_{1m}d_{2n} + d_{1n}d_{2m}).$$

Substituting (15) together with the above series expansion for $(P_\lambda^\varepsilon)_{11} + (P_\lambda^\varepsilon)_{22}$ into equation (12) and then dividing by ε^2, we obtain

$$B_{1\lambda}^2 d_1^2 + B_{2\lambda}^2 d_2^2 + \left(R_\lambda + \frac{1}{R_\lambda}\right)B_{1\lambda}B_{2\lambda}d_1 d_2 + \left(\frac{\pi k}{L}\right)^2 +$$
$$+\varepsilon^2\left(P_{4\lambda}^\varepsilon - 2\sum_{n=2}^{\infty}\frac{\varepsilon^{2n-4}}{(2n)!}\left(\frac{\pi k}{L}\right)^{2n}\right) = 0, \quad k = 1, \ldots, N-1.$$

For a fixed k, we pass to the limit in this equation as $\varepsilon \to 0$, assuming that the sequence of its roots $\lambda_k(\varepsilon)$ is bounded and $\lambda_k(\varepsilon) \to \lambda_k < \infty$. As a result, we see that the limit point λ_k is a root of the equation

$$B_{1\lambda}^2 d_1^2 + B_{2\lambda}^2 d_2^2 + \left(R_\lambda + \frac{1}{R_\lambda}\right) B_{1\lambda} B_{2\lambda} d_1 d_2 + \left(\frac{\pi k}{L}\right)^2 = 0, \quad k = 1, \ldots, N-1,$$

which can be rewritten as

$$\lambda^2 \left(\frac{\rho_1 d_1^2}{A_{1\lambda}} + \frac{\rho_2 d_2^2}{A_{2\lambda}} + d_1 d_2 \left(\frac{\rho_1}{A_{2\lambda}} + \frac{\rho_2}{A_{1\lambda}}\right)\right) + \left(\frac{\pi k}{L}\right)^2 = 0, \quad k = 1, \ldots, N-1.$$

The last equation, in turn, is equivalent to the rational equation

$$\begin{gathered} b_{12}\lambda^3 + \left(a_{12} - \sum_{s=1}^{2} R_s d_{3-s} \left(k_s + \frac{2}{3}\right) \sum_{n=1}^{N_s} \frac{v_n^{(s)}}{\lambda + \gamma_n^{(s)}}\right) \lambda^2 + \\ + C_k \prod_{s=1}^{2} \left(a_s + b_s\lambda - R_s \left(k_s + \frac{2}{3}\right) \sum_{n=1}^{N_s} \frac{v_n^{(s)}}{\lambda + \gamma_n^{(s)}}\right) = 0, \quad k = 1, \ldots, N-1, \end{gathered} \tag{21}$$

where $C_k = \pi^2 k^2/(\rho_0 L^2)$, $\rho_0 = \rho_1 d_1 + \rho_2 d_2$. In particular, if $d^{(1)}(t) = d^{(2)}(t) = 0$ then equation (21) becomes

$$b_{12}\lambda^3 + (a_{12} + b_1 b_2 C_k)\lambda^2 + (a_1 b_2 + a_2 b_1) C_k \lambda + a_1 a_2 C_k = 0. \tag{22}$$

It is important to emphasize that the coefficients of equations (20) and (21), and, therefore, all finite limits of sequences $\lambda(\varepsilon) \in S_\varepsilon$ depend on the volume fractions d_1 and d_2 of the phases, but are independent of the number M and the locations $x_1 = \varepsilon h_1$, $x_1 = \varepsilon h_2$, ..., $x_1 = \varepsilon h_{2M-1}$ of the layers boundaries inside the period Y_ε.

We end this section by concluding that the spectrum S_ε converges in the sense of Hausdorff to the union of roots of equations (20) and (21). Indeed, we just proved that the second condition of this convergence is satisfied. On the other hand, it follows from Rouche's theorem that for any root of equation (20) there exists a sequence of roots of equations (11) which converges to it as $\varepsilon \to 0$. Similarly, for any root of equation (21) with a fixed k there exists a sequence of roots of the k-th equations (12) which converges to it as $\varepsilon \to 0$. This means that the first condition of the above-mentioned convergence is also satisfied.

4. Homogenized problem and its spectrum

In order to clarify the meaning of the roots of equations (20) and (21), we consider the corresponding homogenized problem constructed as $\varepsilon \to 0$, which describes one-dimensional oscillations of a homogeneous solid material along the Ox_1 axis. It is known from [11] that this problem has the form

$$\begin{gathered} \rho_0 \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_1}{\partial x_1} + f_1(x_1, t), \quad x_1 \in (0, L), \quad t > 0, \\ u_1(0,t) = u_1(L,t) = 0, \quad t > 0; \quad u_1(x_1, 0) = \frac{\partial u_1}{\partial x_1}(x_1, 0) = 0, \quad x_1 \in (0, L) \end{gathered} \tag{23}$$

with

$$\sigma_1 = \alpha_1 \frac{\partial u_1}{\partial x_1} + \beta_1 \frac{\partial^2 u_1}{\partial x_1 \partial t} - \left(\sum_{n=1}^{N_3} q_n e^{-\xi_n t}\right) * \frac{\partial u_1}{\partial x_1},$$

where $\xi_1, \ldots, \xi_{N_3}$ are the roots of the equation

$$b_{12}\xi - a_{12} = \sum_{s=1}^{2} R_s d_{3-s}\left(k_s + \frac{2}{3}\right)\sum_{n=1}^{N_s}\frac{v_n^{(s)}}{\xi - \gamma_n^{(s)}}, \tag{24}$$

and the coefficients q_n are defined by

$$q_n = d_2 p_n\left(a_2 - a_1 - (b_2 - b_1)\xi_n + \sum_{s=1}^{2}(-1)^s R_s\left(k_s + \frac{2}{3}\right)\sum_{n=1}^{N_s}\frac{v_n^{(s)}}{\xi_n - \gamma_n^{(s)}}\right).$$

If $b^{(1)} + b^{(2)} \neq 0$, then

$$N_3 = R_1N_1 + R_2N_2 + 1, \quad \alpha_1 = \frac{a_1 b_2^2 d_1 + a_2 b_1^2 d_2}{b_{12}^2}, \quad \beta_1 = \frac{b_1 b_2}{b_{12}},$$

while $p_1, \ldots, p_{N_3}$ is the solution of the linear system

$$R_1\left(\sum_{n=1}^{N_3}\frac{p_n}{\xi_n - \gamma_{n_1}^{(1)}} + \frac{b_2 d_1}{b_{12} d_2}\right) = 0, \quad R_2\left(\sum_{n=1}^{N_3}\frac{p_n}{\xi_n - \gamma_{n_2}^{(2)}} - \frac{b_1}{b_{12}}\right) = 0,$$

$$\sum_{n=1}^{N_3} p_n = \frac{d_1}{b_{12}^2}(b_1 a_2 - b_2 a_1), \quad n_s = 1, \ldots, N_s, \quad s = 1, 2.$$

If $b^{(1)} + b^{(2)} = 0$, then

$$N_3 = R_1N_1 + R_2N_2, \quad \alpha_1 = \frac{a_1 a_2}{a_{12}}, \quad \beta_1 = 0,$$

while $p_1, \ldots, p_{N_3}$ is the solution of the linear system

$$R_1\left(\sum_{n=1}^{N_3}\frac{p_n}{\xi_n - \gamma_{n_1}^{(1)}} + \frac{a_2 d_1}{a_{12} d_2}\right) = 0, \quad R_2\left(\sum_{n=1}^{N_3}\frac{p_n}{\xi_n - \gamma_{n_2}^{(2)}} - \frac{a_1}{a_{12}}\right) = 0,$$

$$n_s = 1, \ldots, N_s, \quad s = 1, 2.$$

Applying the Laplace transform to problem (23) with $f_1(x_1, t) \equiv 0$, we obtain

$$\begin{gathered}\rho_0\lambda^2 u_{1\lambda} = \left(\alpha_1 + \beta_1\lambda - \sum_{n=1}^{N_3}\frac{q_n}{\lambda + \xi_n}\right)\frac{d^2 u_{1\lambda}}{dx_1^2}, \quad x_1 \in (0, L), \\ u_{1\lambda}(0) = u_{1\lambda}(L) = 0.\end{gathered} \tag{25}$$

By definition, the spectrum S of one-dimensional eigenoscillations of the homogenized material along the Ox_1 axis is the set of eigenvalues of the spectral problem (25), i.e., $S = \{\lambda \in \mathbb{C}:\ u_{1\lambda}(x_1) \not\equiv 0\}$. It is known (see, for example, [7, 8]) that the set S has the following structure:

$$S = \bigcup_{n=1}^{N_3+2}\{\lambda_{nk}\}_{k=1}^{\infty}, \quad \lambda_{ik} \in \mathbb{R}, \quad \lambda_{ik} < 0, \quad i = 1, \ldots, N_3,$$

$$\lambda_{jk} \in \mathbb{C}, \quad \mathrm{Re}\lambda_{jk} \leqslant 0, \quad j = N_3 + 1, N_3 + 2,$$

where $\lambda_{1k}, \ldots, \lambda_{(N_3+2)k}$ are the roots of the equations

$$\lambda^2 + \beta_1 C_k \lambda + \alpha_1 C_k = C_k \sum_{n=1}^{N_3} \frac{q_n}{\lambda + \xi_n}. \tag{26}$$

We claim that equations (21) and (26) are equivalent one another. Indeed, in the case of two elastic phases, (26) becomes the quadratic equation

$$\lambda^2 + \alpha_1 C_k = 0,$$

which is the same equation as (21) with $b_1 = b_2 = b_{12} = 0$ and $R_1 = R_2 = 0$.

Further, if $b^{(1)} + b^{(2)} \neq 0$ and $d^{(1)}(t) = d^{(2)}(t) = 0$, then $N_3 = 1$ and equation (26) can be reduced to the cubic equation

$$\lambda^3 + (\xi_1 + \beta_1 C_k)\lambda^2 + (\alpha_1 + \beta_1 \xi_1) C_k \lambda + (\alpha_1 \xi_1 - q_1) C_k = 0. \tag{27}$$

Substituting

$$\xi_1 = \frac{a_{12}}{b_{12}}, \quad p_1 = \frac{d_1(b_1 a_2 - b_2 a_1)}{b_{12}^2},$$

$$q_1 = d_2 p_1 (a_2 - a_1 + (b_2 - b_1)\xi_1) = \frac{d_1 d_2}{b_{12}^3}(a_1 b_2 - a_2 b_1)^2,$$

and the above expressions for α_1 and β_1 into (27), after straightforward calculations we arrive exactly at equation (22).

In the case when at least one of the tensors $d^{(1)}(t)$ and $d^{(2)}(t)$ is nonzero, the equivalence of the rational equations (21) and (26) is proved by applying Vieta's formulas for coefficients and roots of the algebraic equations corresponding to them (see for details [8]).

It can be easily seen that the roots of equations (20) and (24) differ from each other only in the sign. Since the roots $\xi_1, \ldots, \xi_{N_3}$ of (24) are positive real numbers (when both phases are not elastic), the roots of (20) are negative real numbers [6–8]. Therefore, taking into account the results of the previous section, we conclude that $S_\varepsilon \to S \cup V$ in the sense of Hausdorff, where we denote $V = \{-\xi_1, \ldots, -\xi_{N_3}\}$. In particular, V is an empty set when both phases are elastic.

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Спектр одномерных собственных колебаний двухфазных слоистых композитов

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Аннотация. Изучен спектр одномерных собственных колебаний двухфазных композитов с периодической структурой. Их фазами являются изотропные упругие или вязкоупругие материалы, а период состоит из $2M$ чередующихся плоских слоев первой и второй фаз. Выведены уравнения, корни которых образуют спектр, и исследовано их асимптотическое поведение. В частности, установлено, что все конечные пределы последовательностей точек спектра зависят от объемных долей фаз и не зависят от числа M и расстояний между границами слоев внутри периода.

Ключевые слова: спектр, собственные колебания, слоистый композит.